A New Expression of the Fractional Fourier Transformation

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Abstract With the help of su(2) algebra technique, a new equivalent form of the fractional Fourier transformation is given. Two examples are illustrated for their physical application in quantum optics.

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There are many literatures concentrated on the fractional Fourier transformation (FFT).^[1] The FFT is an extension of the ordinary Fourier transform, and depends on a parameter θ , which can be interpreted as a rotation by an angle θ in the time-frequency plane or in the spacefrequency plane. The FFT is very useful for wave propagation problems and signal processing, etc. Its definition is

$$\Psi_{\theta}(p) = \int \mathrm{d}q K_{\theta}(p,q) \Phi(q) \,, \tag{1}$$

where $K_{\theta}(p,q)$ is called a kernel,^[1] and is a rather complex expression. When $\theta = n\pi$, the kernel is reduced to a δ -function. But the kernel is continuous in the generalized function sense, even when $\theta = n\pi$.^[1]

In this paper, we will show that the transform (1) can be rewritten as a simpler form. The form is convenient to be used to the quantum optics. Now, considering the following transformation

$$|\Psi\rangle = \exp(-iN\theta)|\Phi\rangle, \qquad \theta = real,$$
 (2)

where $N = a^{\dagger}a$ is a photo-number operator. According to the representation theory of quantum mechanics, equation (2) may be rewritten as

$$\Psi_{\theta}(p) = \int \mathrm{d}q \langle p | \exp(-\mathrm{i}N\theta) | q \rangle \Phi(q) , \qquad (3)$$

here

$$\hat{q} = \frac{1}{\sqrt{2}} (a + a^{\dagger}), \qquad \hat{p} = \frac{1}{i\sqrt{2}} (a - a^{\dagger}), \qquad (4a)$$

$$\hat{p}|p\rangle = p|p\rangle, \qquad \qquad \hat{q}|q\rangle = q|q\rangle.$$
(4b)

Comparing Eq. (3) with Eq. (1), we have $K_{\theta}(p,q) = \langle p | \exp(-iN\theta) | q \rangle$. Let

$$Z_{+} = \frac{q^2}{2}, \quad Z_{-} = \frac{p^2}{2}, \quad Z_{0} = \left(\frac{\mathrm{i}\hat{p}\hat{q} - 1/2}{2}\right), \quad (5)$$

from Eq. (4), we can easily show that the operators Z_+ , Z_- and Z_0 satisfy su(2) commutation relations

$$[Z_+, Z_-] = 2Z_0, \qquad [Z_0, Z_\pm] = \pm Z_\pm.$$
 (6)

With the help of su(2) algebra technique, we obtain

$$\begin{split} \langle p|\exp(Z_{-}+Z_{+})\theta|q\rangle &= \langle p|\exp[\operatorname{i}\operatorname{tg}\theta Z_{-}]\exp[-\ln\cos\theta Z_{0}]\exp[\operatorname{i}\operatorname{tg}\theta Z_{+}]|q\rangle \\ &= \sqrt{\frac{1}{\cos\theta}}\exp\left[\operatorname{i}\operatorname{tg}\theta\left(\frac{p^{2}+q^{2}}{2}\right) + \operatorname{i}\left(1-\frac{1}{\cos\theta}\right)pq\right]\langle p|q\rangle \,. \end{split}$$

Because $\langle p|q\rangle = e^{-ipq}/\sqrt{2\pi}$, we have

$$\langle p | \exp(iN\theta) | q \rangle = \sqrt{\frac{e^{i\theta}}{2\pi\cos\theta}} \exp\left[i\operatorname{tg}\theta\left(\frac{p^2+q}{2}\right) - \frac{ipq}{\cos\theta}\right].$$
 (7)

Comparing Eq. (7) with the ordinary form of FFT, we discover that they are the same when $\theta \to \pi/2 - \theta$. Obviously, it is reduced into $\langle p|q \rangle$ when $\theta \to 0$. Consequently equation (3) turned into the common Fourier transform.

The following two examples illustrate the potentialities of the FFT in quantum optics.

First we consider the squeezed state. As is well known that the squeezed state is very important in many problems and applications.^[2] It is defined by

$$|\Phi\rangle = |\lambda\rangle = \exp\left[\frac{\lambda^* a a - \lambda a^{\dagger} a^{\dagger}}{2}\right]|0\rangle, \qquad (8)$$

 λ is a parameter, $|0\rangle$ is the photon vacuum state. According to Eq. (2a), the FFT image of $|\lambda\rangle$ is

$$|\Psi_{\lambda}\rangle = \exp(-iN\theta)|\lambda\rangle.$$
⁽⁹⁾

What is the squeezing effect of $|\Psi_{\lambda}\rangle$? To answer this ques-

tion, we define two operators as follows:

$$Y_1 + iY_2 = e^{-i\phi} \frac{\hat{q} + i\hat{p}}{\sqrt{2}} = e^{-i\phi} a,$$
 (10a)

$$Y_1 - iY_2 = e^{i\phi} \frac{q - ip}{\sqrt{2}} = e^{i\phi} a^{\dagger}.$$
 (10b)

It is easy to show $[Y_1, Y_2] = i/2$. It means that the uncertainty relation of an arbitrary state is $\Delta Y_1 \Delta Y_2 \ge 1/16$.

By making use of the following equalities

$$\exp(-iN\theta)a\exp(iN\theta) = \exp(i\theta)a, \qquad (11a)$$

$$\exp(-iN\theta)a^{\dagger}\exp(iN\theta) = \exp(-i\theta)a^{\dagger}, \qquad (11b)$$

and the well-known expectation values of $a^{\dagger}(a)$ for $|\lambda\rangle$, we have

$$\langle \Psi | Y_1 | \Psi \rangle = \langle \Psi | Y_2 | \Psi \rangle = 0,$$
 (12a)

$$\langle \Psi | Y_1^2 | \Psi \rangle = \frac{1}{4} [\sin^2 \theta \, \mathrm{e}^{2r} + \cos^2 \theta \, \mathrm{e}^{-2r}], \qquad (12\mathrm{b})$$

$$\langle \Psi | Y_2^2 | \Psi \rangle = \frac{1}{4} [\cos^2 \theta \, \mathrm{e}^{2r} + \sin^2 \theta \, \mathrm{e}^{-2r}], \qquad (12c)$$

here $\lambda = r e^{i\phi}$. Consequently,

$$\Delta Y_1 = \frac{1}{4} [\sin^2 \theta \, \mathrm{e}^{2r} + \cos^2 \theta \, \mathrm{e}^{-2r}], \qquad (13a)$$

$$\Delta Y_2 = \frac{1}{4} [\cos^2 \theta \, \mathrm{e}^{2r} + \sin^2 \theta \, \mathrm{e}^{-2r}] \,. \tag{13b}$$

From Eq. (13), we can obtain easily

$$\Delta Y_1 \Delta Y_2 = \frac{1}{16} [\cos^2(2\theta) + \sin^2(2\theta) \cosh^2(2r)]. \quad (14)$$

Obviously, when $\theta = n\pi$, $|\Psi_{\lambda}\rangle$ is an ideal squeezed state, that is to say, Y_1 is squeezed, but Y_2 is not. Generally, Y_1 and Y_2 have no squeezing effects.

Secondly, let us consider the FFT of the Wigner operator. It is well known that the Wigner operator is defined by

$$\Delta = \pi^{-2} \int d^2 \beta D(\beta) \exp(\beta^* \alpha - \beta \alpha^*), \qquad (15)$$

here $D(\beta)|0\rangle$ is the coherent state in quantum optics. Let the FFT of the Δ and ρ (state density operator) operators be

$$\Delta \to \Delta' = e^{-i\theta N} \Delta e^{i\theta N} , \qquad (16a)$$

$$\rho \to \rho' = e^{-i\theta N} \rho e^{i\theta N} , \qquad (16b)$$

then the Wigner function $W = \text{Tr}(\rho \Delta)$ is invariant for the FFT.

Now let α and β in Eq. (15) is rewritten as

$$\alpha = \frac{1}{\sqrt{2}}(q + ip), \qquad \beta = \frac{1}{\sqrt{2}}(q_0 + ip_0), \qquad (17)$$

then equation (15) becomes

$$\Delta = \frac{1}{\pi} \int \mathrm{d}q_0 \exp(\mathrm{i}pq_0) \Big| q + \frac{q_0}{2} \Big\rangle \Big\langle q - \frac{q_0}{2} \Big| \,. \tag{18}$$

Consequently

$$\begin{split} \Delta' &= \frac{1}{\pi} \int \mathrm{d}q_0 \,\mathrm{e}^{\mathrm{i}pq_0} \,\mathrm{e}^{-\mathrm{i}\theta N} \left| q + \frac{q_0}{2} \right\rangle \left\langle q - \frac{q_0}{2} \right| \,\mathrm{e}^{\mathrm{i}\theta N} \\ &= \frac{1}{\pi} \int \mathrm{d}q_0 \,\mathrm{d}p' \,\mathrm{d}p'' \,\mathrm{e}^{\mathrm{i}pq_0} \left\langle p' \right| \,\mathrm{e}^{-\mathrm{i}\theta N} \left| q + \frac{q_0}{2} \right\rangle \left\langle q - \frac{q_0}{2} \right| \,\mathrm{e}^{\mathrm{i}\theta N} \left| p'' \right\rangle \left| p' \right\rangle \left\langle p'' \right| \\ &= \frac{2}{\pi} \int \mathrm{d}p' \,\mathrm{d}p'' \,\mathrm{e}^{\mathrm{i}\mathrm{tg}\,\theta(p''^2 - p'^2)/2 + \mathrm{i}q(p'' - p')/2\cos\theta} \delta(p'' - (2p\cos\theta - 2q\sin\theta - p')) |p'\rangle \left\langle p'' \right| \end{split}$$

Let

$$p' = p\cos\theta + q\sin\theta + g/2, \qquad (19a)$$

$$p'' = p\cos\theta + q\sin\theta - q/2.$$
(19b)

Finally we obtain

$$\Delta' = \frac{1}{\pi} \int \mathrm{d}g \,\mathrm{e}^{\mathrm{i}gQ} \Big| P + \frac{g}{2} \Big\rangle \Big\langle P - \frac{g}{2} \Big| \,, \qquad (20)$$

here

$$Q = q\cos\theta + p\sin\theta, \qquad (21a)$$

$$P = -q\sin\theta + p\cos\theta. \qquad (21b)$$

References

 T. Alieva, V. Lopez, F.G. Lopez and L.B. Almeida, J. Mod. Opt. 41 (1994) 1037, and the references therein. It means that the Wigner operator has a rotation by θ in the phase plane.

From the above discussion, it is clear that our form is more convenient than the ordinary one for applications of FFT to the problems of quantum optics.

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[2] M.O. Scully and M.S. Zubairy, *Quantum Optics*, Cambridge University Press (1996); Zu-Rong YU, Progress in Physics (in Chinese) **19** (1999) 72.