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Multi-solutions with specific geometrical wave structures to a nonlinear evolution equation in the presence of the linear superposition principle

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Abstract

Lump solutions are one of the most common solutions for nonlinear evolution equations. This study aspires to investigate the generalized Hietarintatype equation. We auspiciously provide multiple M-lump waves. On the other hand, collision phenomena to multiple M-lump waves with soliton wave solutions are also provided. During the collision, the amplitude of the lump will change significantly over the processes, whereas the amplitude of the soliton will just minimally alter. As it is of paramount importance, we use suitable values of parameter to put out the physical features of the reported results through three dimensional and contour graphics. The results presented express physical features of lump and lump interaction phenomena of different kinds of nonlinear physical processes. Further, this study serves to enrich nonlinear dynamics and provide insight into how nonlinear waves propagate.

Keywords: the generalized Hietarintatype equation, M-lump waves, interaction phenomena, hirota bilinear method

(Some figures may appear in colour only in the online journal)

1. Introduction

A wave is a dynamic disturbance of one or more quantities that propagates and is frequently expressed by a wave equation in physics, mathematics, and related subjects. At least two field quantities are present in the wave medium for physical waves. When quantities oscillate at a set frequency around an equilibrium value, periodic waves are provided. A standing wave is provided when two superimposed periodic waves travel in opposite directions. A traveling wave is provided when the entire waveform travels in one direction. When the wave amplitude appears to be resized or even nil, a standing wave's vibrational amplitude nulls out at certain points. In mathematics and physics, a soliton, also known as a solitary wave, is a wave packet that keeps its structure while travelling at a constant speed. When dispersive and nonlinear effects in a medium balance out, soliton formation arises. A class of weakly nonlinear dispersive partial differential equations has physical system solutions called solitons [1–6].

Partial differential equations are usually utilized in scientific areas with a powerful quantitative focus. such as physics and engineering. For instance, general relativity, quantum mechanics, electrostatics, electrodynamics, thermodynamics, fluid dynamics, elasticity, and many more areas are all founded on them. The Poincaré conjecture from geometric topology is one of their most notable applications, although they also come from purely mathematical ideas like differential geometry and the calculus of variations [7, 8].

In addition, several studies show that interaction solutions between lumps and other types of solutions to nonlinear integrable equations existed. It should be emphasized, however,

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that numerous approaches such as the Lie symmetric analysis and its application [9–14], the Kudryashov method [15, 16], the tanh-coth method [17], the improved Sardar-subequation [18], the new generalized auxiliary equation method [19], and other techniques [20–24] have been employed for generating interaction phenomena for nonlinear differential equations. Values are assigned to these constant coefficients in particular conditions to ensure that solutions exist [25–41]. Nonetheless, it is worth emphasizing that the problem of dealing with such interactions has yet to be studied. This article provides us with lump collision phenomena to the generalized Hietarintatype equation [42]

$$\alpha_1(6u_xu_{xx}+u_{xxxx})+\alpha_2(3u_tu_{tt}+3u_{xt}v_{tt}+u_{xttt}) \\ +\gamma_1u_{yt}+\gamma_2u_{xx}+\gamma_3u_{xt}+\gamma_4u_{xy}+\gamma_5u_{yy}=0. \tag{1}$$

The multi-soliton solutions, one-lump wave, and mixed one-lump-soliton wave have been studied in [42]. It is worth noting that when $\alpha_2=0$ and $\gamma_1=0$, the aforementioned equation is simplified to the following equation:

$$\alpha(6u_xu_{xx}+u_{xxxx})+\gamma_2u_{xx} \\ +\gamma_3u_{xt}+\gamma_4u_{xy}+\gamma_5u_{yy}=0. \tag{2}$$

In this work, we are interested to study multiple M-lump waves. and constructing interactions of M-lump waves with soliton wave solutions. These interesting solutions are original and have not been proposed in other research papers.

The introduction and primary problem are covered in the first section of the paper. The sorts of M-lump wave solutions are addressed using the long-wave technique in the second section. The third section introduces the phenomenon of collision between 1- and 2-M-lump wave solutions and 1- or 2-soliton solutions. Breather wave, mixed breather-soliton, and mixed breather-M-lump waves are all examined in the fourth section. In the final section, conclusions are made.

2. M-lump wave solutions

Consider equation (2). With the transformation

$$u=2\frac{\partial}{\partial x}(\ln f(x,y,t)). \tag{3}$$

It is easy to know that when f solves equation (2) then u is a solution of equation (1). Usually, the soliton results are provided by utilizing

$$f=f_N=\sum_{\mu=0,1}\exp\left(\sum_{i=1}^{N}\mu_i\Phi_j+\sum_{1\leqslant i<j\leqslant N}^{(N)}\mu_i\mu_jA_{ij}\right). \tag{4}$$

$\sum_{\mu=0,1}$ stands for a summation that comprises of all possible gatherings of $\mu_i=0,1$, for $i=1,2,\ldots,N$. Substituting (2) into equation (1) and integrating it with respect to x provides

$$6\alpha f_{xx}{}^2-2\gamma_5f_y^2-2\gamma_4f_yf_x-2f_x(\gamma_3f_t+\gamma_2f_x+4\alpha f_{xxx}) \\ +2f(\gamma_5f_{yy}+\gamma_3f_{xt}+\gamma_4f_{xy}+\gamma_2f^{(2,0,0)}+\alpha f_{xxxx})=0. \tag{5}$$

The Hirota bilinear form of equation (5) becomes

$$(\alpha D_x^4+\gamma_2D_x^2+\gamma_3D_xD_t+\gamma_4D_xD_y+\gamma_5D_y^2)f.f=0, \tag{6}$$

where D is the Hirota bilinear operator, and provided by Hirota direct method as

$$\prod_{i=1}^{m}D_{x_i}^{n_i}f.g=\prod_{i=1}^{m}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x'_i}\right)^{n_i}f(x)g(x')\bigg|_{x'=x}, \tag{7}$$

where $\boldsymbol{x}=(x_1,x_2,\ldots,x_M)$, $\boldsymbol{x}'=(x'_1,x'_2,\ldots,x'_M)$ are vectors and $n_1,n_2,\ldots,n_M$ are random non-negative integers numbers. Let the dispersion variable Φ_m is defined as

$$\Phi_i=k_i(x+l_iy+w_it)+\alpha_i, \tag{8}$$

where the dispersion relation and the phase shift are

$$w_i=-\frac{k_i{}^2\alpha+\gamma_2+l_i(\gamma_4+l_i\gamma_5)}{\gamma_3}, \tag{9}$$

and

$$e^{A_{ij}}=1-\frac{12k_ik_j\alpha}{3(k_i+k_j)^2\alpha-(l_i-l_j)^2\gamma_5}. \tag{10}$$

Taking $N=1$ in equation (3) and put it into equation (2), we get an equation that presents a one-soliton solution

$$u=\frac{2k_1e^{k_1x+k_1l_1y+\alpha_1}}{e^{k_1x+k_1l_1y+\alpha_1}+e^{\frac{k_1t(k_1^2\alpha+\gamma_2+l_1(\gamma_4+l_1\gamma_5))}{\gamma_3}}}. \tag{11}$$

Utilizing $N=2$ in equation (3), provides

$$f=1+e^{\Phi_1}+e^{\Phi_2}+e^{\Phi_1+\Phi_2+A_{12}}, \tag{12}$$

where $e^{A_{ij}}$ is defined in equation (9). As a result of substituting equations (11) into (2), we obtain an equation that describes a two-soliton solution. Also, by taking $N=3$ in equation (3), a result is

$$f=1+e^{\Phi_1}+e^{\Phi_2}+e^{\Phi_3}+e^{\Phi_1+\Phi_2+A_{12}}+e^{\Phi_1+\Phi_3+A_{13}} \\ +e^{\Phi_2+\Phi_3+A_{23}}+e^{\Phi_1+\Phi_2+\Phi_3+A_{123}}, \tag{13}$$

where $A_{123}=A_{12}A_{13}A_{23}$ and $A_{ij}(i<j)$ are stated in equation (9). As a result of inserting this equation into equation (2), we get an equation that presents a three-soliton solution. Moreover, by taking $n=4$, we can get a 4-soliton solution. In figure 1, we present multiple soliton solutions.

We now provide M-lump waves of the studied equation, using the long-wave method. Plugging $e^{\alpha_i}=-1$ into equations (7), (3) holds the formula

$$f_N=\sum_{\mu=0,1}\prod_{i=1}^{N}(-1)^{\mu_i}e^{\mu_i\Phi_i}\prod_{i<j}^{(N)}e^{\mu_i\mu_nA_{ij}},$$

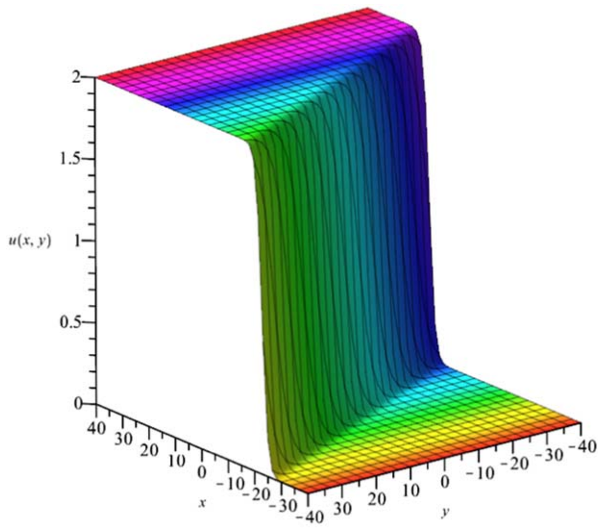
where

$$\Phi_i=k_i\left(x+l_iy-\left(\frac{k_i{}^2\alpha+\gamma_2+l_i(\gamma_4+l_i\gamma_5)}{\gamma_3}\right)t\right)+\alpha_i.$$

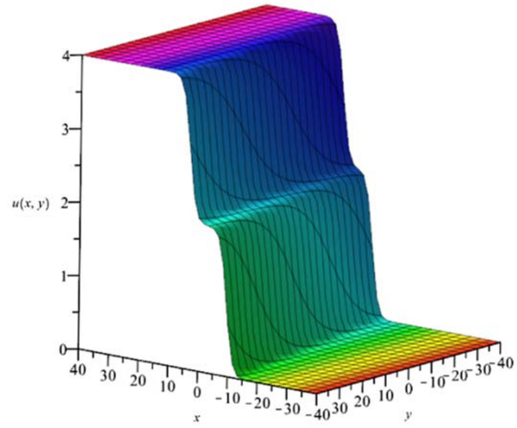
Inserting $k_i\to 0$ into f_N, provides

$$f=f_N=\sum_{\mu=0,1}\prod_{i=1}^{N}(-1)^{\mu_i}(1+\mu_ik_i\Theta_i) \\ \times\prod_{i<j}^{(N)}(1+\mu_ik_i\mu_jk_jB_{ij})+O(k^{N+1}).$$

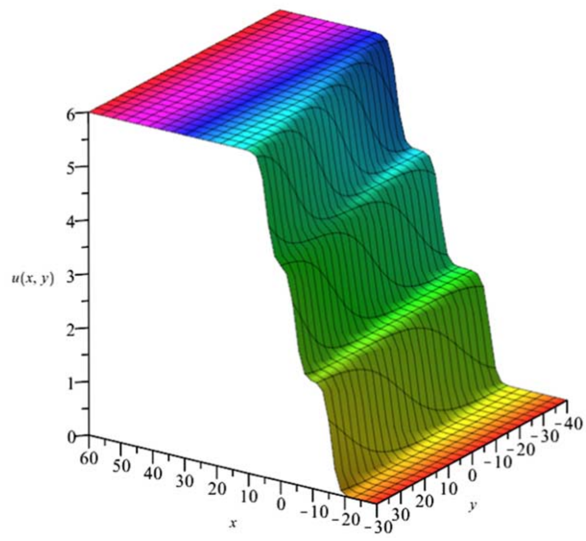
It is obvious that $u=2\ln({}^{f_N}/{}_{\prod_{i=1}^{N}k_i})_x$. To do more about the solution, the term $\prod_{i=1}^{N}k_i$ can be eliminated, and be replaced



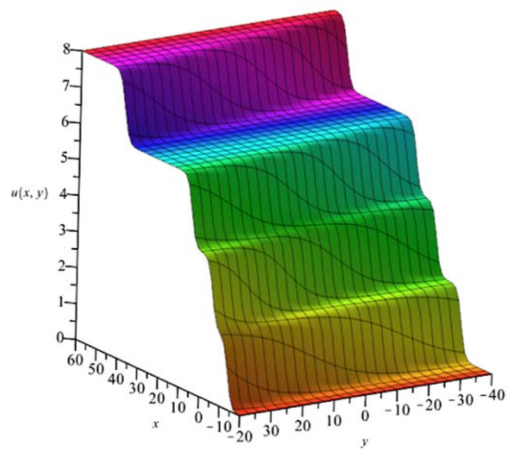
(a) $k_1 = 1, l_1 = 1/2, \alpha_1 = 2$



(b) $k_1 = 1, l_1 = 1/8, k_2 = 1, l_2 = 1/6, \alpha_1 = 0, \alpha_2 = 2$



(c) $k_1 = 1, l_1 = 1/8, k_2 = 1, l_2 = 1/6, k_3 = 1, l_3 = 1/4, \alpha_1 = 10, \alpha_2 = 5, \alpha_3 = 0$



(d) $k_1 = 1, l_1 = 1/10, k_2 = 1, l_2 = 1/8, k_3 = 1, l_3 = 1/6, k_4 = 1, l_4 = 1/4, \alpha_1 = 8, \alpha_2 = 6, \alpha_3 = 4, \alpha_4 = 2$

Figure 1. Multiple soliton wave when $\gamma_2 = 1/2$, $\gamma_3 = -1$, $\gamma_4 = 1/4$, $\gamma 5 = 1/5$, $\alpha = -1/5$, $t = 2$.

by f_N. This leads us to

$$f_N = \prod_{i=1}^{N} \Theta_i + \frac{1}{2}\sum_{i,j}^{(N)} B_{ij} \prod_{r\neq i,j}^{N} \Theta_r + \cdots + \frac{1}{M!2^M} \times \sum_{i,j\cdots,p,q}^{(N)} \overbrace{A_{ij}B_{kl}\cdots B_{pq}}^{M} \prod_{s\neq i,j,k,l,\cdots,p,q}^{N} \Theta_s + \cdots,$$

here, $\Theta_i = x + l_i y - \left(\frac{\gamma_2 + l_i(\gamma_4 + l_i\gamma_5)}{\gamma_3}\right)t$. $\sum_{i,j,\cdots,p,q}^{N}$ is a collection of all feasible gatherings of i, j,...,p, q, that accept different values from 1, 2,...,N. If the parameters l_i affirm the situation $l_{M+i} = l_i^*(i = 1, 2, ..., M)$ with $N = 2M$ and $B_{ij} > 0$.

Here we apply long-wave approach on equation (3) by considering $N = 2$, $k_i \to 0$, $e^{\alpha_i} = -1 (i = 1, 2)$ and $\frac{k_1}{k_2} = O(1)$, then equation (3) becomes

$$f = \Theta_1\Theta_2 + B_{12}, \tag{14}$$

where equation (7) could be rewritten as

$$\Theta_i = x + l_i y - \left(\frac{\gamma_2 + l_i(\gamma_4 + l_i\gamma_5)}{\gamma_3}\right)t, \tag{15}$$

and equation (9) becomes

$$B_{ij} = \frac{12\alpha}{(l_i - l_j)^2\gamma_5},\ l_{\frac{N}{2}+i} = l_i^*, \left(i = 1, 2, ..., \frac{N}{2}\right) \text{and } i < j. \tag{16}$$

Inserting equations (13)–(15) into (2), provides

$$u = 2\frac{\partial}{\partial x}\log\left((x' + ay')^2 + b^2y'^2 - \frac{3\alpha}{b^2\gamma_5}\right) = \frac{4(x' + ay')}{(x' + ay')^2 + b^2y'^2 - \frac{3\alpha}{b^2\gamma_5}} \tag{17}$$

with

$$x' = x - \left(\frac{\gamma_2}{\gamma_3} - a^2\frac{\gamma_5}{\gamma_3} - b^2\frac{\gamma_5}{\gamma_3}\right)t.$$

$$y' = y - \left(\frac{\gamma_4}{\gamma_3} + 2a\frac{\gamma_5}{\gamma_3}\right)t,$$

With a single-M-lump wave for equation (2) as shown in figure 2. This rational result is a permanent wave result decomposing as $O\left(\frac{1}{x^2}, \frac{1}{y^2}\right)$ for $|x|, |y| \to \infty$ and travelling with the speed

$$v_x = \frac{\gamma_2}{\gamma_3} - a^2\frac{\gamma_5}{\gamma_3} - b^2\frac{\gamma_5}{\gamma_3}, \qquad v_y = \frac{\gamma_4}{\gamma_3} + 2a\frac{\gamma_5}{\gamma_3}.$$

Finding out that this wave travels in the following straight line is interesting

$$y = \frac{(\gamma_4 + 2a_1\gamma_5)\left(x + \sqrt{-\frac{3\alpha}{b_1^2\gamma_5}}\right)}{\gamma_2 - (a_1^2 + b_1^2)\gamma_5}.$$

A one-M-lump wave that travels along this line at different times is drawn in figure 1(b).

To derive a 2-M-lump result to equation (2), we let $N = 4$ in equation (3), and considering $e^{\alpha_i} = -1 (i = 1, 2, 3, 4)$, $k_i \to 0$, we provide

$$f = \Theta_1\Theta_2\Theta_3\Theta_4 + B_{12}\Theta_3\Theta_4 + B_{13}\Theta_2\Theta_4 + B_{14}\Theta_2\Theta_3 + B_{23}\Theta_1\Theta_4 + B_{24}\Theta_1\Theta_3 + B_{34}\Theta_1\Theta_2 + B_{12}\Theta_{34} + B_{13}\Theta_{24} + B_{14}B_{23}, \tag{18}$$

where Θ_1, Θ_2, Θ_3, Θ_4, w_i, $B_{ij} (i < j)$ and $l_{\frac{N}{2}+i}$ are defined in equations (14) and (15), respectively.

Inserting equations (17) into (3), provides

$$y_1 = \frac{(\gamma_4 + 2a_1\gamma_5)\left(x + \sqrt{-\frac{3\alpha}{b_1^2\gamma_5}}\right)}{\gamma_2 - (a_1^2 + b_1^2)\gamma_5},$$

and

$$y_2 = \frac{(\gamma_4 + 2a_2\gamma_5)\left(x + \sqrt{-\frac{3\alpha}{b_2^2\gamma_5}}\right)}{\gamma_2 - (a_2^2 + b_2^2)\gamma_5}.$$

The travel of two-M-lump waves alongside the two lines is graphed in figure 3 in different periods.

To provide a 3M-lump result of equation (1), we implement $k_i \to 0$, $e^{\alpha_i} = -1$, $(i = 1, 2, 3, 4, 5, 6)$ and consider $N = 6$ in equation (3), which provides

$$\begin{aligned} f_6 = {} & \Theta_1\Theta_2\Theta_3\Theta_4\Theta_5\Theta_6 + B_{12}B_{34}B_{56} \\ & + B_{12}B_{35}B_{46} + B_{12}B_{45}B_{36} + B_{13}B_{24}B_{56} \\ & + B_{13}B_{25}B_{46} + B_{13}B_{45}B_{26} \\ & + B_{23}B_{14}B_{56} + B_{14}B_{25}B_{36} + B_{14}B_{35}B_{26} + B_{24}B_{15}B_{36} \\ & + B_{34}B_{15}B_{26} + B_{23}B_{15}B_{46} + B_{23}B_{45}B_{16} \\ & + B_{24}B_{35}B_{16} + B_{34}B_{25}B_{16} + \Theta_2\Theta_3\Theta_4\Theta_5B_{16} \\ & + \Theta_2\Theta_3\Theta_5\Theta_6B_{14} \\ & + \Theta_2\Theta_3\Theta_4\Theta_6B_{15} + \Theta_3\Theta_4\Theta_5\Theta_6B_{12} \\ & + \Theta_2\Theta_4\Theta_5\Theta_6B_{13} + \Theta_1\Theta_2\Theta_4\Theta_6B_{35} + \Theta_1\Theta_2\Theta_4\Theta_5B_{36} \\ & + \Theta_1\Theta_4\Theta_5\Theta_6B_{23} + \Theta_1\Theta_3\Theta_5\Theta_6B_{24} + \Theta_1\Theta_3\Theta_4\Theta_6B_{25} \\ & + \Theta_1\Theta_3\Theta_4\Theta_5B_{26} + \Theta_1\Theta_2\Theta_3\Theta_4B_{56} + \Theta_1\Theta_2\Theta_3\Theta_6B_{45} \\ & + \Theta_1\Theta_2\Theta_3\Theta_5B_{46} + \Theta_1\Theta_2\Theta_5\Theta_6B_{34} + \Theta_1\Theta_2B_{34}B_{56} \\ & + \Theta_1\Theta_2B_{35}B_{46} + \Theta_1\Theta_2B_{45}B_{36} + \Theta_1B_{23}\Theta_4B_{56} \\ & + \Theta_1B_{23}B_{45}\Theta_6 + \Theta_1B_{23}\Theta_5B_{46} + \Theta_1\Theta_3B_{24}B_{56} \\ & + \Theta_1\Theta_6B_{24}B_{35} + \Theta_1\Theta_5B_{24}B_{36} + \Theta_1\Theta_3B_{25}B_{46} \\ & + \Theta_1\Theta_6B_{34}B_{25} + \Theta_1\Theta_4B_{25}B_{36} + \Theta_1\Theta_3B_{45}B_{26} \\ & + \Theta_1\Theta_5B_{34}B_{26} + \Theta_1\Theta_4B_{35}B_{26} + \Theta_4\Theta_5B_{12}B_{36} \\ & + \Theta_3\Theta_4B_{12}B_{56} + \Theta_3\Theta_6B_{12}B_{45} + \Theta_3\Theta_5B_{12}B_{46} \\ & + \Theta_5\Theta_6B_{12}B_{34} + \Theta_4\Theta_6B_{12}B_{35} + \Theta_5\Theta_6B_{13}B_{24} \\ & + \Theta_4\Theta_6B_{13}B_{25} + \Theta_4\Theta_5B_{13}B_{26} + \Theta_2\Theta_4B_{13}B_{56} \\ & + \Theta_2\Theta_6B_{13}B_{45} + \Theta_2\Theta_5B_{13}B_{46} + \Theta_2\Theta_3B_{14}B_{56} \\ & + \Theta_2\Theta_6B_{14}B_{35} + \Theta_2\Theta_5B_{14}B_{36} + \Theta_5\Theta_6B_{23}B_{14} \\ & + \Theta_3\Theta_6B_{14}B_{25} + \Theta_3\Theta_5B_{14}B_{26} + \Theta_4\Theta_6B_{23}B_{15} \\ & + \Theta_3\Theta_6B_{24}B_{15} + \Theta_3\Theta_4B_{15}B_{26} + \Theta_2\Theta_3B_{15}B_{46} \\ & + \Theta_2\Theta_6B_{34}B_{15} + \Theta_2\Theta_4B_{15}B_{36} \\ & + \Theta_2\Theta_4B_{35}B_{16} + \Theta_4\Theta_5B_{23}B_{16} \\ & + \Theta_3\Theta_5B_{24}B_{16} + \Theta_3\Theta_4B_{25}B_{16} \\ & + \Theta_2\Theta_3B_{45}B_{16} + \Theta_2\Theta_5B_{34}B_{16}. \end{aligned} \tag{19}$$

We should know that Θ_1, Θ_2, Θ_3, Θ_4, Θ_5, Θ_6, B_{ij} and $l_{\frac{N}{2}+i}$ are provides in equations (14) and (15), respectively. Subsequently, inserting equations (21) into (3), a 3-M-lump result is provided as displayed in figure 4. We should know that $l_1 = a_1 + ib_1$, $l_2 = a_2 + ib_2$, $l_3 = a_3 + ib_3$, $l_4 = l_1^*$, $l_5 = l_2^*$ and $l_6 = l_3^*$. The following straight lines are followed by a three-M-lump wave, which is interesting to know

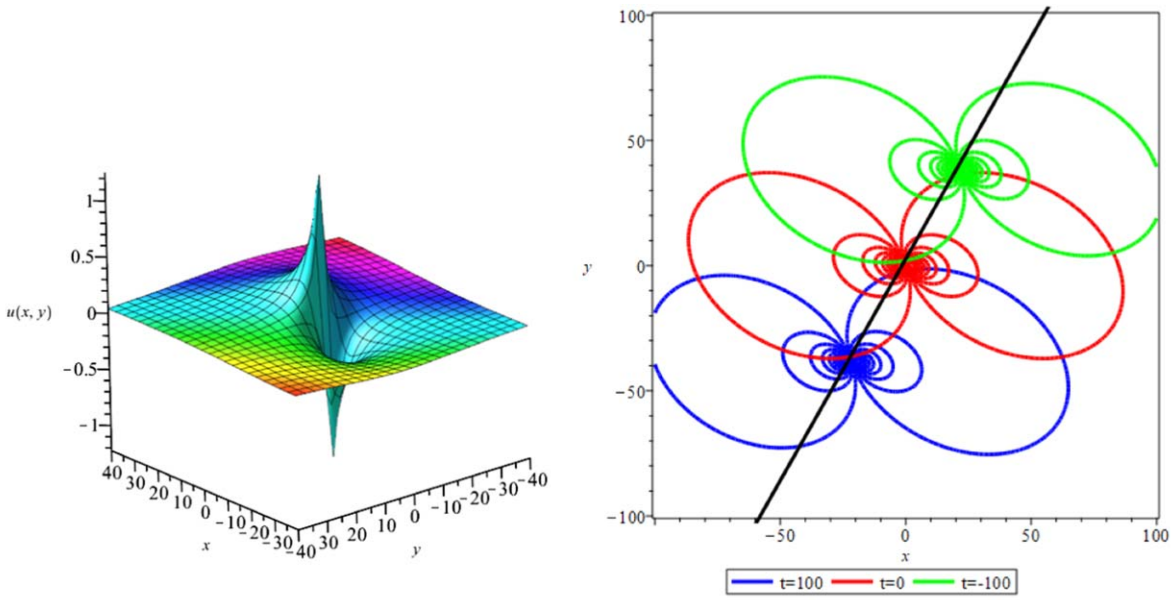
$$y_1 = \frac{(\gamma_4 + 2a_1\gamma_5)\left(x + \sqrt{-\frac{3\alpha}{b_1^2\gamma_5}}\right)}{\gamma_2 - (a_1^2 + b_1^2)\gamma_5},$$

$$y_2 = \frac{(\gamma_4 + 2a_2\gamma_5)\left(x + \sqrt{-\frac{3\alpha}{b_2^2\gamma_5}}\right)}{\gamma_2 - (a_2^2 + b_2^2)\gamma_5},$$

and

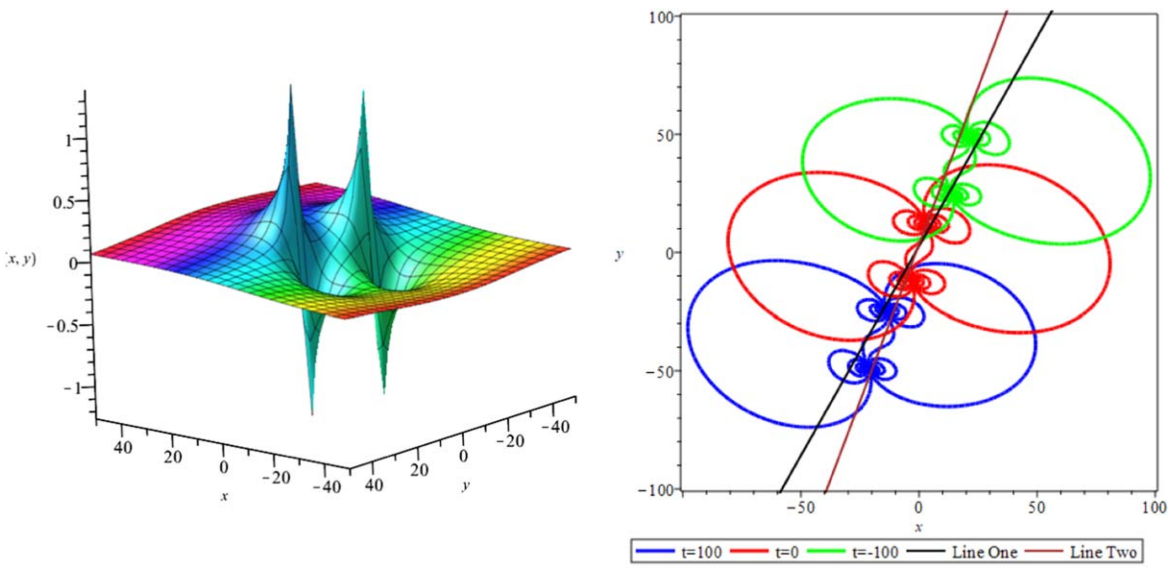
$$y_3 = \frac{(\gamma_4 + 2a_3\gamma_5)\left(x + \sqrt{-\frac{3\alpha}{b_3^2\gamma_5}}\right)}{\gamma_2 - (a_3^2 + b_3^2)\gamma_5}.$$

In figure 5, the three-M-lump waves' path along these three lines is graphed for various times.



(a) t=5 (b) $z = 1, t = 5$

Figure 2. One-M-lump wave when $a_1 = 1/3$, $b_1 = 8/7$, $\gamma_2 = 1/2$, $\gamma_3 = -1$, $\gamma_4 = 1/4$, $\gamma 5 = 1/5$, $\alpha = -1/5$.



(a) t=5 (b) $z = 1, t = 5$

Figure 3. Two-M-lump wave when $a_1 = 1/3$, $b_1 = 8/7$, $a_2 = 1/4$, $b_2 = 4/3$, $\gamma_2 = 1/2$, $\gamma_3 = -1$, $\gamma_4 = 1/4$, $\gamma 5 = 1/5$, $\alpha = -1/5$.

3. Collision phenomena

This section attempts to design the collision of a 1-M-lump result along with one-soliton, two-soliton, and a 2-M-lump wave with a single soliton. To begin with, we set $N = 3$ in equation (3) and take the limit $k_i \to 0, (i = 1, 2)$ and $\frac{k_1}{k_2} = O(1)$, and this provides

$$f = \Theta_1\Theta_2 + B_{12} + \xi_1 e^{\Phi_3}, \tag{20}$$

where

$$\xi_1 = \Theta_1\Theta_2 + B_{12} + C_{23}\Theta_1 + C_{13}\Theta_2 + C_{13}C_{23}.$$

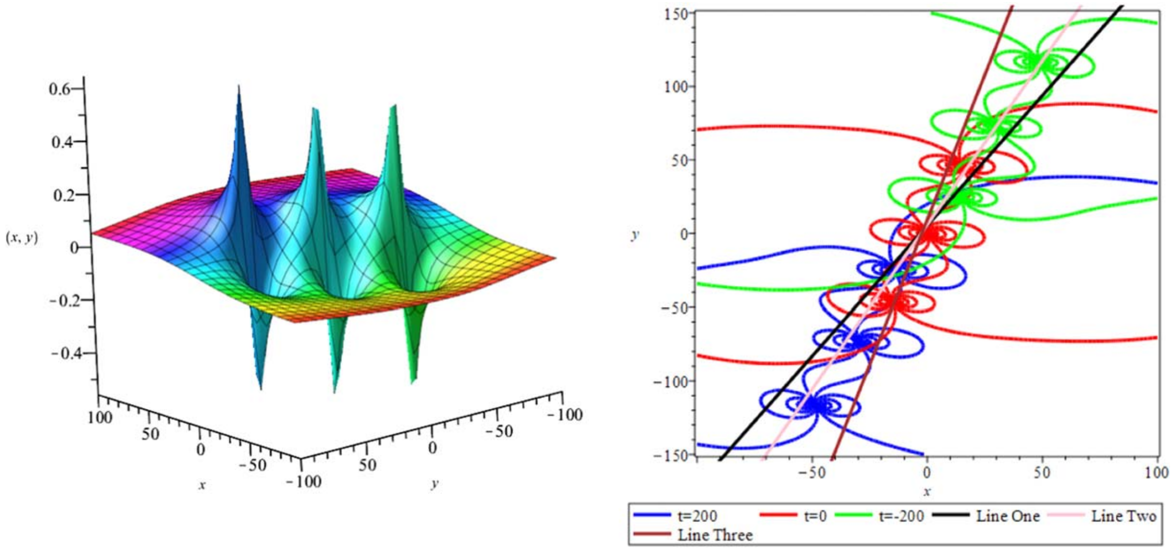
Here Φ_3 is expressed in equation (7), $\Theta_i (i = 1, 2)$ are provided in equation (14), B_{12} is given in equation (15). C_{13}, C_{23} are derived as follows

$$C_{ij} = -\frac{12k_j\alpha}{3{k_j}^2\alpha - (l_i - l_j)^2\gamma_5} \qquad i < j. \tag{21}$$

A result that depicts the interaction between a 1-M-lump solution and a 1-soliton solution is obtained by putting equations (20) into (19), and then into equation (3), as shown in figure 4.

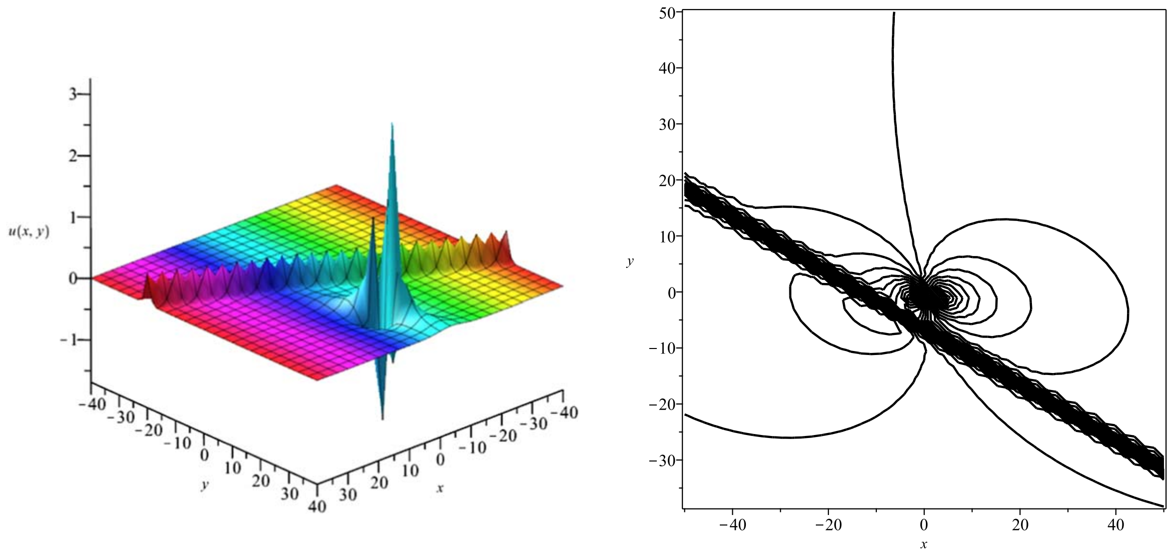
To provide a collision of two-soliton with an M-lump wave, we consider that

$$f = B_{12} + \xi_1 e^{\Phi_3} + \xi_2 e^{\Phi_4} + \Theta_1\Theta_2 - e^{\Phi_3+\Phi_4+A_{34}} \times(\Theta_1\Theta_2 + B_{12} - C_{14}C_{23} - C_{13}C_{24} - \xi_1 - \xi_2), \tag{22}$$



(a) t=5 (b) $z=1, t=5$

Figure 4. Three-M-lump wave when $a_1=1/3$, $b_1=8/7$, $a_2=1/4$, $b_2=9/7$, $a_3=1/5$, $b_3=10/7$, $\gamma_2=1/2$, $\gamma_3=-1$, $\gamma_4=1/4$, $\gamma 5=1/5$, $\alpha=-1/5$.



(a) (b)

Figure 5. Mixed M-lump-soliton wave when $a_1=1/3$, $b_1=3/2$, $k_3=1$, $l_3=2$, $\alpha_3=5$, $\gamma_2=1/2$, $\gamma_3=-1$, $\gamma_4=1/4$, $\gamma 5=1/5$, $\alpha=-1/5$, $t=5$.

where

$$\xi_1=\Theta_1\Theta_2+B_{12}+C_{23}\Theta_1+C_{13}\Theta_2+C_{13}C_{23},$$

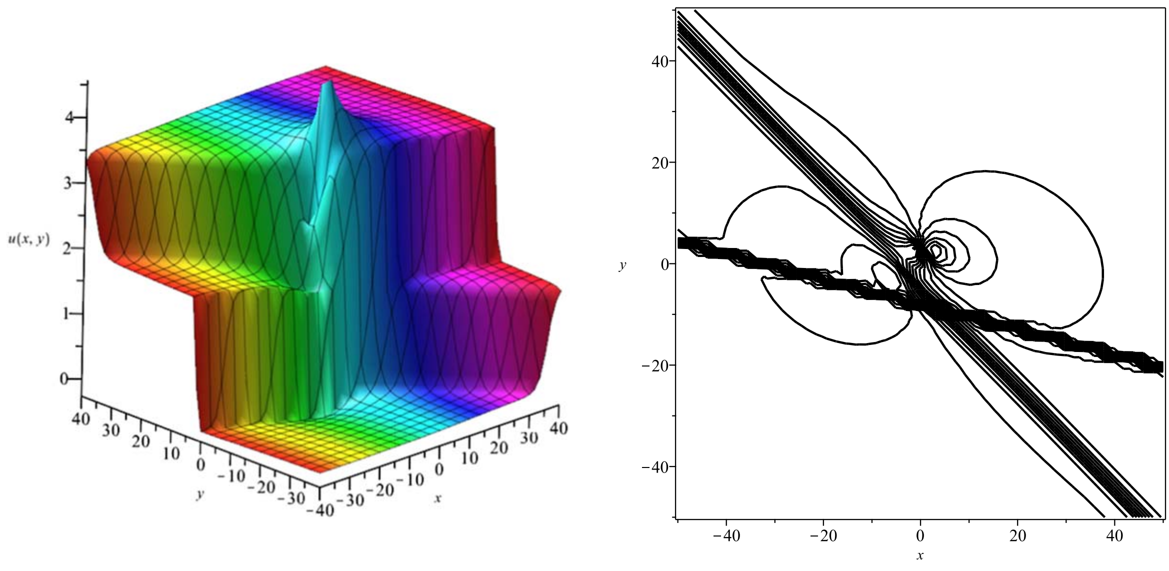
and

$$\xi_2=\Theta_1\Theta_2+B_{12}+C_{24}\Theta_1+C_{14}\Theta_2+C_{14}C_{24}.$$

The constants B_{ij} and C_{ij} are provided in equations (14) and (20). The functions Φ_i and Θ_i are provided in equations (7) and (15). Inserting equations (21) into (2), an outcome is a result that stands for a collision between a one-M-lump wave and a two-soliton solution which display in figure 6.

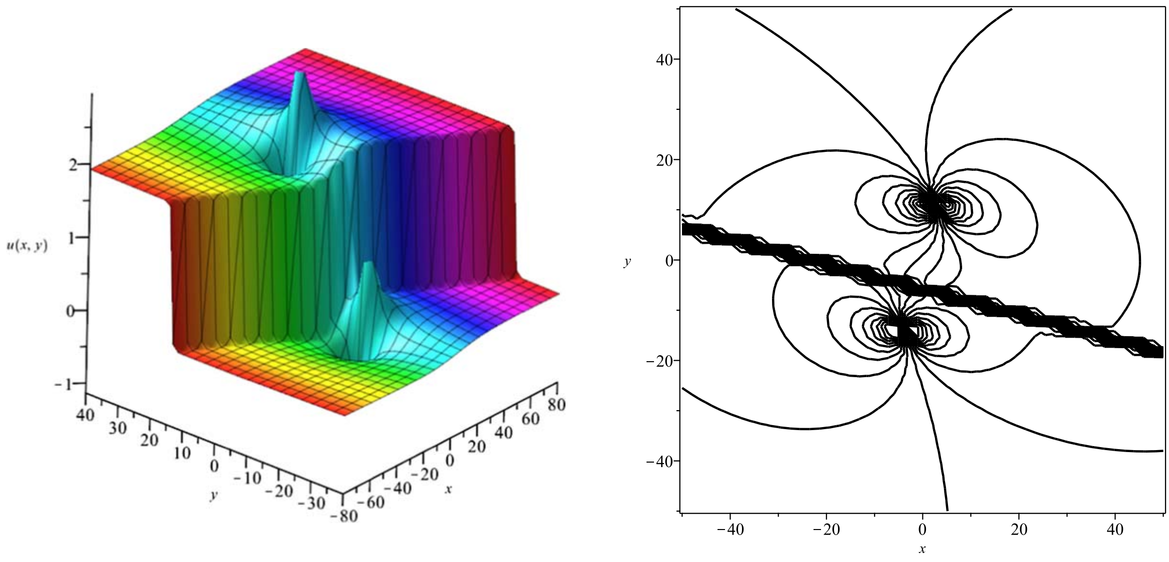
When $N=5$ and considering the limit $k_i\to 0$, $(i=1, 2, 3, 4)$ in equation (3), the collision between two-M-lump wave and one-soliton are

$$\begin{aligned} f=&\Theta_1\Theta_2\Theta_3\Theta_4+B_{34}\Theta_1\Theta_2+B_{24}\Theta_1\Theta_3\\ &+B_{23}\Theta_1\Theta_4+B_{14}\Theta_2\Theta_3+B_{13}\Theta_2\Theta_4\\ &+B_{12}\Theta_3\Theta_4+Q\mathrm{e}^{k_5(x+l_5y+w_5t)+\alpha_5}\\ &+B_{14}B_{23}+B_{13}B_{24}+B_{12}B_{34}, \end{aligned} \tag{23}$$



(a) (b) $z = 1, t = 5$

Figure 6. Mixed M-lump-two-soliton wave when $a_1 = 1/3$, $b_1 = 3/2$, $a_2 = 1/4$, $b_2 = 9/7$, $k_3 = 1$, $l_3 = 2$, $k_4 = 3/4$, $l_4 = 1$, $\alpha_3 = 10$, $\alpha_4 = 0$, $\gamma_2 = 1/2$, $\gamma_3 = -1$, $\gamma_4 = 1/4$, $\gamma 5 = 1/5$, $\alpha = -1/5$, $t = 5$.



(a) (b) $z = 1, t = 5$

Figure 7. Mixed two-M-lump-soliton wave when $a_1 = 1/3$, $b_1 = 8/7$, $a_2 = 1/4$, $b_2 = 4/3$, $k_5 = 1$, $l_5 = 4$, $\alpha_5 = 1$, $\gamma_2 = 1/2$, $\gamma_3 = -1$, $\gamma_4 = 1/4$, $\gamma 5 = 1/5$, $\alpha = -1/5$, $t = 5$.

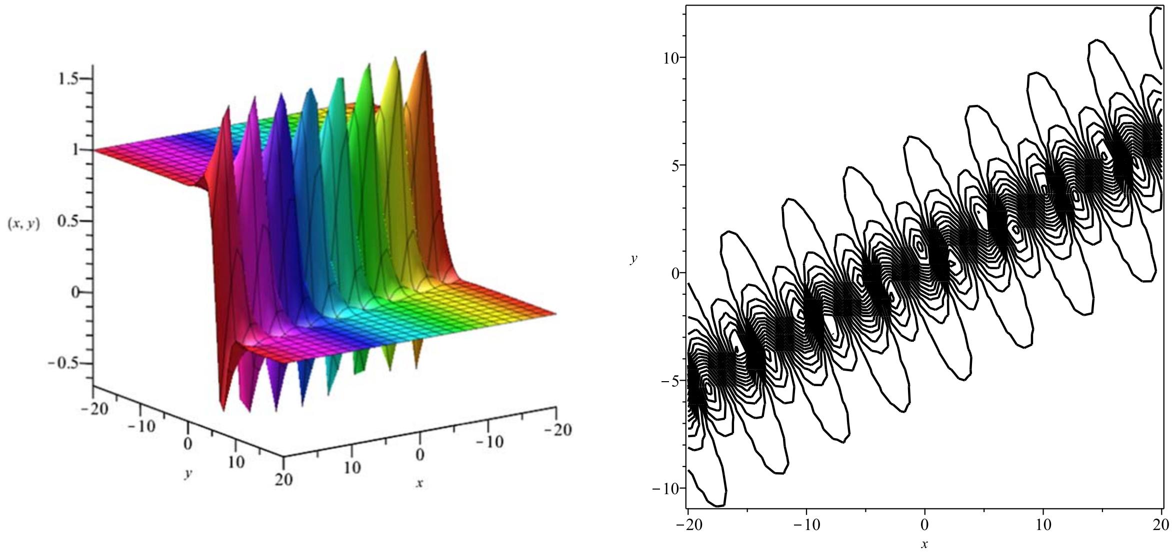
where

$$\begin{aligned} Q = {} & \Theta_1\Theta_2\Theta_3\Theta_4 + C_{45}\Theta_1\Theta_2\Theta_3 + C_{15}\Theta_2\Theta_3\Theta_4 \\ & + C_{25}\Theta_1\Theta_3\Theta_4 \\ & + C_{35}\Theta_1\Theta_2\Theta_4 + (B_{34} + C_{35}C_{45})\Theta_1\Theta_2 \\ & + (B_{24} + C_{25}C_{45})\Theta_1\Theta_3 + (B_{14} + C_{15}C_{45})\Theta_2\Theta_3 \\ & + (B_{23} + C_{25}C_{35})\Theta_1\Theta_4 + (B_{13} + C_{15}C_{35})\Theta_2\Theta_4 \\ & + (B_{12} + C_{15}C_{25})\Theta_3\Theta_4 + (B_{34}C_{25} + B_{24}C_{35} \\ & + B_{23}C_{45} + C_{25}C_{35}C_{45})\Theta_1 \\ & + (B_{34}C_{15} + B_{14}C_{35} + B_{13}C_{45} + C_{15}C_{35}C_{45})\Theta_2 \\ & + (B_{24}C_{15} + B_{14}C_{25} + B_{12}C_{45} + C_{15}C_{25}C_{45})\Theta_3 \\ & + (B_{23}C_{15} + B_{13}C_{25} + B_{12}C_{35} + C_{15}C_{25}C_{35})\Theta_4 \\ & + B_{14}B_{23} + B_{13}B_{24} + B_{12}B_{34} + B_{34}C_{15}C_{25} \\ & + B_{24}C_{15}C_{35} + B_{14}C_{25}C_{35} \\ & + B_{23}C_{15}C_{45} + B_{13}C_{25}C_{45} \\ & + B_{12}C_{35}C_{45} + C_{15}C_{25}C_{35}C_{45}. \end{aligned}$$

In this article, all constants and functions are listed. Now that we have provided a collision between a two-M-lump and a one-soliton solution, inserting equations (22) into (3), and the consequence is the aspect displayed in figure 7.

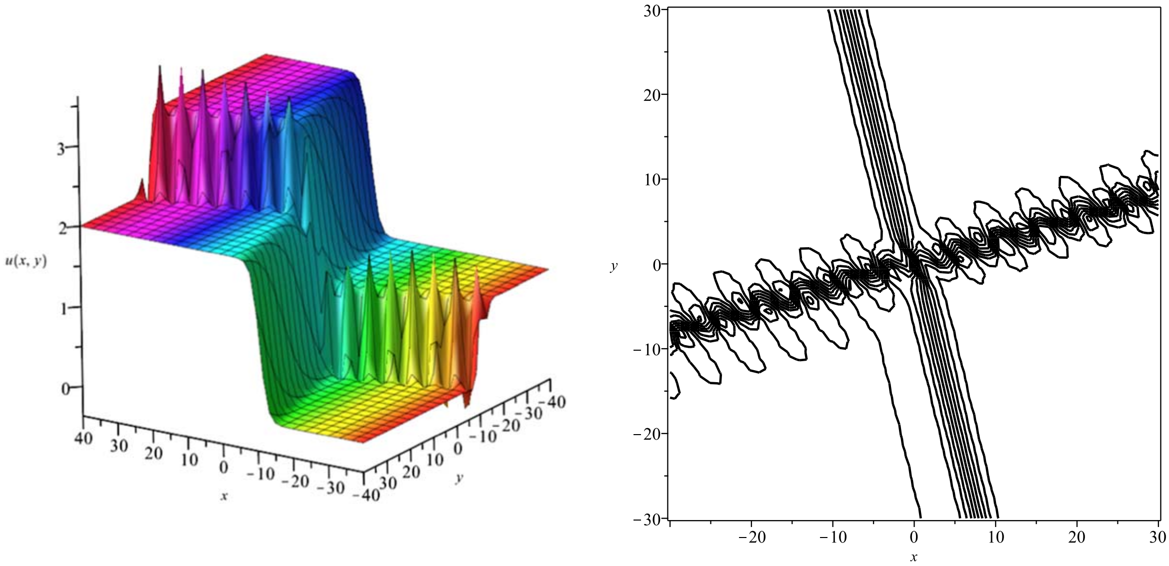
4. Breather solution and its interactions

In this section, we try to explore some new solutions to the suggested equation like breather wave, mixed breather-soliton, and mixed breather-M-lump waves. Taking $k_1 = \frac{1}{4} + \mathrm{i}$, $\mathrm{k}_2 = k_1{}^*$, $l_1 = \frac{1}{2} + \mathrm{i}$, $l_2 = l_1{}^*$ in equation (10) then into equation (2) a result yields a breather wave as presented in figure 8.



(a) (b) $z=1, t=5$

Figure 8. Breather wave when $\alpha_1=0$, $\alpha_2=0$, $\gamma_2=1/2$, $\gamma_3=-1$, $\gamma_4=1/4$, $\gamma 52=1/5$, $\alpha=-1/5$, $t=2$.



(a) (b) $z=1, t=5$

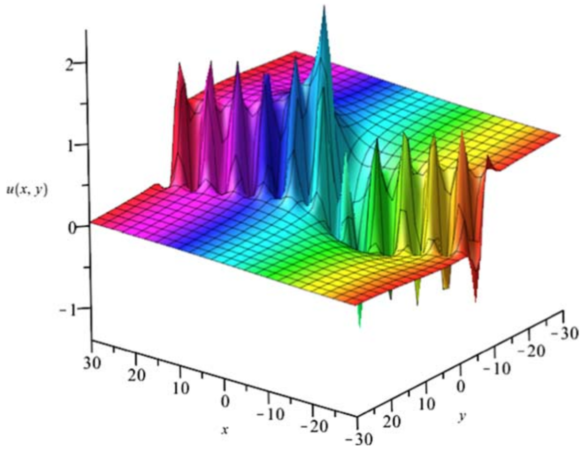
Figure 9. Mixed breather-soliton wave when $k_3=1$, $l_3=1/4$, $\alpha_1=0$, $\alpha_2=0$, $\alpha_3=0$, $\gamma_2=1/2$, $\gamma_3=-1$, $\gamma_4=1/4$, $\gamma 52=1/5$, $\alpha=-1/5$, $t=2$.

An interaction physical phenomena between breather and soliton solution could be constructed by taking $k_1=\frac{1}{4}+$ i, $k_2=k_1{}^*$, $l_1=\frac{1}{2}+\text{i}$, $l_2=l_1{}^*$ in equation (12) then substituting these values into equation (2), an outcome gives us a mixed breather-soliton wave as seen in figure 9.

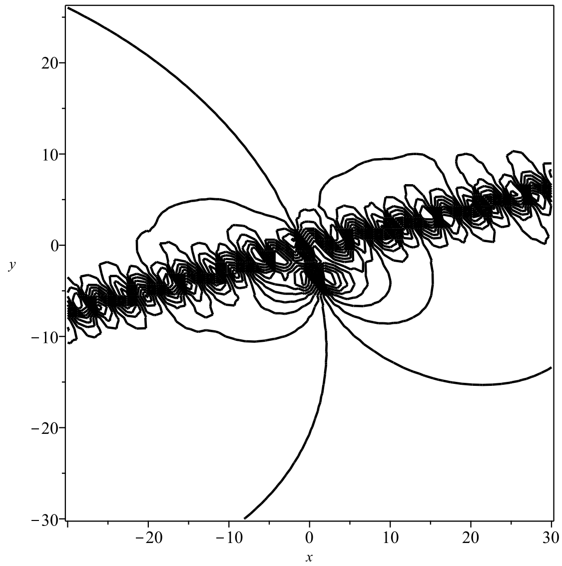
Moreover, to offer a mixed breather-M-lump wave, we let $N=4$, $k_3=\frac{1}{5}+\text{i}$, $k_4=k_3{}^*$, $l_3=\frac{1}{2}+\text{i}$, $l_4=l_3{}^*$ in equation (21) and putting these values into equation (2) gives an interaction between the breather wave and M-lump wave as shown in figure 10.

5. Conclusions

This study investigated the generalized Hietarintatype equation. Multiple M-lump waves alongside their collision phenomena to multiple M-lump waves with soliton wave solutions are auspiciously provided. Using suitable values of parameter, we put out the physical features of the reported results through three dimensional and contour graphics. The results presented express physical features of lump and lump interaction phenomena of different kinds of nonlinear physical processes. Further, Breather solutions and their



(a)



(b) $z = 1, t = 5$

Figure 10. Mixed breather-M-lump wave when $a_1 = 1/3$, $b_1 = 10/7$, $a_2 = 1/4$, $b_2 = 13/7$, $\alpha_3 = 0$, $\alpha_4 = 0$, $\gamma_2 = 1/2$, $\gamma_3 = -1$, $\gamma_4 = 1/4$, $\gamma 52 = 1/5$, $\alpha = -1/5$, $t = 2$.

interactions such as breather waves, mixed breather-solitons, and mixed breather-M-lump waves have been investigated. In a subsequent paper, we analyze the extended Hietarintatype equation with variable coefficients and derive a number of novel conclusions for this model.

Conflict of interest

The authors declare that they have no conflict of interest.

Ethical standard

The author state that this research paper complies with ethical standards. This research paper does not involve either human participants or animals.

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