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# Electrical properties of a generalized 2 × n resistor network

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## Abstract

Any changes in resistor conditions will increase the difficulty of resistor network research. This paper considers a new model of a generalized 2 $\times$ $n$ resistor network with an arbitrary intermediate axis that was previously unsolved. We investigate the potential function and equivalent resistance of the 2 $\times$ $n$ resistor network using the RT-I theory. The RT-I method involves four main steps: (1) establishing difference equations on branch currents, (2) applying a matrix transform to study the general solution of the differential equation, (3) obtaining a current analysis of each branch according to the boundary constraints, and (4) deriving the potential function of any node of the 2 $\times$ $n$ resistor network by matrix transformation, and the equivalent resistance formula between any nodes. The article concludes with a discussion of a series of special results, comparing and verifying the correctness of the conclusions. The work establishes a theoretical basis for related scientific research and application.

Keywords: generalized network, difference equations, matrix transform, potential function, equivalent resistance

## 1. Introduction

The well-known nodal current and loop voltage laws were established by Kirchhoff in 1845; since then, great progress has been made in the research and application of resistor networks [1]. When solving some complex scientific problems, the use of resistor network models can make the problems specific and intuitive, and easy to analyze and study. Therefore, the resistor network model has certain practical value for scientific research. For example, the study of the resistance between two points of an infinite grid and some new ideas and methods were proposed in references [2–4]. In particular, in 2000, Prof. Cserti made great progress and proposed the lattice Green's function to calculate the resistance of infinite resistor networks [3], providing an effective method for many scholars who study the resistance of an infinite resistor network. Green's function is a very important theoretical tool for computing infinite networks, as shown by the excellent applications of the theoretical works in reference [3] on the study of the infinite network. Giordano [4], Asad [5], Asad and Hijjawi [6–9], Owaidat [10–15], and Cserti *et al* [16] have also made new theoretical achievements. The application of Green's function is a very efficient way to find the resistance of an infinitely resistive network. But the infinite resistance network model is always idealized, and the finite resistance network is the actual real-life problem. Obviously, Green's function does not apply to finite resistor networks. However, Wu [17] proposed a different method (called the Laplace matrix method) in 2004 and used the eigenvalues and eigenvectors of the Laplace matrix to calculate the resistance between any two nodes in a resistor network. This method is used to calculate the resistance between any nodes of the finite resistance lattice and has made a significant contribution to the study of the finite resistance network model. Laplace matrix analysis has also been applied to impedance networks in the literature [18]. Later, Chair [19] and Chair and Dannoun [20] used Wu's results to study the simple random network and the single-sum problem of the Möbius $n$-ladder network, simplifying the results of the double sum. After some improvements, the Laplace matrix method was further developed and applied [21, 22]. The Laplace matrix method relies on two matrices in two directions and is not suitable for networks at arbitrary boundaries.

How to solve the problem of the resistance network with an arbitrary boundary is a longstanding scientific problem. However, in 2011, there was a turning point, Tan proposed a new theory to solve this problem. Tan's theory only relies on

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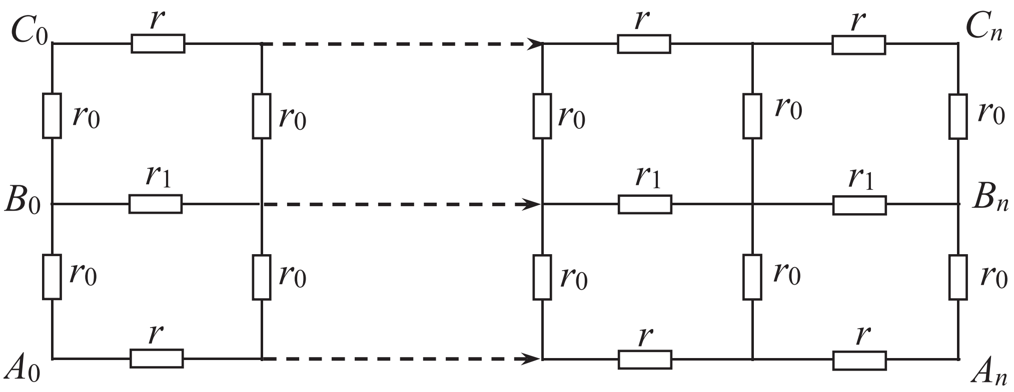

**Figure 1.** A generalized $2 \times n$ resistor network with an arbitrary intermediate horizontal axis; there are three resistance elements $\{r_0, r, r_1\}$ arranged in the network model.

one matrix in one direction [1], which can avoid the eigenvalue problem of solving another matrix, and allows having any element in the other matrix (because the theory does not need to solve it). In 2015, Tan called the method the recursion-transform (RT) method [23–25]. At that time, the RT method took the branch current as the parameter to establish the matrix equation, so it is also called the RT-I method. In 2017, Tan also established the method of establishing a matrix equation with node voltage as a parameter, which is called the RT-V method [26]. The RT theory can not only calculate the equivalent resistance but also conveniently solve the potential function, Tan and Tan *et al* used RT theory to solve many resistor network problems of various complex structures [27–38]. RT theory combined with variable substitution technology can also conveniently study complex impedance problems. In the RT theory, there is a special method called the N-RT method, which specializes in studying the equivalent resistance of *n*-order circuit networks. In practical applications, *n*-order circuit networks have very important research value [39–41]. For example, in the literature [39, 40], equivalent circuits can be used to study the reflection and refraction of electromagnetic waves of the superstructure grating, while reference [41] uses complex impedance circuits to study water transport in plants. At present, researchers have accomplished many achievements in research by using the N-RT method [42–49]. This paper intends to study a class of a generalized $2 \times n$ circuit network model, which is different from previous models in that the resistance on the middle axis is independent of the resistance on the upper and lower boundaries as shown in figure 1. We use the RT-I approach for this problem. We use the RT-I method to study the analytical expression of the electrical characteristics of this circuit network (node potential and equivalent resistance).

This article is organized as follows: in section 2, a series of major results on the $2 \times n$ resistor network are presented (including the potential function formula for any node and the equivalent resistance formula between any nodes). The methods and calculations used in this study are described in section 3. Section 4 presents the main results. And in section 5, we provide a number of special cases, while comparing and verifying their correctness.

## 2. Main results

We define a generalized $2 \times n$ resistor network with an arbitrary intermediate horizontal axis as shown in figure 1. There are three resistance elements $\{r_0, r, r_1\}$ arranged in the network model, and the resistance $r_1$ is located on the middle horizontal axis, the resistance $r$ is located on the upper and lower boundary lines, and the resistance $r_0$ is located on the vertical axis. Set the horizontal line as the $X$-axis, and assume current $J$ from the $A_{x_1}$ input and from the $P_{x_2}(P_{x_2} = C_{x_2}, B_{x_2}, A_{x_2})$ output (without loss of generality let $x_1 \leqslant x_2$). Set the potential function of any node $d(x, y)$ to $U(x, y)$, selecting $U(B_0)$ as the potential reference point. The above assumptions apply throughout this article.

### 2.1. Necessary parameter definitions

The content of this article is complex, and there are many equations and parameters, so in order to facilitate the research and calculations, and to simplify the various expressions given below, the following definitions are given:

$$h = r/r_0,\ h_1 = r_1/r_0, \tag{1}$$

$$t_1 = 2 + h,\ t_2 = 2 + h + 2h_1, \tag{2}$$

$$\lambda_i = \frac{1}{2}\left(t_i + \sqrt{t_i^2 - 4}\right),\ \bar{\lambda}_i = \frac{1}{2}\left(t_i - \sqrt{t_i^2 - 4}\right), \tag{3}$$

$$F_k^{(i)} = (\lambda_i^k - \bar{\lambda}_i^k)/(\lambda_i - \bar{\lambda}_i),\ \Delta F_k^{(i)} = F_{k+1}^{(i)} - F_k^{(i)}, \tag{4}$$

$$\beta_{x_s \vee x}^{(i)} = \begin{cases} \beta_{x_s,x}^{(i)} = \Delta F_{x_s}^{(i)} \Delta F_{n-x}^{(i)}, & \text{if} \quad x_s \leqslant x \\ \beta_{x,x_s}^{(i)} = \Delta F_x^{(i)} \Delta F_{n-x_s}^{(i)}, & \text{if} \quad x_s \geqslant x \end{cases}. \tag{5}$$

The above parameters are important to use in this article and applicable throughout the article.

### 2.2. Analytic expression of the potential function

Consider a generalized $2 \times n$ resistor network shown in figure 1, let current $J$ flow from $A_{x_1}$ to $C_{x_2}$ $(x_1 \leqslant x_2)$. Select

$U(B_0)$ as the potential reference point, and set the horizontal line as the $X$-axis, the potential function formulae of the $2 \times n$ circuit network are as follows:

(a) The potential function of the node $A_x$ on the axis $A_0A_n$ can be expressed as

$$\frac{U(A_x)}{J} = \frac{r_1 r}{r + 2r_1}(x_1 - x_s) + \frac{r}{2h}\left(\frac{\beta^{(1)}_{x_1 \vee x} + \beta^{(1)}_{x_2 \vee x}}{F^{(1)}_{n+1}} + r^2 \frac{\beta^{(2)}_{x_1 \vee x} - \beta^{(2)}_{x_2 \vee x}}{(r + 2r_1)^2 F^{(2)}_{n+1}}\right), \tag{6}$$

where $h = r/r_0$, $h_1 = r_1/r_0$, $\beta^{(i)}_{x_s \vee x}$ is defined in equation (5), and $x_s$ is a piecewise function

$$x_s = \{x_1\colon\ 0 \leqslant x \leqslant x_1\} \cup \{x\colon\ x_1 \leqslant x \leqslant x_2\} \cup \{x_2\colon\ x_2 \leqslant x \leqslant n\}. \tag{7}$$

(b) The potential function of node $B_x$ on the axis $B_0B_n$ can be expressed as

$$\frac{U(B_x)}{J} = \frac{r_1 r}{r + 2r_1}(x_1 - x_s) + \frac{hr_1}{(h + 2h_1)^2}\left(\frac{\beta^{(2)}_{x_2 \vee x} - \beta^{(2)}_{x_1 \vee x}}{F^{(2)}_{n+1}}\right). \tag{8}$$

In particular, when $x = 0$, the potential of the reference point $U(B_0) = U_0$ is obtained as

$$\frac{U(B_0)}{J} = \frac{hr_1}{(h + 2h_1)^2}\left(\frac{\Delta F^{(2)}_{n-x_2} - \Delta F^{(2)}_{n-x_1}}{F^{(2)}_{n+1}}\right). \tag{9}$$

(c) The potential function of node $C_x$ on the axis $C_0C_n$ can be expressed as

$$\frac{U(C_x)}{J} = \frac{r_1 r}{r + 2r_1}(x_1 - x_s) - \frac{r}{2h}\left(\frac{\beta^{(1)}_{x_1 \vee x} + \beta^{(1)}_{x_2 \vee x}}{F^{(1)}_{n+1}} - r^2 \frac{\beta^{(2)}_{x_1 \vee x} - \beta^{(2)}_{x_2 \vee x}}{(r + 2r_1)^2 F^{(2)}_{n+1}}\right). \tag{10}$$

The above potential function formula is given for the first time in this paper, which is a theoretical innovation and discovery. Note that $B_0$ is chosen as the potential reference point in all of the above conclusions, and $U(B_0)$ is given by equation (9).

### 2.3. Equivalent resistance formulae

Consider a generalized $2 \times n$ circuit network as shown in figure 1, set the horizontal line as the $x$-axis, and the equivalent resistance between $A_{x_1}$ and $C_{x_2}$ can be expressed as

$$R_n(A_{x_1}, C_{x_2}) = \frac{rr_1}{r + 2r_1}|x_2 - x_1| + \frac{r_0}{2}\left(\frac{\beta^{(1)}_{1,1} + 2\beta^{(1)}_{1,2} + \beta^{(1)}_{2,2}}{F^{(1)}_{n+1}} + r^2 \frac{\beta^{(2)}_{1,1} - 2\beta^{(2)}_{1,2} + \beta^{(2)}_{2,2}}{(2r_1 + r)^2 F^{(2)}_{n+1}}\right), \tag{11}$$

where $0 \leqslant x_1 \leqslant n$, $0 \leqslant x_2 \leqslant n$, and $\beta^{(i)}_{k,s}$ is short for $\beta^{(i)}_{x_k,x_s}$, which is defined in equation (5).

The equivalent resistance between $A_{x_1}$ and $A_{x_2}$ can be expressed as

$$R_n(A_{x_1}, A_{x_2}) = \frac{rr_1}{r + 2r_1}|x_2 - x_1| + \frac{r_0}{2}\left(\frac{\beta^{(1)}_{1,1} - 2\beta^{(1)}_{1,2} + \beta^{(1)}_{2,2}}{F^{(1)}_{n+1}} + r^2 \frac{\beta^{(2)}_{1,1} - 2\beta^{(2)}_{1,2} + \beta^{(2)}_{2,2}}{(2r_1 + r)^2 F^{(2)}_{n+1}}\right). \tag{12}$$

Formulas (11) and (12) are given for the first time in this article and are applicable to cases where $x_1$, $x_2$, $n$ are any natural numbers, and also cases where $n$ is infinite (this issue will be discussed later in this article).

In the following, we use RT-I theory to give a detailed calculation process of all the above conclusions. RT-I theory is an advanced theory established in the literature [23] to study the resistor network model and mainly consists of four basic steps: first, establishing a matrix equation of the current parameters on three adjacent vertical axes; second, establishing a boundary condition of the current parameter; third, implementing a diagonal matrix transformation to solve the matrix equations; and fourth, implementing the matrix inverse transformation to solve the branch current, calculate the node potential, and find the equivalent resistance.

## 3. Methods and calculations

### 3.1. Modeling general differential equations

As shown in figure 1, below we use RT-I theory to carry out our research. Let current $J$ flow from node $A_{x_1}$ to node $C_{x_2}$(or $A_{x_2}$). For ease of study, we show a network graph in figure 2 with current parameters and their directions. Let the currents passing through the three rows of horizontal axis resistors be $I_{ak}$, $I_{bk}$, $I_{ck}$ $(1 \leqslant k \leqslant n)$, and the currents passing through the vertical resistance $r_0$ are $I_k$ and $I'_k$ $(0 \leqslant k \leqslant n)$.

Using Kirchhoff's loop voltage law, the current equations for the $k$th grid are obtained as

$$I_k r_0 - I_{k-1} r_0 + I_{ak} r - I_{bk} r_1 = 0, \tag{13}$$

$$I'_k r_0 - I'_{k-1} r_0 - I_{ck} r + I_{bk} r_1 = 0. \tag{14}$$

Similarly, the current equation of the $(k+1)$th grid loop can be obtained. Then, taking the difference between the equations obtained and equations (13) and (14), we get

$$(I_{k+1} - 2I_k + I_{k-1}) r_0 + (I_{ak+1} - I_{ak}) r - (I_{bk+1} - I_{bk}) r_1 = 0, \tag{15}$$

$$(I'_{k+1} - 2I'_k + I'_{k-1}) r_0 - (I_{ck+1} - I_{ck}) r + (I_{bk+1} - I_{bk}) r_1 = 0. \tag{16}$$

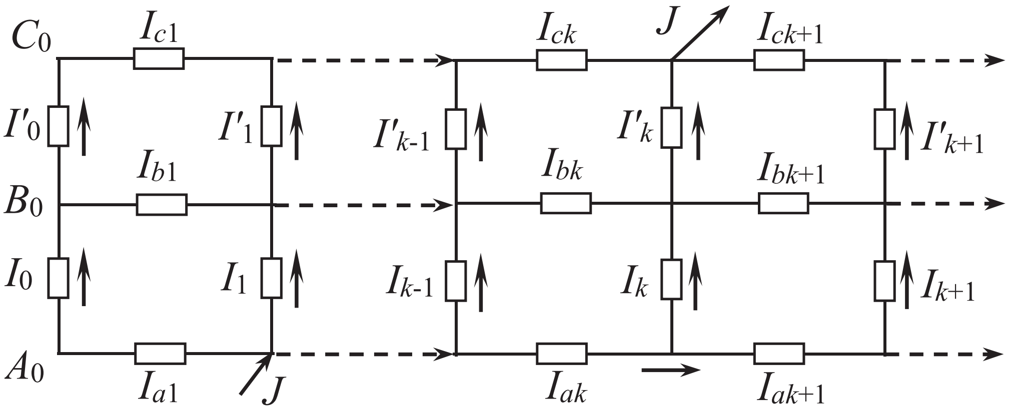

**Figure 2.** Sub-network of a 2 × $n$ resistor network with current parameters.

By the Kirchhoff node current equation shown in figure 2, we can get

$$I_{ck+1} - I_{ck} = I'_k,\ I_{ak+1} - I_{ak} = -I_k,\ I_{bk+1} - I_{bk} = I_k - I'_k. \tag{17}$$

Substituting equations (17) into (15) and (16), respectively, and simplifying them, we can get

$$r_0(I_{k+1} + I_{k-1}) = (r + r_1 + 2r_0)I_k - r_1 I'_k, \tag{18}$$

$$r_0(I'_{k+1} + I'_{k-1}) = (r + r_1 + 2r_0)I'_k - r_1 I_k. \tag{19}$$

Research shows that it is difficult to obtain a direct solution to the systems of equations (18) and (19); thus, we use the RT-I theory to express equations (18) and (19) as the matrix

$$\begin{bmatrix} I_{k+1} \\ I'_{k+1} \end{bmatrix} = \begin{bmatrix} 2+h+h_1 & -h_1 \\ -h_1 & 2+h+h_1 \end{bmatrix} \begin{bmatrix} I_k \\ I'_k \end{bmatrix} - \begin{bmatrix} I_{k-1} \\ I'_{k-1} \end{bmatrix}, \tag{20}$$

where $h = r/r_0$, $h_1 = r_1/r_0$. The key step in RT-I theory is to do a matrix transformation to indirectly study the general solution. The specific transformation method is to implement a diagonal matrix transformation for the second-order matrix in (20). For example,

$$P_{2\times2}\begin{bmatrix} I_{k+1} \\ I'_{k+1} \end{bmatrix} = P_{2\times2}\begin{bmatrix} 2+h+h_1 & -h_1 \\ -h_1 & 2+h+h_1 \end{bmatrix} \begin{bmatrix} I_k \\ I'_k \end{bmatrix} - P_{2\times2}\begin{bmatrix} I_{k-1} \\ I'_{k-1} \end{bmatrix}. \tag{21}$$

Assume that there are constants $t_1$, $t_2$ and elements of $[1, p_i]$, which satisfy

$$\begin{bmatrix} 1 & p_1 \\ 1 & p_2 \end{bmatrix}\begin{bmatrix} 2+h+h_1 & -h_1 \\ -h_1 & 2+h+h_1 \end{bmatrix} = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}\begin{bmatrix} 1 & p_1 \\ 1 & p_2 \end{bmatrix}. \tag{22}$$

Equation (22) is regarded as the identity matrix and is solved to obtain

$$p_i^2 - 1 = 0 \Rightarrow p_1 = 1, p_2 = -1, \tag{23}$$

$$t_1 = 2 + h,\ t_2 = 2 + h + 2h_1. \tag{24}$$

Equation (24) is the definition given in equation (2) above. So equations (23) and (24) can convert the matrix equation (21) into a new matrix equation,

$$\begin{bmatrix} X_{k+1}^{(1)} \\ X_{k+1}^{(2)} \end{bmatrix} = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}\begin{bmatrix} X_k^{(1)} \\ X_k^{(2)} \end{bmatrix} - \begin{bmatrix} X_{k-1}^{(1)} \\ X_{k-1}^{(2)} \end{bmatrix}, \tag{25}$$

where the transformation relationship is

$$\begin{pmatrix} X_k^{(1)} \\ X_k^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} I_k \\ I'_k \end{pmatrix} = \begin{pmatrix} I_k + I'_k \\ I_k - I'_k \end{pmatrix}. \tag{26}$$

Let the two roots of $x^2 - t_i x + 1 = 0$ from (25) be $\lambda_k$, $\bar{\lambda}_k$ $(k = 1, 2)$, by solving the characteristic equation we have equation (3).

Equation (25) is the simple linear difference equation, so solving the difference equation (25) results in a segmented function solution,

$$X_k^{(i)} = X_1^{(i)}F_k^{(i)} - X_0^{(i)}F_{k-1}^{(i)},\ (0 \leqslant k \leqslant x_1), \tag{27}$$

$$X_k^{(i)} = X_{x_1+1}^{(i)}F_{k-x_1}^{(i)} - X_{x_1}^{(i)}F_{k-x_1-1}^{(i)},\ (x_1 \leqslant k \leqslant x_2), \tag{28}$$

$$X_k^{(i)} = X_{x_2+1}^{(i)}F_{k-x_2}^{(i)} - X_{x_2}^{(i)}F_{k-x_2-1}^{(i)},\ (x_2 \leqslant k \leqslant n), \tag{29}$$

where $F_k^{(i)} = (\lambda_i^k - \bar{\lambda}_i^k)/(\lambda_i - \bar{\lambda}_i)$ is defined in equation (4).

### *3.2. Constraint equations on the left and right boundaries*

According to RT-I theory [23], the constraint of the boundary current has four parts, namely, the leftmost and rightmost boundary conditions, and the boundary condition at the input and output current. First, consider the current constraint of the left boundary grid. According to figure 2, the first loop voltage equations can be obtained as

$$\begin{aligned} &I_0 r_0 + I_{b1} r_1 - I_1 r_0 - I_{a1} r = 0, \\ &I'_0 r_0 + I_{c1} r - I'_1 r_0 - I_{b1} r_1 = 0. \end{aligned} \tag{30}$$

And we get the node current equations $I_{b1} = I_0 - I'_0$, $I_{a1} = -I_0$, $I_{c1} = I'_0$. From these equations, one can get

$$\begin{bmatrix} I_1 \\ I'_1 \end{bmatrix} = \begin{bmatrix} h+h_1+1 & -h_1 \\ -h_1 & h+h_1+1 \end{bmatrix}\begin{bmatrix} I_0 \\ I'_0 \end{bmatrix}. \tag{31}$$

Next, we use $P_{2\times2}$ appearing in equation (21) to multiply

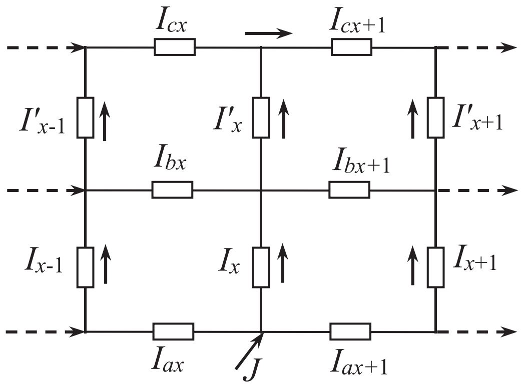

**Figure 3.** Current input image with the current parameters of the $2 \times n$ resistor network.

equation (31) to implement the matrix transformation; then, the matrix equation (31) can be transformed into

$$X_1^{(i)} = (t_i - 1)X_0^{(i)},\ (i = 1, 2), \tag{32}$$

where $t_i$ is given by equation (24).

Secondly, consider the constraint equation of the right boundary grid. Similar to the establishment of equation (31), we have

$$\begin{bmatrix} I_{n-1} \\ I'_{n-1} \end{bmatrix} = \begin{bmatrix} h_1 + h + 1 & -h_1 \\ -h_1 & h_1 + h + 1 \end{bmatrix} \begin{bmatrix} I_n \\ I'_n \end{bmatrix}. \tag{33}$$

Similar to the matrix transformation above, the matrix equation (33) can be transformed into

$$X_{n-1}^{(i)} = (t_i - 1)X_n^{(i)},\ (i = 1, 2). \tag{34}$$

The above equations are general-purpose equations for all cases and can be applied to calculate potential functions in various situations. The following are special current constraint equations based on different current output conditions.

### 3.3. Constraint equation of the current input and output

First, we consider the case of input current $J$ at node $A_{x_1}$. Similar to the network analysis method above, from figure 3, we can obtain a matrix equation in the case of node current input

$$\begin{bmatrix} I_{x_1+1} \\ I'_{x_1+1} \end{bmatrix} = \begin{bmatrix} 2 + h + h_1 & -h_1 \\ -h_1 & 2 + h + h_1 \end{bmatrix} \begin{bmatrix} I_{x_1} \\ I'_{x_1} \end{bmatrix} - \begin{bmatrix} I_{x_1-1} \\ I'_{x_1-1} \end{bmatrix} - \begin{bmatrix} hJ \\ 0 \end{bmatrix}. \tag{35}$$

Let us take the matrix that appeared in (21) times the matrix (35) from the left to do the matrix transformation, then the matrix equation (35) can be transformed into

$$X_{x_1+1}^{(i)} = t_i X_{x_1}^{(i)} - X_{x_1-1}^{(i)} - hJ. \tag{36}$$

Second, we consider the case of the input current at node $C_{x_2}$. The matrix equations are obtained using the same method described above,

$$\begin{bmatrix} I_{x_2+1} \\ I'_{x_2+1} \end{bmatrix} = \begin{bmatrix} 2 + h + h_1 & -h_1 \\ -h_1 & 2 + h + h_1 \end{bmatrix} \begin{bmatrix} I_{x_2} \\ I'_{x_2} \end{bmatrix} - \begin{bmatrix} I_{x_2-1} \\ I'_{x_2-1} \end{bmatrix} - \begin{bmatrix} 0 \\ hJ \end{bmatrix}. \tag{37}$$

Similar to the matrix transformation we did above, the matrix equation (37) can be transformed into

$$X_{x_2+1}^{(i)} = t_i X_{x_2}^{(i)} - X_{x_2-1}^{(i)} - p_i hJ, \tag{38}$$

where $p_i = \{1, -1\}$ is given by equation (23).

In addition, if we consider the case of output current from the node $A_{x_2}$, the difference equation can be rewritten as

$$X_{x_2+1}^{(i)} = t_i X_{x_2}^{(i)} - X_{x_2-1}^{(i)} + p_i hJ. \tag{39}$$

The above series of equations (20)–(39) are all the equations we need, and based on these equations, the analytical formulae (6)–(12) can be derived. The detailed calculation process is given below.

## 4. Derivation of main results

### 4.1. A general solution of the matrix equations

In order to derive the analytical formula of the potential function, a potential reference point needs to be set. Let node $B_0$ be the potential reference point, and set $U(B_0) = U_0$. Because the potential decreases along the current direction, the potential functions on any axis are

$$U(B_x) = U_0 - r_1 \sum_{k=1}^{x} I_{bk}, \tag{40}$$

$$U(A_x) = U(B_x) + r_0 I_x, \tag{41}$$

$$U(C_x) = U(B_x) - r_0 I'_x, \tag{42}$$

where $0 \leqslant x \leqslant n$, and the current $I'_x$ and $I_x$ are shown in figures 2 and 3.

Next, to calculate the current $I_{bk}$, the analytical formula of the matrix equation $X_k^{(i)}$ needs to be solved first. Substituting (32) into (27) to get $(0 \leqslant k \leqslant x_1)$

$$X_k^{(i)} = \Delta F_k^{(i)} X_0^{(i)}. \tag{43}$$

Substituting (43) into (28) to get $(x_1 \leqslant k \leqslant x_2)$

$$X_k^{(i)} = F_{k-x_1+1}^{(i)} X_{x_1}^{(i)} - F_{k-x_1}^{(i)} X_{x_1-1}^{(i)} - hJF_{k-x_1}^{(i)}. \tag{44}$$

Substituting (43) with $k = x_1, x_1 - 1$ into (44), yielding $(x_1 \leqslant k \leqslant x_2)$

$$X_k^{(i)} = \Delta F_k^{(i)} X_0^{(i)} - hJF_{k-x_1}^{(i)}, \tag{45}$$

where $F_{k-x_1+1}^{(i)} \Delta F_{x_1}^{(i)} - F_{k-x_1}^{(i)} \Delta F_{x_1-1}^{(i)} = \Delta F_k^{(i)}$ is used.

Substituting (38) into (29), we can get $(x_2 \leqslant k \leqslant n)$

$$X_k^{(i)} = F_{k-x_2+1}^{(i)} X_{x_2}^{(i)} - F_{k-x_2}^{(i)} X_{x_2-1}^{(i)} - p_i hJF_{k-x_2}^{(i)}. \tag{46}$$

Substituting (45) with $k = x_2,\ x_2 - 1$ into (46), yielding

$(x_2 \leqslant k \leqslant n)$

$$X_k^{(i)} = \Delta F_k^{(i)} X_0^{(i)} - hJ\,(F_{k-x_1}^{(i)} + p_i F_{k-x_2}^{(i)}), \tag{47}$$

where $F_{k-x_2+1}^{(i)} F_{x_2-x_1}^{(i)} - F_{k-x_2}^{(i)} F_{x_2-x_1-1}^{(i)} = F_{k-x_1}^{(i)}$ is used.

Substituting (47) with $k = n - 1$, $n$ into (34), and simplifying it we get

$$X_0^{(i)} = hJ \frac{\Delta F_{n-x_1}^{(i)} + p_i \Delta F_{n-x_2}^{(i)}}{(t_i - 2) F_{n+1}^{(i)}}. \tag{48}$$

This is a key solution on which all ranges of solutions depend. Therefore, substituting equation (48) into equations (43), (45), and (47) to get a general formula,

$$X_k^{(i)} = hJ \frac{\beta_{x_1 \vee k}^{(i)} + p_i \beta_{x_2 \vee k}^{(i)}}{(t_i - 2) F_{n+1}^{(i)}},\ (0 \leqslant k \leqslant n), \tag{49}$$

where $\beta_{x_s \vee k}^{(i)}$ is defined in equation (5).

### 4.2. Calculate the branch currents

Applying the matrix inverse transformation by the matrix equation (26), one can get

$$I_k = \frac{X_k^{(1)} + X_k^{(2)}}{2},\ I_k' = \frac{X_k^{(1)} - X_k^{(2)}}{2}. \tag{50}$$

In addition, the node current equation $I_{bk} - I_{bk-1} = I_{k-1} - I_{k-1}'$ is obtained according to figure 2, and then we use equation (50) to get

$$I_{bk} = \sum_{s=0}^{k-1} (I_s - I_{ss}') = \sum_{s=0}^{k-1} X_s^{(2)}. \tag{51}$$

For the following potential function to be derived, we need to calculate equation (54) in sections.

When $0 \leqslant k \leqslant x_1$, substituting equation (43) into equation (51) we get

$$I_{bk} = F_k^{(2)} X_0^{(2)},\ (0 \leqslant k \leqslant x_1). \tag{52}$$

Summing over equation (52) again, one can get $(0 \leqslant k \leqslant x_1)$

$$\sum_{k=1}^{x} I_{bk} = \frac{\Delta F_x^{(2)} - 1}{t_2 - 2} X_0^{(2)} = \frac{X_x^{(2)} - X_0^{(2)}}{t_2 - 2}, \tag{53}$$

where $X_k^{(i)}$ is given by equation (49).

When $x_1 \leqslant k \leqslant x_2$, substituting equation (45) into equation (51), we get

$$I_{bk} = F_k^{(2)} X_0^{(2)} - hJ \frac{\Delta F_{k-1-x_1}^{(2)} - 1}{t_2 - 2}. \tag{54}$$

Further, the sum is calculated as

$$\sum_{k=1}^{x} I_{bk} = hJ \frac{x - x_1}{h + 2h_1} + \frac{X_x^{(2)} - X_0^{((2)2)}}{t_2 - 2}. \tag{55}$$

When $x_2 \leqslant k \leqslant n$, substituting equation (47) into equation (51), we get

$$I_{bk} = F_k^{(2)} X_0^{(2)} - hJ \left( \frac{\Delta F_{k-1-x_1}^{(2)} - 1}{t_2 - 2} - \frac{\Delta F_{k-1-x_2}^{(2)} - 1}{t_2 - 2} \right). \tag{56}$$

Utilize equation (55) and further calculate equation (56) to obtain $(x_2 \leqslant k \leqslant n)$

$$\sum_{k=1}^{x} I_{bk} = hJ \frac{x_2 - x_1}{h + 2h_1} + \frac{X_x^{(2)} - X_0^{(2)}}{t_2 - 2}, \tag{57}$$

where $X_k^{(i)}$ is given by equation (49).

Since $t_2 - 2 = h + 2h_1$, the above equations can be rewritten uniformly as

$$\sum_{k=1}^{x} I_{bk} = Jh \frac{x_s - x_1}{h + 2h_1} + \frac{X_x^{(2)} - X_0^{(2)}}{h + 2h_1}, \tag{58}$$

where $x_s = \{x_1, x \leqslant x_1\} \cup \{x, x_1 \leqslant x \leqslant x_2\} \cup \{x_2, x \geqslant x_2\}$ is defined in equation (7). Equation (58) is a key equation for calculating voltages, which can be conveniently applied to equation (40) to solve the potential function.

### 4.3. Derivation of the potential functions

Substituting equation (58) into equation (40), we get

$$U(B_x) = U_0 - r_1 Jh \frac{x_s - x_1}{h + 2h_1} - r_1 \frac{X_x^{(2)} - X_0^{(2)}}{h + 2h_1}. \tag{59}$$

Let the potential reference point be $B_0$, and the reference potential is $U(B_0) = U_0 = \frac{-r_1}{h + 2h_1} X_0^{(2)}$, so equation (59) can be rewritten as

$$U(B_x) = -r_1 Jh \frac{x_s - x_1}{h + 2h_1} - \frac{r_1}{h + 2h_1} X_x^{(2)}, \tag{60}$$

where $X_k^{(i)}$ is given by equation (49). Note that equation (60) is self-consistent with the assumption of $U(B_0) = U_0$, because there must be $U(B_0) = U_0$ when $x = 0$.

We can immediately derive the potential function (8) when substituting equation (49) into equation (60). Here we first prove the potential function (8).

In addition, the potential function (6) can be obtained by equation (41) and equation (50) as

$$U(A_x) = U(B_x) + \frac{r_0}{2} (X_x^{(1)} + X_x^{(2)}). \tag{61}$$

Substituting equation (60) into equation (61) and simplifying it

$$U(A_x) = -r_1 Jh \frac{x_s - x_1}{h + 2h_1} + \frac{r}{2} \left( \frac{X_x^{(1)}}{h} + \frac{X_x^{(2)}}{h + 2h_1} \right), \tag{62}$$

where $X_k^{(i)}$ is given by equation (49). Substituting equation (49) into equation (62), the potential function (6) is derived.

Finally, by calculating the potential $U(C_x)$ by equations (42) and (50), one can get

$$U(C_x) = U(B_x) - \frac{r_0}{2} (X_x^{(1)} - X_x^{(2)}). \tag{63}$$

Substituting equation (60) into equation (63) and simplifying it, we have

$$U(C_x) = -r_1 Jh \frac{x_s - x_1}{h + 2h_1} - \frac{r}{2} \left( \frac{X_x^{(1)}}{h} - \frac{X_x^{(2)}}{h + 2h_1} \right). \tag{64}$$

Substituting equation (49) into (64), we immediately derive the potential function (10).

So far, the analytical formulae of the potential function are fully proven. The above proof process is logical and the results are self-consistent.

### *4.4. Derivation of equivalent resistance*

Next, we derive the equivalent resistance of equations (11) and (12). When calculating the equivalent resistance $R(A_{x_1}C_{x_2})$, equations (6) and (10) can be used directly. For example, taking $x = x_1$ in equation (6) and $x = x_2$ in equation (10), respectively, one can get

$$\frac{U(A_{x_1})}{J} = \frac{r_0}{2h} \times \left(\frac{\beta^{(1)}_{x_1,x_1} + \beta^{(1)}_{x_1,x_2}}{F^{(2)}_{n+1}} + \left(\frac{r}{r+2r_1}\right)^2 \frac{\beta^{(2)}_{x_1,x_1} - \beta^{(2)}_{x_1,x_2}}{F^{(2)}_{n+1}}\right), \tag{65}$$

$$\frac{U(C_{x_2})}{J} = \frac{r_1 r}{r+2r_1}(x_1 - x_2) - \frac{r_0}{2h}\left(\frac{\beta^{(1)}_{x_1,x_2} + \beta^{(1)}_{x_2,x_2}}{F^{(2)}_{n+1}} + \left(\frac{r}{r+2r_1}\right)^2 \frac{\beta^{(2)}_{x_1,x_2} - \beta^{(2)}_{x_2,x_2}}{F^{(2)}_{n+1}}\right). \tag{66}$$

Using Ohm's law $R(A_{x_1}C_{x_2}) = [U(A_{x_1}) - U(C_{x_2})]/J$, then by equations (65) and (66) derive equation (11).

In addition, when deriving the equivalent resistance $R(A_{x_1}A_{x_2})$, it is necessary to reconsider the situation in which current $J$ flows from $A_{x_1}$ to node $A_{x_2}$, in this case, except for equations (37) and (38), the remaining equations (13)–(36) and equations (40) and (42) are general equations suitable for all cases. To calculate the equivalent resistance $R(A_{x_1}A_{x_2})$, simply replace equation (38) with equation (39).

Similar to the derivation of equations (48) and (49) above, by equations (27)–(29), (32)–(36), and (39), one can get

$$X^{(i)}_0 = hJ\frac{\Delta F^{(i)}_{n-x_1} - p_i \Delta F^{(i)}_{n-x_2}}{(t_i - 2)F^{(i)}_{n+1}}, \tag{67}$$

and obtain ($x_1 \leqslant k \leqslant x_2$)

$$X^{(i)}_k = hJ\frac{\Delta F^{(i)}_{x_1}\Delta F^{(i)}_{n-k} - p_i \Delta F^{(i)}_{k}\Delta F^{(i)}_{n-x_2}}{(t_i - 2)F^{(i)}_{n+1}}. \tag{68}$$

By figure 2, one can compute the voltages $U(A_{x_1}, A_{x_2})$ that go through three different paths,

$$U(A_{x_1}, A_{x_2}) = \sum_{i=x_1+1}^{x_2} I_{ai} r,\ U(A_{x_1}, A_{x_2}) = \sum_{i=x_1+1}^{x_2} I_{bi} r_1 + I_{x_1} r_0 - I_{x_2} r_0,$$

$$U(A_{x_1}, A_{x_2}) = \sum_{i=x_1+1}^{x_2} I_{ci} r + (I_{x_1} + I'_{x_1} - I'_{x_2} - I_{x_2}) r_0. \tag{69}$$

The upper three forms are appropriately deformed and summed to get

$$\left(\frac{2}{r} + \frac{1}{r_1}\right) U(A_{x_1}, A_{x_2}) = \sum_{i=x_1+1}^{x_2} (I_{ai} + I_{bi} + I_{ci}) + \frac{r_0}{r_1}(I_{x_1} - I_{x_2}) + \frac{r_0}{r}(I_{x_1} + I'_{x_1} - I'_{x_2} - I_{x_2}). \tag{70}$$

According to the continuity equation of the current $I_{ai} + I_{bi} + I_{ci} = J$, substituting it into equation (70), one can get

$$U(A_{x_1}, A_{x_2}) = \frac{r r_1 J}{r + 2r_1}|x_2 - x_1| + \frac{r_0 r_0}{r+2r_1}[(h + h_1)(I_{x_1} - I_{x_2}) + h_1(I'_{x_1} - I'_{x_2})]. \tag{71}$$

Substitute equation (50) with $k = x_1, x_2$ into equation (71) to yield

$$U(A_{x_1}, A_{x_2}) = \frac{r r_1 I}{r + 2r_1}|x_2 - x_1| + \frac{r_0}{2}\left[(X^{(1)}_{x_1} - X^{(1)}_{x_2}) + \frac{h}{h+2h_1}(X^{(2)}_{x_1} - X^{(2)}_{x_2})\right]. \tag{72}$$

When equation (68) is taken as $k = x_1, x_2$, respectively, substituting equation (68) into equation (72), one can get

$$\frac{U(A_{x_1}, A_{x_2})}{J} = \frac{r r_1}{r + 2r_1}|x_2 - x_1| + \frac{r_0}{2}\left(\frac{\beta^{(1)}_{1,1} - 2\beta^{(1)}_{1,2} + \beta^{(1)}_{2,2}}{F^{(1)}_{n+1}} + r^2\frac{\beta^{(2)}_{1,1} - 2\beta^{(2)}_{1,2} + \beta^{(2)}_{2,2}}{(2r_1 + r)^2 F^{(2)}_{n+1}}\right). \tag{73}$$

When applying equation (73) to Ohm's law $R_n(A_{x_1}, A_{x_2}) = U(A_{x_1}, A_{x_2})/J$, the resistance formula (12) is derived.

So far formulae (11) and (12) are proved. The following are several special cases of electrical characteristics, also compared and studied for their correctness. Of course, since all the calculations in this article are rigorous and precise, and all the results are self-consistent, the results presented must be correct.

## 5. Special cases and comparisons

The potential functions and two equivalent resistance formulae of a generalized 2 × $n$ resistor network are given above. Since the model shown in figure 1 has three arbitrary independent parameters of $\{r, r_0, r_1\}$, all the formulas presented in this article have universal applicability and represent all situations. Some of their special cases are given below, while comparing and verifying their correctness.

Example 1. In the network shown in figure 1, let current $J$ flows from $A_{x_1}$ to $C_{x_2}$, select $U(B_0)$ as the potential reference point, then by equations (6) and (10), one can get a potential function relationship,

$$\frac{U(A_x)}{J}+\frac{U(C_x)}{J}=\frac{2r_1r}{r+2r_1}(x_1-x_s)+hr\frac{\beta^{(2)}_{x_1\vee x}-\beta^{(2)}_{x_2\vee x}}{(h+2h_1)^2F^{(2)}_{n+1}}, \tag{74}$$

where $h=r/r_0$, $h_1=r_1/r_0$, and $x_s$ is a piecewise function given by equation (7).

Example 2. From equations (8) and (74), we get a potential function relationship ($0\leqslant x\leqslant n$)

$$U(A_x)+U(C_x)+\frac{r}{r_1}U(B_x)=rJ(x_1-x_s), \tag{75}$$

Equation (75) is an interesting function relation, particularly when $0\leqslant x\leqslant x_1$, using equation (75) to get

$$U(A_x)+U(C_x)+\frac{r}{r_1}U(B_x)=0,\ (0\leqslant x\leqslant x_1). \tag{76}$$

Example 3. In the network shown in figure 1, let current $J$ flow from $A_{x_1}$ to $C_{x_2}$, then equations (6) and (10) are subtracted to give the potential function relationship

$$U(A_x)-U(C_x)=Jr_0\frac{\beta^{(1)}_{x_1\vee x}+\beta^{(1)}_{x_2\vee x}}{F^{(1)}_{n+1}}, \tag{77}$$

Equation (77) reveals a simple relationship between two voltages.

Example 4. In the network shown in figure 1, from equations (11) and (12), one can get a formula relationship between two different equivalent resistances,

$$R_n(A_{x_1},C_{x_2})-R_n(A_{x_1},A_{x_2})=2r_0\frac{\Delta F^{(1)}_{x_1}\Delta F^{(1)}_{n-x_2}}{F^{(1)}_{n+1}}. \tag{78}$$

Equation (78) is an interesting relational equation that reveals the intrinsic relationship between two equivalent resistances. In particular, when $n\to\infty$ but $x_1, x_2$ are finite, from equation (78) we have

$$R_\infty(A_{x_1},C_{x_2})-R_\infty(A_{x_1},A_{x_2})=2r_0\Delta F^{(1)}_{x_1}(1-\bar\lambda_1)\bar\lambda_1^{x_2}. \tag{79}$$

Example 5. When $r_1=r$, the equivalent resistances equations (11) and (12) degrade to the following simple results:

$$R_n(A_{x_1},C_{x_2})=\frac{r}{3}|x_2-x_1|+\frac{r_0}{2}\left(\frac{\beta^{(1)}_{1,1}+2\beta^{(1)}_{1,2}+\beta^{(1)}_{2,2}}{F^{(1)}_{n+1}}+\frac{\beta^{(2)}_{1,1}-2\beta^{(2)}_{1,2}+\beta^{(2)}_{2,2}}{9F^{(2)}_{n+1}}\right), \tag{80}$$

$$R_n(A_{x_1},A_{x_2})=\frac{r}{3}|x_2-x_1|+\frac{r_0}{2}\left(\frac{\beta^{(1)}_{1,1}-2\beta^{(1)}_{1,2}+\beta^{(1)}_{2,2}}{F^{(1)}_{n+1}}+\frac{\beta^{(2)}_{1,1}-2\beta^{(2)}_{1,2}+\beta^{(2)}_{2,2}}{9F^{(2)}_{n+1}}\right). \tag{81}$$

Reference [1] has studied the equivalent resistance. When $r_1=r$, comparing the results in [1] with our results of equations (80) and (81), it is not difficult to find that they are exactly the same, which verifies the correctness of the conclusions obtained in this paper.

Example 6. Consider the case of $x_2=x_1$, and $x_1=0$, $x_2=n$, three formulae are derived from equations (11) and (12), respectively

$$R_n(A_{x_1},C_{x_1})=2r_0\frac{\Delta F^{(1)}_{x_1}\Delta F^{(1)}_{n-x_1}}{F^{(1)}_{n+1}}, \tag{82}$$

$$R_n(A_0,C_n)=\frac{rr_1}{r+2r_1}n+r_0\left(\frac{\Delta F^{(1)}_n+1}{F^{(1)}_{n+1}}+r^2\frac{\Delta F^{(2)}_n-1}{(2r_1+r)^2F^{(2)}_{n+1}}\right), \tag{83}$$

$$R_n(A_0,A_n)=\frac{rr_1}{r+2r_1}n+r_0\left(\frac{\Delta F^{(1)}_n-1}{F^{(1)}_{n+1}}+r^2\frac{\Delta F^{(2)}_n-1}{(2r_1+r)^2F^{(2)}_{n+1}}\right). \tag{84}$$

Example 7. When $r_1=\infty$, from equations (11) and (12), one can get

$$R_n(A_{x_1},C_{x_2})=\frac{r}{2}|x_2-x_1|+r_0\frac{\beta^{(1)}_{1,1}+2\beta^{(1)}_{1,2}+\beta^{(1)}_{2,2}}{2F^{(1)}_{n+1}}, \tag{85}$$

$$R_n(A_{x_1},A_{x_2})=\frac{r}{2}|x_2-x_1|+r_0\frac{\beta^{(1)}_{1,1}-2\beta^{(1)}_{1,2}+\beta^{(1)}_{2,2}}{2F^{(1)}_{n+1}}. \tag{86}$$

Example 8. When $r_1=0$, it leads to $h_1=0$, from equation (2) to get $t_2=t_1=2+h$, so $\lambda_2=\lambda_1$, and $\beta^{(2)}_{x_1,x_k}=\beta^{(1)}_{x_1,x_k}$, then equations (11) and (12) degrade to

$$R_n(A_{x_1},C_{x_2})=r_0\frac{\beta^{(1)}_{1,1}+\beta^{(1)}_{2,2}}{F^{(1)}_{n+1}}, \tag{87}$$

$$R_n(A_{x_1},A_{x_2})=r_0\frac{\beta^{(1)}_{1,1}-2\beta^{(1)}_{1,2}+\beta^{(1)}_{2,2}}{F^{(1)}_{n+1}}. \tag{88}$$

Example 9. Consider the case of a half-infinite resistor network. When $x_1, x_2$ is finite but $n\to\infty$, there is a limit of

$$\lim_{n\to\infty}\frac{\Delta F^{(i)}_{x_1}\Delta F^{(i)}_{n-x_2}}{F^{(i)}_{n+1}}=\left(\frac{t_i-2}{\lambda_i-\bar\lambda_i}\right)(\lambda_i^{x_1-x_2}+\bar\lambda_i^{x_1+x_2+1}). \tag{89}$$

When $r_1=r$, taking the limits of equations (11) and (12), one can get the equivalent resistance formulas for two infinite networks:

$$R_\infty(A_{x_1},C_{x_2})=\frac{r}{3}|x_2-x_1|+\frac{r}{2}\left[g_1(x_1,x_2)+\frac{1}{3}\bar g_2(x_1,x_2)\right], \tag{90}$$

$$R_\infty(A_{x_1},A_{x_2})=\frac{r}{3}|x_2-x_1|+\frac{r}{2}\left[\bar g_1(x_1,x_2)+\frac{1}{3}\bar g_2(x_1,x_2)\right], \tag{91}$$

where we define

$$g_i(x_1, x_2) = \frac{2 + \bar{\lambda}_i^{2x_1+1} + \bar{\lambda}_i^{2x_2+1} + 2(\lambda_i^{x_1-x_2} + \bar{\lambda}_i^{x_1+x_2+1})}{\lambda_1 - \bar{\lambda}_1},$$

$$\bar{g}_i(x_1, x_2) = \frac{2 + \bar{\lambda}_i^{2x_1+1} + \bar{\lambda}_i^{2x_2+1} - 2(\lambda_i^{x_1-x_2} + \bar{\lambda}_i^{x_1+x_2+1})}{\lambda_1 - \bar{\lambda}_1}. \quad (92)$$

In particular, when $x_1 = x_2 = x$, equation (90) degrades to

$$R_\infty(A_x, C_x) = 2r\frac{1 + \bar{\lambda}_1^{2x+1}}{\lambda_1 - \bar{\lambda}_1} = 2r\left(\frac{1 + \bar{\lambda}_1^{2x+1}}{\sqrt{h^2 + 4h}}\right), \quad (93)$$

where $h = r/r_0$, $\bar{\lambda}_1 = \frac{1}{2}(2 + h - \sqrt{h^2 + 4h})$.

## 6. Summary and comments

This paper considers a generalized $2 \times n$ network shown in figure 1, and uses RT-I theory to study the electrical properties (potential function and equivalent resistance) of a class of $2 \times n$ resistor networks with an arbitrary intermediate resistance axis. This is of great significance for the study of the $2 \times n$ resistor network model with multiple parameters. The main results of the research presented in this paper are potential functions of any node of a $2 \times n$ resistor network such as equations (6), (8), and (10), and two equivalent resistance formulae such as equations (11) and (12). Equations (6)–(12) are given for the first time in this paper and are a theoretical innovation and discovery. Besides, they are universal, applicable to cases where $x_1, x_2, n$ are any natural numbers, and also to cases where $n$ is infinite. In addition to studying the generally applicable cases, this paper also gives some special situations and compares and verifies the correctness of the conclusions. By discussing a series of special results, several interesting relational formulas were also obtained. In general, the work in this paper provides several new and universally applicable theoretical formulas for the study of the resistor network model.

Also, the basic formula obtained in this paper is also applicable to the structure of the complex impedance network shown in figure 1, and only the complex impedance needs to be replaced with the complex impedance, so that the general expression of the equivalent complex impedance of any node of the $2 \times n$ impedance network can be obtained directly.

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## Author contributions

S.Z. investigated the research background of this problem; Z-X.W. drew images from the text and checked the syntax; Y-Q.Z. checked and validated the correctness of the calculations; Z-Z.T. designed the research and performed the theoretical calculation and analysis, and also wrote the first draft of the article.

## Data availability statement

No Data associated in the manuscript.

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