 Communications in Theoretical Physics

Commun. Theor. Phys. **73** (2021) 025103 (5pp) https://doi.org/10.1088/1572-9494/abd0e4

Relativistic Landau quantization in the spiral dislocation spacetime

A V D M Maia and K Bakke

Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58051-900, João Pessoa, PB, Brazil

E-mail: kbakke@fisica.ufpb.br

Received 18 August 2020, revised 26 November 2020
Accepted for publication 6 December 2020
Published 25 January 2021

Abstract

We analyse the interaction of a relativistic electron with a uniform magnetic field in the spiral dislocation spacetime. We show that analytical solutions to the Dirac equation can be obtained, where the spectrum of energy corresponds to the relativistic Landau levels. We also analyse the influence of the spiral dislocation on the relativistic Landau levels by showing that there exists an analogue of the Aharonov–Bohm effect for bound states.

Keywords: spiral dislocation spacetime, relativistic Landau quantization, Aharonov–Bohm effect, relativistic wave equations, Dirac equation

1. Introduction

In recent decades, some materials described by the Dirac equation have drawn a great deal of attention in the literature. Examples of these materials are graphene [1–6], fullerenes [7–9] and topological insulators [10]. By using graphene as an example, quantum effects like the quantum Hall effect [11, 12], the Aharonov–Bohm effect [6, 13–15] and the Klein paradox [16, 17] have been investigated. In particular, the interaction of an electron in graphene with a uniform magnetic field can yield a spectrum of energy known as the relativistic Landau levels [13, 18–26]. From the perspective of achieving relativistic bound states, the confinement of electrons to a quantum dot or a quantum ring in a graphene layer, in the presence of a uniform magnetic field, has also been dealt with in [14, 15, 27]. There, some characteristics of this material have been analysed, such as the magnetization [14, 27]. Furthermore, quantum effects associated with the presence of topological defects in graphene have also been investigated [2, 6, 13–15, 27, 28]. It is worth noting that the description of a topological defect in a graphene layer follows the same mathematical description of topological defects in gravitation [6, 14, 18, 27, 29–74]. This geometric description of topological defects in materials described by the Dirac equation is known in the literature as the Katanaev-Volovich approach [75–78]. In this model, linear topological defects in solids can be described by using the differential geometry. The information about the strain and stress produced by the defect in the elastic medium is described by geometric quantities, such as the metric and the curvature tensor. Besides, the presence of topological defects in an elastic medium shows the possibility of finding analogue effects of the Aharonov–Bohm effect in solids [79–83].

An interesting point raised in [28, 84] is the possibility of describing an edge dislocation in graphene as a pair of pentagon–heptagon disclinations. In particular, this pair of pentagon–heptagon disclinations in graphene could be described by considering a spiral dislocation in the graphene layer [74]. Therefore, inspired by these studies of graphene, in this work we analyse a relativistic electron that interacts with a uniform magnetic field in the presence of a spiral dislocation. In the context of gravitation, this interaction can be viewed as the relativistic Landau quantization in the spiral dislocation spacetime [69, 74]. We show that analytical solutions to the Dirac equation can be obtained. Besides, we show that an analogue of the Aharonov–Bohm effect [79–83] exists.

The structure of this paper is as follows: in section 2, we introduce the line element of the spiral dislocation spacetime. Then, we analyse the interaction of a relativistic electron with a uniform magnetic field by searching for relativistic bound state solutions to the Dirac equation. In section 3, we present our conclusions.

2. Relativistic Landau quantization

Let us analyse the interaction of a relativistic electron with a uniform magnetic field in the spacetime with a spiral dislocation

[69, 74, 78]. By using the units $\hbar = 1$ and $c = 1$, therefore, the spiral dislocation spacetime is described as the line element [69, 74, 78]:

$$ds^2 = -dt^2 + dr^2 + 2\beta\, dr\, d\varphi + (\beta^2 + r^2)d\varphi^2 + dz^2, \tag{1}$$

where the constant β is the parameter related to the distortion of the topological defect. Note that the spatial part of the line element (1) describes the distortion of a circle into a spiral. In this case, the dislocation is parallel to the plane $z = 0$; hence, it corresponds to an edge dislocation in the context to the description of topological defects in solids [70, 74, 78, 85]. Even though we are dealing with a topological defect in the spacetime, we can consider the parameter β to be defined in the range $0 < \beta < 1$ in the same way as the description of topological defects in solids [70, 74, 78, 85].

Recently, we have discussed the behaviour of a Dirac particle confined to a hard-wall confining potential in the spacetime with a spiral dislocation [69]. We have shown that the Dirac equation is dealt with based on the spinor theory in curved spacetime [86]. In short, the spinors are defined in the local reference frame of the observers through a non-coordinate basis $\hat{\theta}^a = e^a_{\ \mu}(x)dx^\mu$, where the Latin indices $a, b, c = 0, 1, 2, 3$ indicate the local reference frame. Moreover, the components $e^a_{\ \mu}(x)$ are called *tetrads* and satisfy the relation [86, 87]: $g_{\mu\nu}(x) = e^a_{\ \mu}(x)e^b_{\ \nu}(x)\eta_{ab}$, where $\eta_{ab} = \text{diag}(-+++)$ is the Minkowski tensor. Note that the tetrads have an inverse, which is defined as $dx^\mu = e^\mu_{\ a}(x)\hat{\theta}^a$, and they are related through $e^a_{\ \mu}(x)e^\mu_{\ b}(x) = \delta^a_{\ b}$ and $e^\mu_{\ a}(x)e^a_{\ \nu}(x) = \delta^\mu_{\ \nu}$. Therefore, let us write the tetrads and the inverse as [69, 74]:

$$e^a_{\ \mu}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad e^\mu_{\ a}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{\beta}{r} & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

Thereby, by solving the Maurer-Cartan structure equations [87] $T^a = d\hat{\theta}^a + \omega^a_{\ b} \wedge \hat{\theta}^b$ (where the operator d corresponds to the exterior derivative, the symbol $\wedge$ means the wedge product, $T^a = T^a_{\ \mu\nu}\, dx^\mu \wedge dx^\nu$ is the torsion 2-form and $\omega^a_{\ b} = \omega_\mu{}^a{}_b(x)dx^\mu$ is the connection 1-form), we obtain [74]

$$T^1 = 2\pi\, \beta\, \delta(r)\delta(\varphi)dr \wedge d\varphi; \quad \omega_\varphi{}^2{}_1(x) = -\omega_\varphi{}^1{}_2(x) = 1, \tag{3}$$

where $\delta(r)\delta(\varphi) = \delta_2(\vec{r})$ is the two-dimensional delta function [81, 88].

Hence, we have a spacetime with the presence of torsion. As shown in [89], the information about the torsion of the spacetime can be introduced into the Dirac equation through the irreducible components of the torsion tensor. In particular, with the torsion 2-form given in equation (3), the only non-null component of the irreducible components of the torsion tensor is [74]:

$$T_\varphi = T^r_{\ \varphi r} = -T^r_{\ r\varphi} = -2\pi\, \beta\, \delta(r)\delta(\varphi), \tag{4}$$

which is a component of the trace four-vector T_μ. Observe that the trace four-vector T_μ does not couple with fermions as shown in [89]. Therefore, the trace four-vector T_μ can be introduced into the Dirac equation through a non-minimal coupling given by:

$$i\gamma^\mu\, \nabla_\mu \to i\gamma^\mu\, \nabla_\mu + \mu\, \gamma^\mu\, T_\mu, \tag{5}$$

where μ is an arbitrary non-minimal coupling parameter (dimensionless) and

$$\nabla_\mu = \partial_\mu + \frac{i}{4}\, \omega_{\mu ab}(x)\Sigma^{ab}. \tag{6}$$

Furthermore, we have that $\Sigma^{ab} = \frac{i}{2}[\gamma^a, \gamma^b]$, where the γ^a matrices are defined in the local reference frame, i.e. they are the Dirac matrices in the Minkowski spacetime [90]:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \gamma^i = \gamma^0\, \hat{\alpha}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}; \quad \Sigma^i = \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \tag{7}$$

Observe that $\vec{\Sigma}$ is the spin vector and the matrices σ^i are the Pauli matrices, which in turn satisfies the relation $(\sigma^i\, \sigma^j + \sigma^j\, \sigma^i) = 2\, \eta^{ij}$.

From now on, let us consider the presence of a uniform magnetic field $\vec{B} = B_0\, \hat{z}$. Then, we can write the electromagnetic four-vector potential in the local reference frame of the observers as $A_2 = B_0\, r/2$. Thereby, in the presence of the uniform magnetic field, the covariant form of the Dirac equation is

$$i\gamma^\mu\, \nabla_\mu\psi + \mu\, \gamma^\mu\, T_\mu\psi - q\, \gamma^\mu\, A_\mu\, \psi = m\, \psi. \tag{8}$$

Since we are interested in the region $r \neq 0$, the contribution associated with T_μ can be neglected. Hence, by using the tetrads field (2), the Dirac equation (8) becomes

$$i\frac{\partial\psi}{\partial t} = m\gamma^0\psi - i\hat{\alpha}^1\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)\psi - i\frac{\hat{\alpha}^2}{r}\left(\frac{\partial}{\partial\varphi} - \beta\frac{\partial}{\partial r}\right)\psi - i\hat{\alpha}^3\frac{\partial\psi}{\partial z} + \frac{q\, B_0\, r}{2}\, \hat{\alpha}^2\psi. \tag{9}$$

We can write the solution to the Dirac equation (9) in the form

$$\psi = e^{-i\mathcal{E}t}\begin{pmatrix} \phi \\ \xi \end{pmatrix}, \tag{10}$$

where $\phi = \phi(r, \varphi, z)$ and $\xi = \xi(r, \varphi, z)$ are two-spinors. Then, by substituting (10) into the Dirac equation (9), we obtain two coupled equations of ϕ and ξ. The first coupled equation is

$$(\mathcal{E} - m)\phi = \left[-i\sigma^1\frac{\partial}{\partial r} - \frac{i\sigma^1}{2r} + i\frac{\beta\, \sigma^2}{r}\frac{\partial}{\partial r} - \frac{i\sigma^2}{r}\frac{\partial}{\partial\varphi} - i\sigma^3\frac{\partial}{\partial z} + \frac{q\, B_0\, r}{2}\, \sigma^2\right]\xi. \tag{11}$$

The second coupled equation is

$$(\mathcal{E} + m)\xi = \left[-i\sigma^1\frac{\partial}{\partial r} - \frac{i\sigma^1}{2r} + i\frac{\beta\, \sigma^2}{r}\frac{\partial}{\partial r} - \frac{i\sigma^2}{r}\frac{\partial}{\partial\varphi} - i\sigma^3\frac{\partial}{\partial z} + \frac{q\, B_0\, r}{2}\, \sigma^2\right]\phi. \tag{12}$$

Therefore, by eliminating ξ in equation (12) and by substituting it into equation (11), we obtain the following second-order differential equation:

$$(\mathcal{E}^2 - m^2)\phi = -\left[1 + \frac{\beta^2}{r^2}\right]\frac{\partial^2\phi}{\partial r^2} - \left[\frac{1}{r} - \frac{\beta^2}{r^3} + \frac{\mathrm{i}\,\beta\,\sigma^3}{r^2} - \mathrm{i}q\,B_0\,\beta\right]\frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\mathrm{i}\sigma^3}{r^2}\frac{\partial\phi}{\partial\varphi} + \frac{1}{4r^2}\phi + \frac{\mathrm{i}\beta\,\sigma^3}{2r^3}\phi - \frac{\beta}{r^3}\frac{\partial\phi}{\partial\varphi} + \frac{2\beta}{r^2}\frac{\partial^2\phi}{\partial\varphi\partial r} - \frac{\partial^2\phi}{\partial z^2} + \frac{qB_0\,\sigma^3}{2}\phi - \mathrm{i}qB_0\frac{\partial\phi}{\partial\varphi} + \frac{\mathrm{i}qB_0}{2}\frac{\beta}{r}\phi + \frac{q^2B_0^2\,r^2}{4}. \quad (13)$$

In search of a solution to equation (13), we need to observe that ϕ is an eigenfunction of σ^3, where $\sigma^3\phi = \pm\phi = s\,\phi$ $(s = \pm 1)$. Besides, this solution can be written in terms of the eigenvalues of the z-component of the total angular momentum and the linear momentum operators as

$$\phi(r,\,\varphi,\,z) = \mathrm{e}^{\mathrm{i}(l+1/2)\varphi + \mathrm{i}kz}\,u(r), \quad (14)$$

where $l = 0, \pm 1, \pm 2, \pm 3, \pm 4\ldots$ and k is a constant. Henceforth, we simplify our discussion by taking $k = 0$. In this way, by substituting the solution (14) into equation (13), we obtain the radial equation

$$\left(1 + \frac{\beta^2}{r^2}\right)u'' + \left(\frac{1}{r} - \frac{\beta^2}{r^3} - \mathrm{i}\frac{2\beta\,\nu}{r^2} - \mathrm{i}m\,\omega\,\beta\right)u' - \frac{\nu^2}{r^2}\,u + \mathrm{i}\frac{\beta\,\nu}{r^3}\,u - \frac{\mathrm{i}\,m\,\omega\,\beta}{2r}u - \frac{m^2\omega^2\,r^2}{4}u + \tau\,u = 0, \quad (15)$$

where we have defined the parameters

$$\nu = l + \frac{1}{2}(1 - s); \quad \omega = \frac{q\,B_0}{m}; \quad \tau = \mathcal{E}^2 - m^2 - m\,\omega\,\nu - s\,m\,\omega. \quad (16)$$

Let us search for a solution to the radial equation (15); therefore, let us take [70, 85]

$$u(r) = \exp\left(\mathrm{i}\left[\nu - \frac{m\omega\beta^2}{2}\right]\tan^{-1}\left(\frac{r}{\beta}\right)\right) \times \mathrm{e}^{\mathrm{i}\,m\omega\,\beta\,r/2} \times f(r), \quad (17)$$

where $f(r)$ is an unknown function. Then, by substituting the radial wave function (17) into equation (15), we obtain the second-order differential equation

$$\left(1 + \frac{\beta^2}{r^2}\right)f'' + \left(\frac{1}{r} - \frac{\beta^2}{r^3}\right)f' - \frac{(\nu^2 - m\omega\nu\beta^2)}{(r^2 + \beta^2)}f - \frac{m^2\omega^2\,r^4}{4(r^2 + \beta^2)}f + \tau\,f = 0. \quad (18)$$

Further, let us define $x = \frac{m\,\omega}{2}(r^2 + \beta^2)$, and thus, we can rewrite equation (18) in the form

$$x\,f'' + f' - \frac{\zeta^2}{4\,x}f - \frac{x}{4}f + \bar{\tau}\,f = 0, \quad (19)$$

where

$$\zeta = \nu - \frac{m\omega\,\beta^2}{2}; \quad \bar{\tau} = \frac{1}{2m\omega}\left[\tau + \frac{m^2\omega^2\beta^2}{2}\right]. \quad (20)$$

Before going further, we must observe that when $r \to \infty$, then $x \to \infty$. However, when $r = 0$, we have $x = \frac{m\,\omega}{2}\,\beta^2$. Since we have considered the parameter β to be defined in the range $0 < \beta < 1$, therefore, we can consider $\beta^2 \ll 1$. Thereby, when $r \to 0$ we can consider x to be very small, and thus, we can assume that the wave function vanishes when $r \to 0$, without loss of generality [70]. Therefore, with the purpose of having the radial wave function well behaved at $r \to \infty$ and $r \to 0$, we can write a solution to equation (19) in the form

$$f(x) = x^{|\zeta|/2}\,e^{-x/2}\,{}_1F_1\left(\frac{|\zeta|}{2} + \frac{1}{2} - \bar{\tau},\,|\zeta| + 1;\,x\right), \quad (21)$$

where ${}_1F_1\left(\frac{|\zeta|}{2} + \frac{1}{2} - \bar{\tau},\,|\zeta| + 1;\,x\right)$ is the confluent hypergeometric function [88, 91]. The asymptotic behaviour of a confluent hypergeometric function for large values of its argument is given by [91]

$${}_1F_1(a,\,b;\,x) \approx \frac{\Gamma(b)}{\Gamma(a)}\,\mathrm{e}^x\,x^{a-b}[1 + \mathcal{O}(|x|^{-1})], \quad (22)$$

Therefore, it diverges when $x \to \infty$. In search of bound state solutions to the Dirac equation, we must impose that $a = -n$ $(n = 0, 1, 2, 3,\ldots)$. With this condition, the confluent hypergeometric function becomes well behaved when $x \to \infty$. With $a = \frac{|\zeta|}{2} + \frac{1}{2} - \bar{\tau}$, hence, we obtain

$$\mathcal{E}_{n,l} = \pm\sqrt{m^2 + 2m\omega\left[n + \frac{|\zeta|}{2} + \frac{\zeta}{2} + \frac{1}{2}(1 - s)\right]}. \quad (23)$$

The spectrum of energy (23) stems from the interaction of a relativistic electron with a uniform magnetic field. Therefore, it corresponds to the relativistic Landau levels [13, 19, 20] in the spiral dislocation spacetime. The effects of torsion of this spacetime yield the presence of the effective angular momentum ζ in the relativistic energy levels (23) even though no interaction between the electron and the topological defect exists. This kind of contribution is analogous to that raised by Peshkin and Tonomura [80]. By considering a point charge that moves in a circular ring of radius R in the presence of a long solenoid of radius $r_0 < R$, concentric to the ring, Peshkin and Tonomura [80] showed that the angular momentum quantum number is modified by $\ell_{\text{eff}} = l - e\Phi/2\pi$ (where Φ is the magnetic flux through the solenoid and e is the electric charge). In addition, they showed that the eigenvalues of energy are determined by $\ell_{\text{eff}} = l - e\Phi/2\pi$ even though no interaction between the point charge and the magnetic field inside the solenoid exists.

This quantum effect characterized by the influence of the magnetic flux on the eigenvalues of energy is known as the Aharonov–Bohm effect for bound states [80]. In the present case, the effective angular momentum ζ shows a shift in the angular momentum quantum number analogous to that obtained by Peshkin and Tonomura [80]. Hence, the influence of the topology of the spacetime on the relativistic Landau levels gives rise to an Aharonov–Bohm-type effect for bound states [82]. Besides, the influence of the topology of the spacetime on the interaction of the electron with the uniform magnetic field modifies the degeneracy of the relativistic Landau levels. Observe that the term m^2 is the contribution to the energy levels that stems from the rest mass of the relativistic electron, while the $\pm$ signs indicate the energy associated with the positive and negative solutions to the Dirac equation [90]. Note that by taking $\beta = 0$, the relativistic Landau levels (23) become those given in the Minkowski spacetime [13, 19, 20].

Next, let us apply the binomial expansion up to terms of order m^{-1} in the relativistic Landau levels (23). Then, we obtain

$$\mathcal{E}_{n,l} \approx m + \omega\left[n + \frac{|\zeta|}{2} + \frac{\zeta}{2} + \frac{1}{2}(1 - s)\right], \tag{24}$$

where the first term of equation (24) is the contribution that stems from the rest mass of the particle. The second term of the energy levels (24) corresponds to the (nonrelativistic) Landau levels in the presence of a spiral dislocation. Note that the energy levels (24) are analogous to the Landau levels for a spinless quantum particle obtained in [70]. Therefore, we can see in the nonrelativistic limit that the degeneracy of the Landau levels [92] is broken by the effects of the topology of the spiral dislocation. Moreover, since there is no interaction between the electron and the topological defect, we also have an analogue of the Aharonov–Bohm effect for bound states [80, 82].

3. Conclusions

We have analysed the interaction of a relativistic electron with a uniform magnetic field in the spiral dislocation spacetime. We have seen that the effects of torsion of this spacetime modify degeneracy of the relativistic Landau levels. This break of degeneracy is given by the appearance of an effective angular momentum $\zeta = l + \frac{1}{2}(1 - s) - \frac{m\omega\,\beta^2}{2}$ that stems from the topology of the spacetime, even though no interaction between the electron and the topological defect exists. Furthermore, we have seen that the presence of this effective angular momentum in the relativistic Landau levels, without the interaction between the electron and the topological defect, yields an analogue effect of the Aharonov–Bohm-type effect for bound states [80, 82]. We have also shown by taking $\beta = 0$ in the relativistic energy levels (23) that we can recover the relativistic Landau levels in the Minkowski spacetime [13, 19, 20]. Finally, by applying the binomial expansion up to terms of order m^{-1} in the relativistic Landau levels (23), we have shown that the nonrelativistic Landau levels in the presence of a spiral dislocation [70] can be obtained.

Acknowledgments

The authors would like to thank CNPq for financial support.

References

[1] Novoselov K S *et al* 2005 *Nature* **348** 197
[2] Cortijo A and Vozmediano M A H 2007 *EPL* **77** 47002
[3] Geim A K 2007 *Nature Mater* **6** 183
[4] Pachos J K 2009 *Contemp. Phys.* **50** 375
[5] Castro Neto A H *et al* 2009 *Rev. Mod. Phys.* **81** 109
[6] Furtado C, Moraes F and de M Carvalho A M 2008 *Phys. Lett.* A **372** 5368
[7] Cavalcante E and Furtado C 2014 *J. Phys. Chem. Solid.* **75** 1265
[8] González J, Guinea F and Vozmediano M A H 1992 *Phys. Rev. Lett.* **69** 172
[9] González J, Guinea F and Vozmediano M A H 1993 *Nucl. Phys.* B **406** 771
[10] Hasan M Z and Kane C L 2010 *Rev. Mod. Phys.* **82** 3045
[11] Zhang Y *et al* 2005 *Nature* **438** 201
[12] Prange R E and Girvin S M (ed) 1990 *The Quantum Hall Effect* (New York: Springer)
[13] Bueno M J *et al* 2012 *Eur. Phys. J.* B **85** 53
[14] Amaro Neto J *et al* 2018 *Eur. Phys. J. Plus* **133** 185
[15] Amaro Neto J *et al* 2016 *Ann. Phys. (NY)* **373** 273
[16] Katsnelson M I *et al* 2006 *Nat. Phys.* **2** 620
[17] Young A F and Kim P 2009 *Nat. Phys.* **5** 222
[18] Oliveira J R S *et al* 2017 *Ann. Phys. (NY)* **383** 610
[19] Rabi I I 1928 *Z. Phys.* **49** 507
[20] Cunha M S *et al* 2016 *Eur. Phys. J.* C **76** 512
[21] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1982 *Quantum Electrodynamics* 2nd edn (Oxford: Pergamon)
[22] Jackiw R 1984 *Phys. Rev.* D **29** 2375
[23] Balatsky A V, Volovik G E and Konyshev A V 1986 *Zh. Eksp. Teor. Fiz.* **90** 2038
[24] Haldane F D M 1988 *Phys. Rev. Lett.* **61** 2015
[25] Schakel A M J 1991 *Phys. Rev.* D **43** 1428
[26] Figueiredo Medeiros E R and Bezerra de Mello E R 2012 *Eur. Phys. J.* C **72** 2051
[27] Bueno M J *et al* 2014 *Eur. Phys. J. Plus* **129** 201
[28] de Juan F *et al* 2010 *Nucl. Phys.* B **828** 625
[29] Bezerra V B 1997 *J. Math. Phys.* **38** 2553
[30] Furtado C, Moraes F and Bezerra V B 1999 *Phys. Rev.* D **59** 107504
[31] de A Marques G, Bezerra V B, Furtado C and Moraes F 2005 *Int. J. Mod. Phys.* A **20** 6051
[32] Salazar-Ramírez M, Ojeda-Guillén D, Morales-González A and García-Ortega V H 2019 *Eur. Phys. J. Plus* **134** 8
[33] Hosseinpour M, Hassanabadi H and de Montigny M 2018 *Int. J. Geom. Methods Mod. Phys.* **15** 1850165
[34] Wang B-Q, Long Z-W, Long C-Y and Wu S-R 2018 *Mod. Phys. Lett.* A **33** 1850025
[35] Deng L-F, Long C-Y, Long Z-W and Xu T 2018 *Adv. High Energy Phys.* **2018** 2741694
[36] Hassanabadi H, Hosseinpour M and de Montigny M 2017 *Eur. Phys. J. Plus* **132** 541
[37] de Montigny M, Hosseinpour M and Hassanabadi H 2016 *Int. J. Mod. Phys.* A **31** 1650191

[38] Ma K, Wang J-H, Yang H-X and Fan H-W 2017 *Chinese Phys. Lett.* **33** 100401
[39] Carvalho J, de M Carvalho A M, Cavalcante E and Furtado C 2016 *Eur. Phys. J.* C **76** 365
[40] Hosseinpour M and Hassanabadi H 2015 *Eur. Phys. J. Plus* **130** 236
[41] Wang J-H, Ma K, Li K and Fan H-W 2015 *Ann. Phys. (NY)* **362** 327
[42] Messai N and Boumali A 2015 *Eur. Phys. J. Plus* **130** 140
[43] Hassanabadi H, Afshardoost A and Zarrinkamar S 2015 *Ann. Phys. (NY)* **356** 346
[44] Wang Z, Cai H and Xu C 2019 *Int. J. Mod. Phys.* A **34** 1950023
[45] Cai H and Ren Z 2018 *Class. Quantum Grav.* **35** 235014
[46] Cai H *et al* 2018 *Class. Quantum Grav.* **35** 155016
[47] Jusufi K, Rahaman F and Banerjee A 2018 *Ann. Phys. (NY)* **389** 219
[48] Wang B-Q *et al* 2016 *Eur. Phys. J. Plus* **131** 378
[49] Hassanabadi H and Hosseinpour M 2016 *Eur. Phys. J.* C **76** 553
[50] Garcia G Q *et al* 2017 *Eur. Phys. J. Plus* **132** 183
[51] Yeşiltaş Ö 2015 *Eur. Phys. J. Plus* **130** 128
[52] Özdemir F and Özdemir N 2004 *Gen. Relativ. Gravit.* **36** 2053
[53] Pitelli P M and Letelier P S 2007 *J. Math. Phys.* **48** 092501
[54] de M Carvalho A M, de Lima Ribeiro C A and Furtado C 2009 *Int. J. Mod. Phys.* D **18** 237
[55] Li X and Liu X 2018 *Int. J. Mod. Phys.* A **33** 1850156
[56] Salazar-Ramírez M *et al* 2016 *Ann. Phys. (NY)* **372** 283
[57] Silva E O *et al* 2015 *Ann. Phys. (NY)* **362** 739
[58] Hosseinpour M, Hassanabadi H and de Montigny M 2019 *Eur. Phys. J.* C **79** 311
[59] Vitória R L L 2019 *Eur. Phys. J.* C **79** 844
[60] Bragança E A F *et al* 2020 *Eur. Phys. J.* C **80** 206
[61] Vitória R L L and Belich H 2019 *Phys. Scr.* **94** 125301
[62] Ahmed F 2019 *Eur. Phys. J.* C **79** 104
[63] de Montigny M, Zare S and Hassanabadi H 2018 *Gen. Relativ. Gravit.* **50** 47
[64] Carvalho J, de M Carvalho A M and Furtado C 2014 *Eur. Phys. J.* C **74** 2935
[65] Ahmed F 2018 *Eur. Phys. J.* C **78** 598
[66] Hassanabadi H, Hosseini M, Zare S and Hosseinpour M 2019 *Few-Body Syst* **60** 12
[67] Cavalcante E *et al* 2016 *Eur. Phys. J. Plus* **131** 288
[68] Wang Z *et al* 2015 *Eur. Phys. J. Plus* **130** 36
[69] Maia A V D M and Bakke K 2019 *Eur. Phys. J.* C **79** 551
[70] Maia A V D M and Bakke K 2019 *Int. J. Mod. Phys.* A **34** 1950153
[71] Wang B-Q, Long Z-W, Long C-Y and Wu S-R 2018 *Int. J. Mod. Phys.* A **33** 1850158
[72] Santos L C N and Barros C C Jr 2018 *Int. J. Mod. Phys.* A **33** 1850122
[73] Santos L C N and Barros C C Jr 2018 *Eur. Phys. J.* C **78** 13
[74] Bakke K and Furtado C 2013 *Phys. Rev.* A **87** 012130
[75] Kleinert H 1989 *Gauge Fields in Condensed Matter* vol 2 (Singapore: World Scientific)
[76] Katanaev M O and Volovich I V 1992 *Ann. Phys. (NY)* **216** 1
[77] Puntigam R A and Soleng H H 1997 *Class. Quantum Grav.* **14** 1129
[78] Valanis K C and Panoskaltsis E V P 2005 *Acta Mech.* **175** 77
[79] Aharonov Y and Bohm D 1959 *Phys. Rev.* **115** 485
[80] Peshkin M and Tonomura A 1989 *The Aharonov-Bohm Effect* (*Lect. Notes Phys.* vol 340) (Berlin: Springer)
[81] Furtado C, Bezerra V B and Moraes F 2001 *Phys. Lett.* A **289** 160
[82] Silva Netto A L, Chesman C and Furtado C 2008 *Phys. Lett.* A **372** 3894
[83] Furtado C, de M Carvalho A M and de Lima Ribeiro C A 2006 *Mod. Phys. Lett.* A **21** 1393
[84] Mesaros A *et al* 2010 *Phys. Rev.* B **82** 073405
[85] Maia A V D M and Bakke K 2018 *Physica* B **531** 213
[86] Birrel N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)
[87] Nakahara M 2003 *Geometry, Topology and Physics* (Bristol: Institute of Physics Publishing)
[88] Arfken G B and Weber H J 2005 *Mathematical Methods for Physicists* 6th edn (New York: Elsevier)
[89] Shapiro I L 2002 *Phys. Rept.* **357** 113
[90] Greiner W 2000 *Relativistic Quantum Mechanics: Wave Equations* 3rd edn (Berlin: Springer)
[91] Abramowitz M and Stegum I A 1965 *Handbook of Mathematical Functions* (New York: Dover Publications Inc)
[92] Landau L D and Lifshitz E M 1977 *Quantum Mechanics, the Nonrelativistic Theory* 3rd edn (Oxford: Pergamon)