

The masking condition for the quantum state in two-dimensional Hilbert space

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Abstract

This paper focuses on quantum information masking for the quantum state in two-dimensional Hilbert space. We present a system of equations as the condition of quantum information masking. It is shown that quantum information contained in a single qubit state can be masked, if and only if the coefficients of the quantum state satisfy the given system of equations. By observing the characteristics of non-orthogonal maskable quantum states, we obtain a related conclusion, namely, if two non-orthogonal two-qubit quantum states can mask a single qubit state, they have the same number of terms and the same basis. Finally, for maskable orthogonal quantum states, we analyze two special examples and give their images for an intuitive description.

Keywords: quantum information masking, limited conditions, two-qubit

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum mechanics is one of the most important inventions in the field of physics in the 20th century [1, 2]. As a crucial research object of quantum mechanics, quantum information has very great significance. In quantum information theory, the evolution in a closed quantum system is unitary and linear [3]. Therefore, unlike the classical world, there are some no-go theorems in the quantum world, such as the no-deleting theorem [4–6], the no-cloning theorem [7–9], the no-broadcasting theorem [10–12] and so on. As one of the no-go theorems, the no-masking theorem [13] plays a vital role in the development of quantum information. Quantum information masking requires that the information contained in subsystems can be encoded into their correlation by some unitary operations, such that the marginal states are identical [14]. In other words, one can hide the initial information contained in the subsystems [15]. This principle greatly improves the security of quantum communication [16, 17]. As a consequence, the masking of quantum information can be used for quantum secret sharing [18–20]. Besides, it also plays a major role in quantum teleportation [21, 22] and key distribution [23]. Therefore, it is necessary to study the quantum states whether can be masked. At present, the

research method for this problem is mainly based on the definition of quantum information masking.

In recent works, many scholars have come up with a lot of new discoveries about quantum masking. Modi *et al* first proposed the no-masking theorem, namely, a unitary operator cannot mask any pure state [13]. Based on this theorem, Ghosh *et al* considered two special kinds of quantum states to analyze the masking condition [24]. Li and Wang showed that tripartite quantum systems can mask arbitrary quantum pure states [25]. In addition, the relationship between the maskable state set and the hyperdisk was studied by Ding *et al* [26]. Li and Modi certified that probabilistic universal masking is impossible and derived a necessary condition to make the ϵ -approximate universal masking possible [27]. Lie *et al* demonstrated that the maximum number of qubits that can be masked is related to the entropy of the quantum state [28]. Moreover, using time-correlated photons, Liu *et al* designed a quantum information masking machine [29]. These above papers are important for the development of quantum information masking.

Here, we mainly study the masking of quantum information in two-dimensional Hilbert space. We express the masking conditions by the coefficients of quantum states. As a result, we obtain a system of equations as the masking conditions. Afterward, by observing some concrete maskable non-orthogonal quantum states, we find a common property

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of these states, i.e. they have the same number of terms and the same basis. Eventually, we analyze two special kinds of quantum states with an orthogonal basis and draw some images for an intuitive description of these examples.

2. No-masking theorem conditions

In this section, we review the definition of masking [13] and discuss the general masking condition of quantum information in \mathbb{C}^2 .

Let \mathcal{H}_X be an n -dimensional Hilbert space associated with the system X . Suppose the states $|a_k\rangle_A$ in \mathcal{H}_A contain the quantum information. We say that the quantum information contained in states $|a_k\rangle_A$ can be masked, if and only if there is an operator F that maps the states to $|\Psi_k\rangle_{AB}$ which belong to $\mathcal{H}_A \otimes \mathcal{H}_B$, such that the marginal states of $|\Psi_k\rangle_{AB}$ satisfy the following two conditions

$$\begin{aligned} \rho_A &= \text{Tr}_B(|\Psi_k\rangle_{AB} \langle \Psi_k|), \\ \rho_B &= \text{Tr}_A(|\Psi_k\rangle_{AB} \langle \Psi_k|). \end{aligned}$$

That is to say, we cannot tell what the value of k is or what information it carries by looking at the states. Since this is a physical process, we can also write it as $F: |a_k\rangle_A \otimes |b_k\rangle_B \rightarrow |\Psi_k\rangle_{AB}$, where the operator F is unitary and is called the masker, $|b_k\rangle_B \in \mathcal{H}_B$. Moreover, in order to complete the difference between the dimensionality of two systems, there is a unitary operator U acting on systems A and B .

Assume that the quantum information in the state $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ can be masked. Then there exist two quantum states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ which belong to $\mathcal{H}_A \otimes \mathcal{H}_B$, such that

$$\begin{aligned} |b\rangle &= \alpha_0|0\rangle + \alpha_1|1\rangle \rightarrow |\Psi\rangle \\ &= \alpha_0|\Psi_0\rangle + \alpha_1|\Psi_1\rangle, \end{aligned}$$

where $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

For the subsystems A and B , we take the partial trace respectively, therefore, we get

$$\begin{aligned} \rho_B &= \text{Tr}_A(|\Psi\rangle \langle \Psi|) = |\alpha_0|^2 \text{Tr}_A(|\Psi_0\rangle \langle \Psi_0|) \\ &\quad + |\alpha_1|^2 \text{Tr}_A(|\Psi_1\rangle \langle \Psi_1|) \\ &\quad + \alpha_0 \alpha_1^* \text{Tr}_A(|\Psi_0\rangle \langle \Psi_1|) \\ &\quad + \alpha_0^* \alpha_1 \text{Tr}_A(|\Psi_1\rangle \langle \Psi_0|), \end{aligned} \tag{1}$$

$$\begin{aligned} \rho_A &= \text{Tr}_B(|\Psi\rangle \langle \Psi|) = |\alpha_0|^2 \text{Tr}_B(|\Psi_0\rangle \langle \Psi_0|) \\ &\quad + |\alpha_1|^2 \text{Tr}_B(|\Psi_1\rangle \langle \Psi_1|) \\ &\quad + \alpha_0 \alpha_1^* \text{Tr}_B(|\Psi_0\rangle \langle \Psi_1|) \\ &\quad + \alpha_0^* \alpha_1 \text{Tr}_B(|\Psi_1\rangle \langle \Psi_0|). \end{aligned} \tag{2}$$

By the definition of masking, the masking conditions are

$$\begin{aligned} \rho_A &= \text{Tr}_B(|\Psi_0\rangle \langle \Psi_0|) = \text{Tr}_B(|\Psi_1\rangle \langle \Psi_1|) \\ &= \text{Tr}_B(|\Psi\rangle \langle \Psi|), \\ \rho_B &= \text{Tr}_A(|\Psi_0\rangle \langle \Psi_0|) = \text{Tr}_A(|\Psi_1\rangle \langle \Psi_1|) \\ &= \text{Tr}_A(|\Psi\rangle \langle \Psi|). \end{aligned}$$

In order to fulfill the above masking conditions, the cross-terms in equations (1) and (2) must vanish, i.e.

$$\begin{aligned} \alpha_0 \alpha_1^* \text{Tr}_A(|\Psi_0\rangle \langle \Psi_1|) \\ + \alpha_0^* \alpha_1 \text{Tr}_A(|\Psi_1\rangle \langle \Psi_0|) &= 0, \end{aligned} \tag{3}$$

$$\begin{aligned} \alpha_0 \alpha_1^* \text{Tr}_B(|\Psi_0\rangle \langle \Psi_1|) \\ + \alpha_0^* \alpha_1 \text{Tr}_B(|\Psi_1\rangle \langle \Psi_0|) &= 0. \end{aligned} \tag{4}$$

In this paper, we mainly discuss the masking of quantum states in two-dimensional Hilbert space. Without loss of generality, suppose that

$$\begin{aligned} |\Psi_0\rangle &= a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle, \\ |\Psi_1\rangle &= b_0|00\rangle + b_1|01\rangle + b_2|10\rangle + b_3|11\rangle, \end{aligned}$$

where $\sum_{i=0}^3 |a_i|^2 = \sum_{i=0}^3 |b_i|^2 = 1$.

When $\text{Tr}_x(|\Psi_0\rangle \langle \Psi_0|) = \text{Tr}_x(|\Psi_1\rangle \langle \Psi_1|)$, where $x \in \{A, B\}$, the coefficients of $|\Psi_0\rangle$ and $|\Psi_1\rangle$ should meet the following system of equations

$$\begin{cases} |a_0|^2 + |a_1|^2 - |b_0|^2 - |b_1|^2 = 0, \\ |a_0|^2 + |a_2|^2 - |b_0|^2 - |b_2|^2 = 0, \\ |a_1|^2 + |a_3|^2 - |b_1|^2 - |b_3|^2 = 0, \\ |a_2|^2 + |a_3|^2 - |b_2|^2 - |b_3|^2 = 0, \\ a_0 a_1^* + a_2 a_3^* - b_0 b_1^* - b_2 b_3^* = 0, \\ a_0 a_2^* + a_1 a_3^* - b_0 b_2^* - b_1 b_3^* = 0. \end{cases} \tag{5}$$

In addition, we take the partial trace with respect to A for $|\Psi_0\rangle \langle \Psi_1|$ and $|\Psi_1\rangle \langle \Psi_0|$, then we gain

$$\begin{aligned} \text{Tr}_A(|\Psi_0\rangle \langle \Psi_1|) &= (a_0 b_0^* + a_2 b_2^*) \\ &|0\rangle \langle 0| + (a_0 b_1^* + a_2 b_3^*)|0\rangle \langle 1| \\ &+ (a_1 b_0^* + a_3 b_2^*)|1\rangle \langle 0| \\ &+ (a_1 b_1^* + a_3 b_3^*)|1\rangle \langle 1|, \end{aligned} \tag{6}$$

$$\begin{aligned} \text{Tr}_A(|\Psi_1\rangle \langle \Psi_0|) &= (a_0^* b_0 + a_2^* b_2) \\ &|0\rangle \langle 0| + (a_1^* b_0 + a_3^* b_2)|0\rangle \langle 1| \\ &+ (a_0^* b_1 + a_2^* b_3)|1\rangle \langle 0| \\ &+ (a_1^* b_1 + a_3^* b_3)|1\rangle \langle 1|. \end{aligned} \tag{7}$$

For the convenience of discussion, we denote

$$\begin{aligned} M &= a_0 b_0^* + a_2 b_2^*, \\ N &= a_0 b_1^* + a_2 b_3^*, \\ P &= a_1 b_0^* + a_3 b_2^*, \\ Q &= a_1^* b_1 + a_3^* b_3. \end{aligned}$$

Substituting equations (6) and (7) into equation (3), we obtain

$$\begin{aligned} (\alpha_1 \alpha_2^* M + \alpha_1^* \alpha_2 M^*)|0\rangle \langle 0| \\ + (\alpha_1 \alpha_2^* N + \alpha_1^* \alpha_2 P^*)|0\rangle \langle 1| \\ + (\alpha_1 \alpha_2^* P + \alpha_1^* \alpha_2 N^*)|1\rangle \langle 0| \\ + (\alpha_1 \alpha_2^* Q + \alpha_1^* \alpha_2 Q^*)|1\rangle \langle 1| &= 0. \end{aligned}$$

As a result, the conditions for the establishment of the above formula can be expressed as

$$\begin{cases} \alpha_1\alpha_2^*M + \alpha_1^*\alpha_2M^* = 0, \\ \alpha_1\alpha_2^*N + \alpha_1^*\alpha_2P^* = 0, \\ \alpha_1\alpha_2^*P + \alpha_1^*\alpha_2N^* = 0, \\ \alpha_1\alpha_2^*Q + \alpha_1^*\alpha_2Q^* = 0. \end{cases}$$

This set of conditions can be further simplified as

$$\begin{cases} \text{Re}(\alpha_1\alpha_2^*M) = 0, \\ \text{Re}(\alpha_1\alpha_2^*Q) = 0, \\ \alpha_1\alpha_2^*N + \alpha_1^*\alpha_2P^* = 0. \end{cases} \tag{8}$$

Similar to the above method, for the subsystem B, we acquire that

$$\begin{cases} \text{Re}(\alpha_1\alpha_2^*M') = 0, \\ \text{Re}(\alpha_1\alpha_2^*Q') = 0, \\ \alpha_1\alpha_2^*N' + \alpha_1^*\alpha_2P'^* = 0, \end{cases} \tag{9}$$

where

$$\begin{aligned} M' &= a_0b_0^* + a_1b_1^*, \\ N' &= a_0b_2^* + a_1b_3^*, \\ P' &= a_2b_0^* + a_3b_1^*, \\ Q' &= a_2b_2^* + a_3b_3^*. \end{aligned}$$

In summary, the masking conditions can be represented by the coefficients of $|\Psi\rangle$, $|\Psi_0\rangle$ and $|\Psi_1\rangle$, i.e. when their coefficients meet the requirements of equations (5), (8) and (9) at the same time, a single qubit state $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ can be masked.

Modi *et al* have shown that masking the quantum information of the two systems is generally impossible, but they have also stated that there are a large number of special quantum states which can be masked [13]. These quantum states which can be masked and the corresponding masker have potential applications in quantum secret sharing and quantum communication protocols. The conditions equations (5), (8) and (9) obtained by us can be used to judge given quantum states in \mathbb{C}^2 whether can be masked. It is also useful for subsequent applications in quantum secret sharing to find masking quantum states more quickly by the masking conditions.

3. Masking of two-qubit quantum states

Here, we consider some non-orthogonal quantum states and orthogonal quantum states respectively, then we get some corresponding conclusions as follows.

3.1. Masking of non-orthogonal quantum states

By observing some maskable non-orthogonal quantum states in \mathbb{C}^2 , we obtain the result that if two non-orthogonal quantum states can mask quantum information contained in a single qubit state, they have the same number of terms and the same basis. Below, we present the analysis process of this result.

Assume that $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are two non-orthogonal quantum states in $\mathbb{C}^2 \otimes \mathbb{C}^2$, and $|\Psi\rangle = \alpha_0|\Psi_0\rangle + \alpha_1|\Psi_1\rangle$ can mask the quantum information in the state

Table 1. $|\Psi_0\rangle$ consists of one term (The duplicate parts have been deleted).

$ \Psi_0\rangle = i_1i_2\rangle$	$ \Psi_1\rangle = b i_1i_2\rangle + c j_1j_2\rangle$	
	$ i_1i_2\rangle$	$ j_1j_2\rangle$
$ 00\rangle$	$ 00\rangle$	$ 00\rangle$
$ 01\rangle$	$ 01\rangle$	$ 01\rangle$
$ 10\rangle$	$ 10\rangle$	$ 10\rangle$
$ 11\rangle$	$ 11\rangle$	$ 11\rangle$

$|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, where $\sum_{i=0}^1|\alpha_i|^2 = 1$. Hence, we have

$$\begin{aligned} \text{Tr}_A(|\Psi\rangle\langle\Psi|) &= \text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) \\ &= \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|), \\ \text{Tr}_B(|\Psi\rangle\langle\Psi|) &= \text{Tr}_B(|\Psi_0\rangle\langle\Psi_0|) \\ &= \text{Tr}_B(|\Psi_1\rangle\langle\Psi_1|). \end{aligned}$$

We discuss four possible cases of $|\Psi_0\rangle$ respectively to make an analysis as follows.

Case 1. When $|\Psi_0\rangle$ consists of one term, i.e. $|\Psi_0\rangle = |i_1i_2\rangle$, where $i_1, i_2 \in \{0, 1\}$. Assume that $|\Psi_1\rangle = b|i_1i_2\rangle + c|j_1j_2\rangle$, where $b \neq 0, c \neq 0, |b|^2 + |c|^2 = 1$ and $i_1, i_2, j_1, j_2 \in \{0, 1\}$. The specific classification of $|\Psi_0\rangle, |\Psi_1\rangle$ and the corresponding masking situation are shown in table 1. Considering the compactness, the table lists only situations that can be masked.

For the cases that cannot be masked and are not listed in table 1, we take the case of $i_1 = j_1$ and $i_2 \neq j_2$ as an example. It can be seen that

$$\begin{aligned} \text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) &= |i_2\rangle\langle i_2|, \\ \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|) &= |b|^2|i_2\rangle\langle i_2| + bc^*|i_2\rangle\langle j_2| \\ &\quad + b^*c|j_2\rangle\langle i_2| + |c|^2|j_2\rangle\langle j_2|. \end{aligned}$$

Since $b, c \neq 0$, then $\text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) \neq \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|)$, which contradicts the masking condition.

Case 2. Suppose $|\Psi_0\rangle = a_0|i_1i_2\rangle + a_1|j_1j_2\rangle$, where $a_0, a_1 \neq 0, |a_0|^2 + |a_1|^2 = 1$ and $i_1, i_2, j_1, j_2 \in \{0, 1\}$.

Assume $|\Psi_1\rangle = b_0|i_1i_2\rangle + b_1|k_1k_2\rangle + b_2|l_1l_2\rangle$, where $\sum_{i=0}^2|b_i|^2 = 1, b_0, b_1, b_2 \neq 0$ and $i_1, i_2, k_1, k_2, l_1, l_2 \in \{0, 1\}$. By classifying and discussing the relationship among $i_1, i_2, j_1, j_2, k_1, k_2, l_1, l_2$, we take the partial trace of $|\Psi_0\rangle$ and $|\Psi_1\rangle$ respect to both system A and system B, through which we can know whether $|\Psi_0\rangle$ and $|\Psi_1\rangle$ can mask the single qubit state. The situations that can be masked are shown in table 2.

Choose the case where $|\Psi_0\rangle = a_0|00\rangle + a_1|01\rangle$ and $|\Psi_1\rangle = b_0|00\rangle + b_1|01\rangle + b_2|10\rangle$ to prove that it cannot be masked.

We obtain that

$$\begin{aligned} \text{Tr}_B(|\Psi_0\rangle\langle\Psi_0|) &= |0\rangle\langle 0|, \\ \text{Tr}_B(|\Psi_1\rangle\langle\Psi_1|) &= (|b_0|^2 + |b_1|^2)|0\rangle\langle 0| \\ &\quad + b_0b_2^*|0\rangle\langle 1| + b_0^*b_2|1\rangle\langle 0| + |b_1|^2|1\rangle\langle 1|, \end{aligned}$$

where $b_0, b_1, b_2 \neq 0$. Then $\text{Tr}_B(|\Psi_0\rangle\langle\Psi_0|) \neq \text{Tr}_B(|\Psi_1\rangle\langle\Psi_1|)$, which contradicts the condition of masking.

Case 3. When $|\Psi_0\rangle$ is constitutive of three terms, namely, $|\Psi_0\rangle = a_0|i_1i_2\rangle + a_1|j_1j_2\rangle + a_2|k_1k_2\rangle$, where $a_0, a_1, a_2 \neq 0, \sum_{i=0}^2|a_i|^2 = 1$ and $i_1, i_2, j_1, j_2, k_1, k_2 \in \{0, 1\}$.

Table 2. $|\Psi_0\rangle$ consists of two terms (The duplicate parts have been deleted).

$ \Psi_0\rangle = a_0 i_1i_2\rangle + a_1 j_1j_2\rangle$		$ \Psi_1\rangle = b_0 i_1i_2\rangle + b_1 k_1k_2\rangle + b_2 l_1l_2\rangle$		
$ i_1i_2\rangle$	$ j_1j_2\rangle$	$ i_1i_2\rangle$	$ k_1k_2\rangle$	$ l_1l_2\rangle$
$ 00\rangle$	$ 01\rangle$	$ 00\rangle$	$ 00\rangle$	$ 01\rangle$
$ 00\rangle$	$ 10\rangle$	$ 00\rangle$	$ 00\rangle$	$ 10\rangle$
$ 00\rangle$	$ 11\rangle$	$ 00\rangle$	$ 00\rangle$	$ 11\rangle$
$ 01\rangle$	$ 10\rangle$	$ 01\rangle$	$ 01\rangle$	$ 10\rangle$
$ 01\rangle$	$ 11\rangle$	$ 01\rangle$	$ 01\rangle$	$ 11\rangle$
$ 10\rangle$	$ 11\rangle$	$ 10\rangle$	$ 10\rangle$	$ 11\rangle$

Table 3. $|\Psi_0\rangle$ consists of three terms (The duplicate parts have been deleted).

$ \Psi_0\rangle = a_0 i_1i_2\rangle + a_1 j_1j_2\rangle + a_2 k_1k_2\rangle$			$ \Psi_1\rangle = b_0 i_1i_2\rangle + b_1 l_1l_2\rangle + b_2 m_1m_2\rangle + b_3 n_1n_2\rangle$			
$ i_1i_2\rangle$	$ j_1j_2\rangle$	$ k_1k_2\rangle$	$ i_1i_2\rangle$	$ l_1l_2\rangle$	$ m_1m_2\rangle$	$ n_1n_2\rangle$
$ 00\rangle$	$ 01\rangle$	$ 10\rangle$	$ 00\rangle$	$ 00\rangle$	$ 01\rangle$	$ 10\rangle$
$ 00\rangle$	$ 01\rangle$	$ 11\rangle$	$ 00\rangle$	$ 00\rangle$	$ 01\rangle$	$ 11\rangle$
$ 00\rangle$	$ 10\rangle$	$ 11\rangle$	$ 00\rangle$	$ 00\rangle$	$ 10\rangle$	$ 11\rangle$
$ 01\rangle$	$ 10\rangle$	$ 11\rangle$	$ 01\rangle$	$ 01\rangle$	$ 10\rangle$	$ 11\rangle$

Suppose that $|\Psi_1\rangle = b_0|i_1i_2\rangle + b_1|l_1l_2\rangle + b_2|m_1m_2\rangle + b_3|n_1n_2\rangle$, where $b_0, b_1, b_2, b_3 \neq 0, \sum_{i=0}^3 |b_i|^2 = 1$ and $i_1, i_2, l_1, l_2, m_1, m_2, n_1, n_2 \in \{0, 1\}$. We get the table 3, similar to the above.

We choose the case where $|\Psi_0\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle$ and $|\Psi_1\rangle = b_0|00\rangle + b_1|01\rangle + b_2|10\rangle + b_3|11\rangle$ to analyze the situation that cannot be masked. Then we have

$$\begin{aligned} \text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) &= (|a_0|^2 + |a_2|^2)|0\rangle\langle 0| \\ &+ |a_1|^2|1\rangle\langle 1| + a_0a_1^*|0\rangle\langle 1| + a_0^*a_1|1\rangle\langle 0|, \\ \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|) &= (|b_0|^2 + |b_2|^2)|0\rangle\langle 0| \\ &+ (|b_1|^2 + |b_3|^2)|1\rangle\langle 1| \\ &+ (b_0b_1^* + b_2b_3^*)|0\rangle\langle 1| + b_0^*b_1|1\rangle\langle 0|. \end{aligned}$$

Since $\text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) = \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|)$, it can be seen that

$$\begin{cases} |a_0|^2 + |a_2|^2 - |b_0|^2 - |b_2|^2 = 0, \\ |a_1|^2 - |b_1|^2 - |b_3|^2 = 0, \\ a_0a_1^* - b_0b_1^* - b_2b_3^* = 0, \\ a_0^*a_1 - b_0^*b_1 = 0. \end{cases}$$

After the calculation, we get $b_2b_3^* = 0$, i.e. $b_2 = 0$ or $b_3 = 0$. This is in contradiction with the assumption of $b_2, b_3 \neq 0$.

Case 4. $|\Psi_0\rangle$ consists of four terms, that is, $|\Psi_0\rangle = a_0|00\rangle + a_1|01\rangle + a_2|10\rangle + a_3|11\rangle$, where $a_0, a_1, a_2, a_3 \neq 0$ and $\sum_{i=0}^3 |a_i|^2 = 1$. So we acquire

$$\begin{aligned} \text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) &= (|a_0|^2 + |a_2|^2)|0\rangle\langle 0| \\ &+ (|a_1|^2 + |a_3|^2)|1\rangle\langle 1| \\ &+ (a_0a_1^* + a_2a_3^*)|0\rangle\langle 1| + a_0^*a_1|1\rangle\langle 0|, \\ \text{Tr}_B(|\Psi_0\rangle\langle\Psi_0|) &= (|a_0|^2 + |a_1|^2)|0\rangle\langle 0| \\ &+ (|a_2|^2 + |a_3|^2)|1\rangle\langle 1| \\ &+ (a_0a_2^* + a_1a_3^*)|0\rangle\langle 1| \\ &+ (a_0^*a_2 + a_1^*a_3)|1\rangle\langle 0|. \end{aligned}$$

Suppose that $|\Psi_1\rangle = b_0|i_1i_2\rangle + b_1|j_1j_2\rangle + b_2|k_1k_2\rangle + b_3|l_1l_2\rangle$, where $\sum_{i=0}^3 |b_i|^2 = 1, b_0, b_1, b_2, b_3 \neq 0$ and $i_1, i_2, j_1, j_2, k_1, k_2, l_1, l_2 \in \{0, 1\}$. We make a similar argument as the proof of Case 3 and find that $|\Psi\rangle = \alpha_0|\Psi_0\rangle + \alpha_1|\Psi_1\rangle$ can mask the quantum information in the state $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ only if $|\Psi_1\rangle = b_0|00\rangle + b_1|01\rangle + b_2|10\rangle + b_3|11\rangle$.

If not, considering the case of $|\Psi_1\rangle = b_0|00\rangle + b_1|00\rangle + b_2|10\rangle + b_3|11\rangle$, we obtain

$$\begin{aligned} \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|) &= (|b_0|^2 \\ &+ b_0b_1^* + b_0^*b_1 + |b_1|^2 + |b_2|^2)|0\rangle\langle 0| \\ &+ |b_3|^2|1\rangle\langle 1| + b_2b_3^*|0\rangle\langle 1| + b_2^*b_3|1\rangle\langle 0|. \end{aligned}$$

Due to $\text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) = \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|)$, we know that

$$\begin{cases} |a_0|^2 + |a_2|^2 - |b_0|^2 - b_0b_1^* - b_0^*b_1 - |b_1|^2|b_2|^2 = 0, \\ |a_1|^2 + |a_3|^2 - |b_3|^2 = 0, \\ a_0a_1^* + a_2a_3^* - b_2b_3^* = 0, \\ a_0^*a_1 - b_2^*b_3 = 0. \end{cases}$$

As a result, we acquire that $a_2 = 0$ or $a_3 = 0$, which contradicts the condition $a_2, a_3 \neq 0$.

In summary, the conclusion is correct, i.e. a single qubit state can be masked by two non-orthogonal quantum states, which have the same number of terms and the same basis.

Based on the general masking conditions of the quantum state obtained in the previous section, we further discuss the masking of non-orthogonal quantum states and derive a conclusion, which provides a more efficient method to judge whether the quantum states can be masked. In other words, for non-orthogonal quantum states in \mathbb{C}^2 , they cannot be used to mask the quantum information in a single qubit state as long as do not have the same number of terms or the same basis. Therefore, the conclusion is useful for the study of quantum information masking.

3.2. Masking of orthogonal quantum states

In this section, we discuss the further optimization of the masking conditions when $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are orthogonal.

Let $|\Psi_0\rangle = a_0|00\rangle + a_1|11\rangle$, $|\Psi_1\rangle = b_0|01\rangle + b_1|10\rangle$ and $|\Psi\rangle = \alpha_0|\Psi_0\rangle + \alpha_1|\Psi_1\rangle$, where $\sum_{i=0}^1|a_i|^2 = \sum_{i=0}^1|b_i|^2 = \sum_{i=0}^1|\alpha_i|^2 = 1$. Under this assumption, the masking conditions equations (5), (8) and (9) can be further simplified as

$$\begin{cases} |a_0|^2 = |a_1|^2 = |b_0|^2 = |b_1|^2 = \frac{1}{2}, \\ \alpha_0\alpha_1^*a_0b_0^* + \alpha_0^*\alpha_1a_1^*b_1 = 0, \\ \alpha_0\alpha_1^*a_0b_1^* + \alpha_0^*\alpha_1a_1^*b_0 = 0, \end{cases} \quad (10)$$

and the quantum states $|\Psi_0\rangle$, $|\Psi_1\rangle$ and $|\Psi\rangle$ satisfy the masking definition

$$\begin{aligned} \text{Tr}_A(|\Psi\rangle\langle\Psi|) &= \text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) \\ &= \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}, \\ \text{Tr}_B(|\Psi\rangle\langle\Psi|) &= \text{Tr}_B(|\Psi_0\rangle\langle\Psi_0|) \\ &= \text{Tr}_B(|\Psi_1\rangle\langle\Psi_1|) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}. \end{aligned}$$

We show that when the coefficients of $|\Psi_0\rangle$, $|\Psi_1\rangle$ and $|\Psi\rangle$ satisfy equation (10), quantum information contained in the quantum state $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ can be masked. To demonstrate this claim, we consider the following two examples.

Example 1. For equation (10), we analyze the case that a_0, a_1, b_0 and b_1 are given. Then we discuss what kind of state $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ can be masked by $|\Psi_0\rangle$ and $|\Psi_1\rangle$.

Given

$$\begin{aligned} |\Psi_0\rangle &= \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|11\rangle, \\ |\Psi_1\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle, \end{aligned}$$

we obtain

$$\begin{aligned} |\Psi\rangle = \alpha_0|\Psi_0\rangle + \alpha_1|\Psi_1\rangle &= \alpha_0\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|11\rangle\right) \\ &+ \alpha_1\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle\right). \end{aligned}$$

Therefore, equation (10) can be simplified to

$$\frac{1}{2}\alpha_0\alpha_1^* - \frac{i}{2}\alpha_0^*\alpha_1 = 0. \quad (11)$$

Let $\alpha_0 = x_0 + y_0i$ and $\alpha_1 = x_1 + y_1i$. Due to $\sum_{i=0}^1|\alpha_i|^2 = 1$, we acquire

$$y_1 = \pm\sqrt{1 - (x_0^2 + y_0^2 + x_1^2)},$$

where $x_0, x_1, y_0 \in [-1, 1]$ and $x_0^2 + y_0^2 + x_1^2 \in [0, 1]$.

Then equation (11) is reformulated as

$$\begin{aligned} x_0x_1 + y_0\sqrt{1 - (x_0^2 + y_0^2 + x_1^2)} \\ + x_0\sqrt{1 - (x_0^2 + y_0^2 + x_1^2)} - x_1y_0 = 0, \end{aligned} \quad (12)$$

or

$$\begin{aligned} x_0x_1 - y_0\sqrt{1 - (x_0^2 + y_0^2 + x_1^2)} \\ - x_0\sqrt{1 - (x_0^2 + y_0^2 + x_1^2)} - x_1y_0 = 0. \end{aligned} \quad (13)$$

As shown in figure 1, we find a kind of state $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ which can be masked, i.e. when the coefficients of the state $|b\rangle$ satisfy the relationship in figure 1, the state $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ can be masked by given quantum states $|\Psi_0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|11\rangle$ and $|\Psi_1\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$.

In particular, according to figure 1(b), we consider the case of $x_0 = -\frac{1}{5}, x_1 = 0$ and $y_0 = -\frac{1}{5}$, namely, $\alpha_0 = \frac{1-i}{5}$ and $\alpha_1 = \frac{\sqrt{23}}{5}$.

Through calculation, we obtain

$$\begin{aligned} \text{Tr}_A(|\Psi_0\rangle\langle\Psi_0|) &= \text{Tr}_A(|\Psi_1\rangle\langle\Psi_1|) \\ &= \text{Tr}_A(|\Psi\rangle\langle\Psi|) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}, \\ \text{Tr}_B(|\Psi_0\rangle\langle\Psi_0|) &= \text{Tr}_B(|\Psi_1\rangle\langle\Psi_1|) \\ &= \text{Tr}_B(|\Psi\rangle\langle\Psi|) = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2}. \end{aligned}$$

In other words, we know that the state $|b\rangle = \frac{1-i}{5}|0\rangle + \frac{\sqrt{23}}{5}|1\rangle$ can be masked by the given quantum states $|\Psi_0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|11\rangle$ and $|\Psi_1\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$.

Example 2. By equation (10), we consider that if α_0 and α_1 are certain, what kind of quantum states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ can mask the quantum information in $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$.

Let $\alpha_0 = \frac{1}{\sqrt{1+\lambda^2}}$ and $\alpha_1 = \left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)i$, where $\lambda \in \mathbb{R}$, and denote

$$\begin{aligned} a_0 &= x_0 + y_0i, & a_1 &= x_1 + y_1i, \\ b_0 &= x_2 + y_2i, & b_1 &= x_3 + y_3i. \end{aligned}$$

Then, equation (10) can be reduced to

$$\begin{cases} x_0^2 + y_0^2 = x_1^2 + y_1^2 = x_2^2 + y_2^2 = x_3^2 + y_3^2 = \frac{1}{2}, \\ -x_0x_2 + x_3x_1 - y_0y_2 + y_3y_1 = 0, \\ -x_0x_3 + x_2x_1 - y_0y_3 + y_2y_1 = 0, \\ -x_0y_2 + x_2y_0 + x_3y_1 - x_1y_3 = 0, \\ x_3y_0 - x_0y_3 + x_2y_1 - x_1y_2 = 0. \end{cases} \quad (14)$$

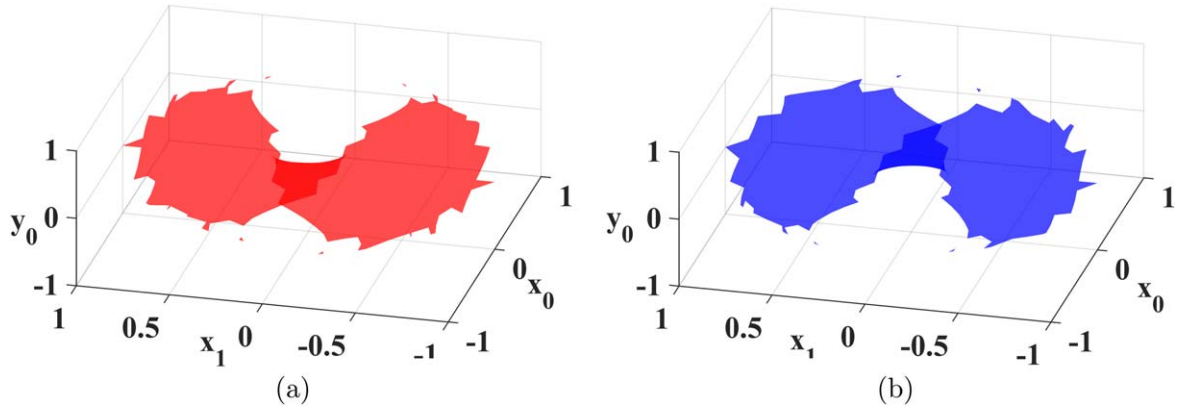


Figure 1. (a) $|\Psi_0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|11\rangle$ and $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ can mask information in $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, if and only if $\alpha_0 = x_0 + y_0i$ and $\alpha_1 = x_1 + \sqrt{1 - (x_0^2 + y_0^2 + x_1^2)}i$ satisfy this figure. (b) $|\Psi_0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{i}{\sqrt{2}}|11\rangle$ and $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ can mask information in $|b\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$, if and only if $\alpha_0 = x_0 + y_0i$ and $\alpha_1 = x_1 - \sqrt{1 - (x_0^2 + y_0^2 + x_1^2)}i$ satisfy this figure.

Particularly, when $x_0 = x_2$ and $y_0 = y_2$, equation (14) is further changed to

$$\begin{cases} x_3x_1 + y_3y_1 - \frac{1}{2} = 0, \\ x_3y_1 - x_1y_3 = 0, \\ (x_0 + y_0)(x_3 - x_1) + (y_0 - x_0)(y_3 - y_1) = 0. \end{cases} \quad (15)$$

Since $x_i^2 + y_i^2 = \frac{1}{2}$, where $i \in \{0, 1, 2, 3\}$, denoting

$$\begin{cases} y_0 = \pm\sqrt{\frac{1}{2} - x_0^2}, \\ y_1 = \pm\sqrt{\frac{1}{2} - x_1^2}, \\ y_2 = y_0 = \pm\sqrt{\frac{1}{2} - x_0^2}, \\ y_3 = \pm\sqrt{\frac{1}{2} - x_3^2}. \end{cases}$$

Through the above equations, we get the solutions to equation (15), that is, $x_1 = x_3 = \frac{\sqrt{2}}{2}$, $x_0 \in \mathbb{R}$ or $x_1 = x_3 = -\frac{\sqrt{2}}{2}$, $x_0 \in \mathbb{R}$. Indeed, we obtain that

$$\begin{cases} |\Psi_0\rangle = (x_0 + y_0i)|00\rangle + \frac{\sqrt{2}}{2}|11\rangle, \\ |\Psi_1\rangle = (x_0 + y_0i)|01\rangle + \frac{\sqrt{2}}{2}|10\rangle, \end{cases} \quad (16)$$

or

$$\begin{cases} |\Psi_0\rangle = (x_0 + y_0i)|00\rangle - \frac{\sqrt{2}}{2}|11\rangle, \\ |\Psi_1\rangle = (x_0 + y_0i)|01\rangle - \frac{\sqrt{2}}{2}|10\rangle. \end{cases} \quad (17)$$

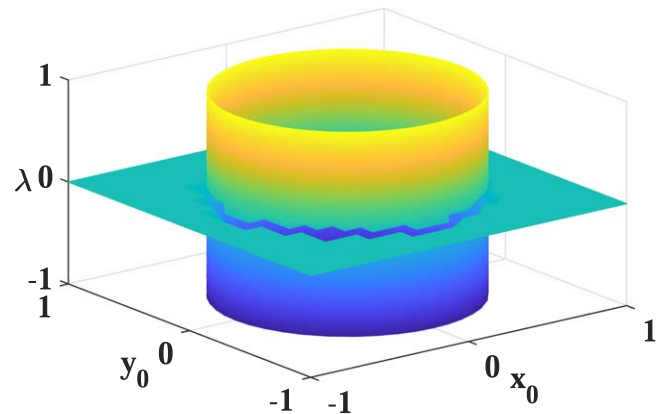


Figure 2. $|b\rangle = (\frac{1}{\sqrt{1+\lambda^2}})|0\rangle + (\frac{\lambda i}{\sqrt{1+\lambda^2}})|1\rangle$ can be masked by $|\Psi_0\rangle = (x_0 + y_0i)|00\rangle + \frac{\sqrt{2}}{2}|11\rangle$ and $|\Psi_1\rangle = (x_0 + y_0i)|01\rangle + \frac{\sqrt{2}}{2}|10\rangle$ or $|\Psi_0\rangle = (x_0 + y_0i)|00\rangle - \frac{\sqrt{2}}{2}|11\rangle$ and $|\Psi_1\rangle = (x_0 + y_0i)|01\rangle - \frac{\sqrt{2}}{2}|10\rangle$, if and only if their coefficients λ, x_0, y_0 satisfy the relationship represented in this image.

According to equations (16) and (17), respectively, equation (10) would be simplified as

$$\left(-x_0^2 - y_0^2 + \frac{1}{2}\right) \frac{\lambda}{1 + \lambda^2} = 0. \quad (18)$$

Moreover, let $\lambda \in [-1, 1]$, then we draw figure 2 for an intuitive description.

As shown in figure 2, for the state $|b\rangle = (\frac{1}{\sqrt{1+\lambda^2}})|0\rangle + (\frac{\lambda i}{\sqrt{1+\lambda^2}})|1\rangle$, there are quantum states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ in equation (16) or equation (17), which can mask the state $|b\rangle$, if and only if their coefficients satisfy the relationship shown in figure 2.

4. Conclusion

In two-dimensional Hilbert space, we expressed the quantum information masking conditions by a system of equations for

the coefficients of quantum states. Moreover, by observing the characteristic of the maskable non-orthogonal two-qubit quantum states, we obtained the conclusion that if two non-orthogonal quantum states can mask a single qubit state, they have the same number of terms and the same basis. Furthermore, we considered two examples of orthogonal quantum states and calculated the masking conditions. Finally, we gave the corresponding images for an intuitive description. Our results would provide a foundation idea for further study on the masking of higher dimensional quantum states.

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