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Two model equations with a second degree logarithmic nonlinearity and their Gaussian solutions

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Abstract

In the paper, we try to study the mechanism of the existence of Gaussian waves in high degree logarithmic nonlinear wave motions. We first construct two model equations which include the high order dispersion and a second degree logarithmic nonlinearity. And then we prove that the Gaussian waves can exist for high degree logarithmic nonlinear wave equations if the balance between the dispersion and logarithmic nonlinearity is kept. Our mathematical tool is the logarithmic trial equation method.

Keywords: Gaussian solitary wave, Gausson, logarithmic nonlinearity, wave equation, trial equation method

1. Introduction

Gaussian waves which include Gaussian solitary wave and Gausson [1, 2], are different to the solitons like sech-function solution in integrable systems. Among those, Gaussian solitary wave is for real equations, while Gausson is for complex equations. Due to the pulse shape, and the shape of Gaussian wave being invariant by the balance between dispersion and logarithmic nonlinearity, some researchers are attracted to study this kind of wave equations from several aspects, such as the constructions of Gaussian solitary waves, stability of solutions, and dynamics of wave propagation and so on.

In 1976, Bialynicki-Birula and Mycielsk [3] proposed the following first degree logarithmic nonlinear Schrödinger equation (LNSE for simplicity)

$$\mathrm{i}u_t + \alpha u_{xx} + 2\beta u \ln|u| = 0, \tag{1}$$

and obtained an analytic soliton-like solution namely Gausson for being of Gaussian wave shape. This is one of the most remarkable features of the LNSE. It also had been shown that the Gaussons can exist for any number of dimensions, and numerically are stable under collisions over a wide range of energies in one and two dimensional cases [4]. Therefore, the model is applied to different areas such as superfluidity, open quantum systems, quantum liquid mixtures, Bose–Einstein condensates, and so forth [5–12].

On the other hand, some scholars studied real logarithmic nonlinear wave equations [13–19]. For example, Carles and Pelinovsky studied log-KdV equation and the stability of its Gaussian solitary waves [15]. Biswas *et al* solved the Bousinesq equation with first degree logarithmic nonlinearity [17]; Darvishi and Najafi gave the Gaussian waves for some logarithmic ZK equation [18]. In addition, Wazwaz *et al* studied several logarithmic nonlinear evolution equations and obtained Gaussian solitary waves solutions, such as the following log-KdV, log-KG, log-Boussinesq, log-BBM, log-BBM-KP, log-TRLW and so forth [20–23]:

$$\text{log-KdV:}\quad u_t + (u\ln|u|)_x + u_{xxx} = 0, \tag{2}$$

$$\text{log-KG:}\quad u_{tt} - u_{xx} + 2u\ln|u| = 0, \tag{3}$$

$$\begin{aligned}&\text{log-Boussinesq: } u_{tt} - u_{xx}\\ &\quad + n(u\ln|u|)_{xx} + u_{xxxx} = 0,\end{aligned} \tag{4}$$

$$\text{log-BBM:}\quad u_t + u_x + n(u\ln|u|)_x + u_{xxt} = 0, \tag{5}$$

$$\begin{aligned}&\text{log-BBM-KP:}\\ &(u_t + u_x + n(u\ln|u|)_x + u_{xxt})_x + u_{yy} = 0,\end{aligned} \tag{6}$$

$$\begin{aligned}&\text{log-TRLW: } u_t + u_x\\ &\quad + n(u\ln|u|)_x + u_{xtt} = 0.\end{aligned} \tag{7}$$

It is obvious that the logarithmic nonlinear terms in the above equations are only the first degree form of $\ln|u|$, and hence we can get the Gaussian solitary wave and the Gaussson by the elementary integral method. However, if we consider high order derivative dispersion, the shape of the Gaussian solitary wave and Gausson wave will not be kept. Therefore, if we hope there also exist the Gaussian solitary wave or the Gaussons for high dispersive wave motions, we have to consider more high degree nonlinear logarithmic terms like $\ln^2|u|$ so that the balance between them can still keep.

It is well-known that to construct the soliton equations is a meaningful work in integrable systems. These soliton equations play an important role for recovering the mechanism of the existence of solitons. Similarly, in the paper, our aim is to try to find the mechanism of the existence of the Gaussian waves for high degree logarithmic nonlinear wave propagations. We propose two nonlinear wave equations with a second degree logarithmic nonlinearity, and show their Gaussian propagation patterns. Because of the existence of the second degree logarithmic nonlinearity, these equations cannot be solved by the routine direct integral method, and hence we use the trial equation method [24–27] to get their integrable factor equations and give the explicit constructions of the Gaussian waves. In fact we provide a new type of trail equation namely logarithmic trial equation which is different to the usual rational or algebraic trial equations to solve two models. Our results mean that the Gaussian solitary waves and the Gaussons can exist not only for the first degree logarithmic nonlinearity but also for the high degree logarithmic cases, and show that the key for the existence of the Gaussian waves is also due to the balance between the dispersion and logarithmic nonlinearity.

The paper is organized as follows. In section 2, we construct two second degree logarithmic nonlinear wave equations. In sections 3 and 4, we obtain the Gaussian wave solutions for two model equations. Last section is a short conclusion.

2. Model equations

For the usual wave equation

$$u_{tt} = k^2 u_{xx}, \tag{8}$$

there are the traveling wave solutions with fixed velocity $v = \pm k$, and there are no dispersion and dissipation. If we consider dispersive wave equation, such as

$$u_{tt} = k^2 u_{xx} + \gamma u_{xxx}, \tag{9}$$

we will find its traveling wave solutions are the forms of hyperbolic functions or tangent functions which are not the Gaussian solitary wave. If we consider a nonlinear term $(u^2)_{xx}$, the Boussinesq equation

$$u_{tt} = k^2 u_{xx} + \alpha (u^2)_{xx} + \gamma u_{xxx}, \tag{10}$$

will be an integrable equation and hence there exist the soliton solutions like sech-function solution. However, it has no the Gaussian solitary wave solution.

On the other hand, by an elementary integral method, we can find there exists a Gaussian solitary wave solution for the following equations (22), (23)

$$u_{tt} = k^2 u_{xx} + \gamma u_{xxxx} + \mu (u \ln u)_{xx}. \tag{11}$$

Furthermore, Wazwaz studied another wave equation

$$\begin{aligned} u_{tt} = k^2 u_{xx} &+ \gamma_1 u_{xxx} \\ &+ \gamma_2 u_{xxtt} + \mu (u \ln u)_{xx}, \end{aligned} \tag{12}$$

which is a kind of Boussinesq equation with dual dispersion and logarithmic nonlinearity. For this equation and other related logarithmic nonlinear wave equations, all their logarithmic terms are of the first degree on $\ln u$.

If we consider high order dispersive effect, a natural problem is how to restrain dispersion so that the wave profile cannot be destroyed. We find that if a second degree nonlinear logarithmic term is considered, the balance between dispersion and nonlinearity will be achieved. Consequently, the second degree logarithmic nonlinear wave equation is given as follows

$$\begin{aligned} u_{tt} - k^2 u_{xx} &+ \gamma_1 (u \ln u)_{xx} + \gamma_2 (u \ln^2 u)_{xx} \\ &+ \mu_1 u_{xxxx} + \mu_2 u_{xxtt} + \mu_3 u_{xxxxxx} = 0, \end{aligned} \tag{13}$$

where γ_1, γ_2, μ_1, μ_2, μ_3 are coefficients of logarithmic nonlinearities and dispersions and satisfy $\gamma_2 \mu_3 < 0$. In addition, $\ln u$ can be written as $\ln|u|$. If we only consider the Gaussian solitary wave, u will be positive, so we need not to write the absolute value symbol.

As the second model, we consider the high degree logarithmic version of equation (1). Indeed, for the first degree logarithmic nonlinear Schrödinger (1), as mentioned above, there exists a Gausson which is a localized non-spreadable solution, and keeps shape in propagation. This is just due to the balance between the logarithmic term and the second order derivative term. However, in general, Gausson will be destroyed if we consider only the high order dispersive terms or the high degree nonlinear logarithmic terms For the existence of Gaussons, we must keep the balance between dispersion and nonlinearity. And hence we propose the following nonlinear Schrödinger equation

$$\begin{aligned} \mathrm{i} u_t &+ \mathrm{i}(\alpha_1 u_{xxx} + \alpha_2 u_{xxt}) + \beta_1 u_{xx} + \beta_2 u_{xxxx} \\ &+ \gamma_1 u \ln|u| + \gamma_2 u \ln^2|u| = \mu u, \end{aligned} \tag{14}$$

which includes high order dispersive terms and a second degree nonlinear logarithmic term, and where the right side is the detuning term, u is the wave profile, $\beta_2 \gamma_2 < 0$, all parameters are constants depending on the concrete model. When $\alpha_1 = \alpha_2 = \beta_2 = \gamma_2 = 0$, it becomes the equation (1). When $\gamma_1 = \gamma_2 = 0$, it describes the optical propagation in a linear dispersive media. We will prove that there exists the Gausson solutions for it.

The second degree logarithmic nonlinear terms $\ln^2 u$ in equations (13) and (14) show a high order effect of nonlinear interaction by which the balance between the highest order nonlinearity and the high order dispersions such as the sixth order derivative in equation (13) and the fourth order derivative in equation (14) can be kept so that the Gaussian solitary waves can be given. In some real models, the term $\ln^{2u}$ will appear naturally. For example, if we take the nonlinear interaction as u^{s-1} in the chain model of propagation of acoustic wave, then we expand it at $s = 1$ to get $u^{s-1} - 1 = (s-1)\ln u + \frac{1}{2}(s-1)^2 \ln^2 u + \cdots$. We will study it in details in further work.

Unlike the equations (1)–(7) can be solved by elementary integral method, these two model equations (13) and (14) cannot be reduced directly to the integral forms since they contain the high order derivative terms Thus, in next sections, we use the logarithmic trial equation method to solve them.

3. Gaussian wave solution for equation (13)

Firstly, Taking the traveling wave transformation

$$u(x, t) = u(\xi),\ \xi = x + \omega t, \tag{15}$$

and substituting it into equation (13) and integrating twice yield the ODE as follows

$$\begin{aligned}(\omega^2 - k^2)u + \gamma_1 u \ln u + \gamma_2 u (\ln u)^2 \\ + (\mu_1 + \mu_2\omega^2)u'' + \mu_3 u^{(4)} = 0,\end{aligned} \tag{16}$$

where we have taken two integral constants to be zeros.

Then take trial equation

$$u'' = F(u) = u\left(\sum_{j=0}^{m} a_j (\ln u)^j\right), \tag{17}$$

where m and a_j are constants to be determined for $j = 0,\cdots,m$. Setting it into equation (16), by the balance principle we get $m = 1$. Therefore, the trial equation equation is given by

$$u'' = a_1 u + a_2 u \ln u. \tag{18}$$

From the equation (18), we derive out by taking the integral constant be zero

$$(u')^2 = u^2\left(a_1 - \frac{a_2}{2} + a_2 \ln u\right), \tag{19}$$

and

$$\begin{aligned}u^{(4)} = (a_1 + a_2 + a_2 \ln u)(a_1 u + a_2 u \ln u) \\ + a_2\left(a_1 - \frac{a_2}{2}\right)u + a_2^2 u \ln u,\end{aligned} \tag{20}$$

and substitute them into equation (16) to get an identity

$$\begin{aligned}&\left\{\mu_3\left(a_1^2 + 2a_1a_2 - \frac{a_2^2}{2}\right)\right. \\ &\left. + a_1(\mu_1 + \mu_2\omega^2) + \omega^2 - k^2\right\}u \\ &+ \{\mu_3(2a_1a_2 + 2a_2^2) + \gamma_1 + a_2(\mu_1 + \mu_2\omega^2)\} \\ &u \ln u + (\mu_3 a_2^2 + \gamma_2)u \ln^2 u = 0.\end{aligned} \tag{21}$$

Letting all coefficients be zeros yields a system of nonlinear algebraic equations

$$\begin{aligned}\mu_3\left(a_1^2 + 2a_1a_2 - \frac{a_2^2}{2}\right) \\ + a_1(\mu_1 + \mu_2\omega^2) + \omega^2 - k^2 = 0,\end{aligned} \tag{22}$$

$$\mu_3(2a_1a_2 + 2a_2^2) + \gamma_1 + a_2(\mu_1 + \mu_2\omega^2) = 0, \tag{23}$$

$$\mu_3 a_2^2 + \gamma_2 = 0. \tag{24}$$

Solving the equations system, we get

$$a_2 = \pm\sqrt{-\frac{\gamma_2}{\mu_3}}, \tag{25}$$

$$a_1 = \frac{2\gamma_2 - \gamma_1}{2\mu_3 a_2} - \frac{\mu_1 + \mu_2\omega^2}{2\mu_3}, \tag{26}$$

and the parameters condition

$$\begin{aligned}\mu_3(a_1^2 + 2a_1a_2) + \frac{\gamma_2^2}{2} \\ + a_1(\mu_1 + \mu_2\omega^2) + \omega^2 - k^2 = 0.\end{aligned} \tag{27}$$

Therefore we can solve the equation (19), that is

$$\int \frac{\mathrm{d}u}{u\sqrt{a_1 - \frac{1}{2}a_2 + a_2 \ln u}} = \pm(\xi - \xi_0), \tag{28}$$

where ξ_0 is an arbitrary constant. And then we give the desired Gaussian solitary wave solution

$$\begin{aligned}u(x, t) &= \mathrm{e}^{\frac{a_2}{4}(\xi - \xi_0)^2 - \frac{a_1}{a_2} + \frac{1}{2}} \\ &= \mathrm{e}^{-\frac{1}{4}\sqrt{-\frac{\gamma_2}{\mu_3}}(x + \omega t - \xi_0)^2 - \frac{a_1}{a_2} + \frac{1}{2}},\end{aligned} \tag{29}$$

where we have choose $a_2 = -\sqrt{-\frac{\gamma_2}{\mu_3}} < 0$ for the solution being a Gaussian solitary wave.

For concrete parameters, we can obtain concrete Gaussian solitary waves. For example, we take $\gamma_1 = 2,\ \gamma_2 = 1,\ \mu_1 = 1,\ \mu_2 = 1,\ \mu_3 = -1,\ k = \frac{3}{\sqrt{2}},\ \omega = \pm 1$, then we get $a_2 = -1,\ a_1 = 1$. So the Gaussian solitary wave is given as follows:

$$u(x, t) = \mathrm{e}^{-\frac{1}{4}\left(\frac{3}{\sqrt{2}}x \pm t - \xi_0\right)^2 + \frac{3}{2}}. \tag{30}$$

4. Gausson for equation (14)

Firstly, substituting envelope traveling wave transformation

$$\begin{aligned}u(x, t) = p(\xi)\mathrm{e}^{\mathrm{i}\eta}, \\ \xi = k_1 x + \omega_1 t.\ \eta = k_2 x + \omega_2 t,\end{aligned} \tag{31}$$

into equation (14) and separating real and imaginary parts yield two ODEs as follows

$$\begin{aligned}&\{\omega_1 - 3\alpha_1 k_1 k_2^2 - \alpha_2(\omega_1 k_2^2 + 2\omega_2 k_1 k_2) \\ &+ 2\beta_1 k_1 k_2 - 4\beta_2 k_1 k_2^3\}p' \\ &+(\alpha_1 k_1^3 + \alpha_2 k_1^2 \omega_1 + 4\beta_2 k_1^3 k_2)p''' = 0,\end{aligned} \tag{32}$$

$$p^{(4)} + Ap'' + B_1 p + B_2 p \ln p + B_3 p \ln^2 p = 0, \tag{33}$$

where

$$A = \frac{-3\alpha_1 k_1^2 k_2 - \alpha_2(\omega_2 k_1^2 + 2\omega_1 k_1 k_2) + \beta_1 k_2^2 - 6\beta_2 k_1^2 k_2^2}{\beta_2 k_1^4}, \tag{34}$$

$$B_1 = \frac{-\omega_2 + \alpha_1 k_2^3 + \alpha_2 \omega_2 k_2^2 - \beta_1 k_1^2 + \beta_2 k_2^4 - \mu}{\beta_2 k_1^4},$$
$$B_2 = \frac{\gamma_1}{\beta_2 k_1^4},\ B_3 = \frac{\gamma_2}{\beta_2 k_1^4}. \tag{35}$$

In equation (32), we let the coefficients of p' and p''' be zeros to give

$$\omega_1 - 3\alpha_1 k_1 k_2^2 - \alpha_2(\omega_1 k_2^2 + 2\omega_2 k_1 k_2) + 2\beta_1 k_1 k_2 - 4\beta_2 k_1 k_2^3 = 0, \tag{36}$$

$$\alpha_1 k_1^3 + \alpha_2 k_1^2 \omega_1 + 4\beta_2 k_1^3 k_2 = 0. \tag{37}$$

Solving the equations system, we get the parameters relations

$$\omega_1 = -\frac{\alpha_1 k_1 + 4\beta_2 k_1 k_2}{\alpha_2}, \tag{38}$$

$$\omega_2 = \frac{2\beta_1 \alpha_2 k_2 - 4\beta_2 k_2 - 2\alpha_1 \alpha_2 k_2^2 - \alpha_1}{2k_2 \alpha_2^2}. \tag{39}$$

Under the above conditions, we only need to solve equation (33). Since equation (33) has the similar form with equation (16), we can solve it by the same method as above section. However, they have different parameters, for the purpose of completeness, we list the computations.

By the same way with the equation (13), the trial equation equation is given by

$$p'' = a_1 p + a_2 p \ln p. \tag{40}$$

From the equation (40), we have by taking the integral constant be zero

$$(p')^2 = p^2\left(a_1 - \frac{a_2}{2} + a_2 \ln p\right), \tag{41}$$

$$p^{(4)} = (a_1 + a_2 + a_2 \ln p)(a_1 p + a_2 p \ln p) + a_2\left(a_1 - \frac{a_2}{2}\right)p + a_2^2 p \ln p. \tag{42}$$

Furthermore, substituting them into equation (33) gives an identity

$$\left(a_1^2 + 2a_1 a_2 - \frac{a_2^2}{2} + a_1 A + B_1\right)p + (2a_1 a_2 + 2a_2^2 + B_2 + a_2 A)p \ln p + (a_2^2 + B_3)p \ln^2 p = 0. \tag{43}$$

Letting all coefficients be zeros yields a system of nonlinear algebraic equations

$$a_1^2 + 2a_1 a_2 - \frac{a_2^2}{2} + a_1 A + B_1 = 0, \tag{44}$$

$$2a_1 a_2 + 2a_2^2 + B_2 + a_2 A = 0, \tag{45}$$

$$a_2^2 + B_3 = 0. \tag{46}$$

Solving the equations system, we get

$$a_2 = \pm\sqrt{-B_3}, \tag{47}$$

$$a_1 = -\frac{A}{2} + \frac{2B_3 - B_2}{2a_2}, \tag{48}$$

and all undetermined parameters k_1, k_2, ω_1, ω_2 must satisfy the condition

$$6B_3^2 + 4B_1 B_3 - B_2^2 - A^2 B_3 - 4a_2 A B_3 = 0. \tag{49}$$

Next we write the equation (41) as the integral form

$$\int \frac{\mathrm{d}p}{p\sqrt{a_1 - \frac{1}{2}a_2 + a_2 \ln p}} = \pm(\xi - \xi_0), \tag{50}$$

where ξ_0 is an arbitrary constant. And then solving the integral gives

$$p(x, t) = \mathrm{e}^{\frac{a_2}{4}(\xi - \xi_0)^2 - \frac{a_1}{a_2} + \frac{1}{2}} = \mathrm{e}^{-\frac{1}{4}\sqrt{-B_3}(k_1 x + \omega_1 t - \xi_0)^2 - \frac{a_1}{a_2} + \frac{1}{2}}, \tag{51}$$

where we have choose $a_2 = -\sqrt{-B_3} < 0$ for the solution being of Gaussian shape. Therefore, the Gausson is given by

$$u(x, t) = \mathrm{e}^{-\frac{1}{4}\sqrt{-\frac{\gamma_2}{\beta_2}}(k_1 x + \omega_1 t - \xi_0)^2 - \frac{a_1}{a_2} + \frac{1}{2}} \mathrm{e}^{\mathrm{i}(k_2 x + \omega_2 t)}. \tag{52}$$

For concrete parameters, we can give concrete Gaussons. For example, we take $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \mu = 1$, $\gamma_1 = 1$, $\gamma_2 = -1$, $k_1 = 1$, $\omega_2 = -1$, then we get $k_2 = \pm\left(\frac{3}{2}\right)^{\frac{1}{4}}$, $\omega_1 = -\frac{3 + 5k_2}{2}$. So the Gausson solution is given as follows:

$$u(x, t) = \mathrm{e}^{-\frac{1}{4}\left(x - \frac{3 \pm 5\left(\frac{3}{2}\right)^{\frac{1}{4}}}{2}t - \xi_0\right)^2 + \frac{3}{2}} \mathrm{e}^{\mathrm{i}\left(\pm\left(\frac{3}{2}\right)^{\frac{1}{4}}x - t\right)}. \tag{53}$$

Remark 2. For the second degree logarithmic nonlinear Schrödinger equation (14), it is just the third and fourth order dispersions to balance the effect of second degree logarithmic nonlinearity. Here, the third order dispersion is a necessary term for p being dependent with temporal variable t. Indeed, if we remove this term, that is, if we take $\alpha_1 = \alpha_2 = 0$, the model equation becomes

$$\mathrm{i}u_t + \beta_1 u_{xx} + \beta_2 u_{xxxx} + \gamma_1 u \ln|u| + \gamma_2 u \ln^2|u| = \mu u. \tag{54}$$

From equations (36) and (37), we must have $\omega_1 = 0$. Without loss of generality, we can take $k_1 = 1$ and $k_2 = 0$. The degenerate Gausson is given by

$$u(x, t) = \mathrm{e}^{-\frac{1}{4}\sqrt{-\frac{\gamma_2}{\beta_2}}(x - x_0)^2 - \frac{a_1}{a_2} + \frac{1}{2}} \mathrm{e}^{\mathrm{i}\omega_2 t}. \tag{55}$$

It is easy to see that the amplitude of the wave is of Gaussian shape and is independent with the temporal variable, while the phase of the wave is independent with the spatial variable.

5. Conclusion

Due to the balance between dispersion and logarithmic nonlinearity, the Gaussian waves can exist for two second degree logarithmic nonlinear model equations. These equations include both real and complex cases. In addition, for all model equations (2)–(7) in introduction, we can give the second degree logarithmic forms For example,

$$u_t + \alpha(u\ln|u|)_x + \beta(u\ln^2|u|)_x + \gamma u_{xxx} + \delta u_{xxxxx} = 0, \tag{56}$$

$$u_{tt} - u_{xx} + \alpha u_{xxxx} + \beta u\ln|u|^2 + \gamma u\ln^2|u|^2 = 0, \tag{57}$$

$$u_t + u_x + \alpha(u\ln|u|^n)_x + \beta(u\ln^2|u|^n)_x + \gamma u_{xxt} + \delta u_{xxxxx} = 0, \tag{58}$$

$$(u_t + u_x + \alpha(u\ln|u|^n)_x + \beta(u\ln^2|u|^n)_x + \gamma u_{xxt} + \delta u_{xxxxx})_x + u_{yy} = 0, \tag{59}$$

$$u_t + u_x + \alpha(u\ln|u|^n)_x + \beta(u\ln^2|u|^n)_x + \gamma u_{xtt} + \delta u_{xxxtt} = 0. \tag{60}$$

The above these high degree logarithmic nonlinear equations show the balance between logarithmic nonlinear and dispersion, and hence can be solved by the trial equation method, and hence the Gaussian waves can be given. This should be a basic principle for the existences of Gaussian solitary waves or Gaussons for the propagation patterns in high degree nonlinear logarithmic media.

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