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Schrödinger and Klein–Gordon theories of black holes from the quantization of the Oppenheimer and Snyder gravitational collapse

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Abstract

The Schrödinger equation of the Schwarzschild black hole (BH) has been recently derived by the author and collaborators. The BH is composed of a particle, the 'electron', interacting with a central field, the 'nucleus'. Via de Broglie's hypothesis, one interprets the 'electron' in terms of BH horizon's modes. Quantum gravity effects modify the BH semi-classical structure at the Schwarzschild scale rather than at the Planck scale. The analogy between this BH Schrödinger equation and the Schrödinger equation of the s states of the hydrogen atom permits us to solve the same equation. The quantum gravitational quantities analogous of the fine structure constant and of the Rydberg constant are not constants, but the dynamical quantities have well-defined discrete spectra. The spectrum of the 'gravitational fine structure constant' is the set of non-zero natural numbers. Therefore, BHs are well-defined quantum gravitational systems obeying Schrödinger's theory: the 'gravitational hydrogen atoms'. By identifying the potential energy in the BH Schrödinger equation as being the gravitational energy of a spherically symmetric shell, a different nature of the quantum BH seems to surface. BHs are self-interacting, highly excited, spherically symmetric, massive quantum shells generated by matter condensing on the apparent horizon, concretely realizing the membrane paradigm. The quantum BH described as a 'gravitational hydrogen atom' is a fictitious mathematical representation of the real, quantum BH, a quantum massive shell having a radius equal to the oscillating gravitational radius. Nontrivial consequences emerge from this result: (i) BHs have neither horizons nor singularities; (ii) there is neither information loss in BH evaporation, nor BH complementarity, nor firewall paradox. These results are consistent with previous ones by Hawking, Vaz, Mitra and others. Finally, the special relativistic corrections to the BH Schrödinger equation give the BH Klein–Gordon equation and the corresponding eigenvalues.

Keywords: quantum black holes, information paradox, quantum shells, quantum levels, Schrodinger and Klein–Gordon theories

1. Introduction

It is a general conviction, which arises from the famous, pioneering works of Bekenstein [1] and Hawking [2], that the role and the importance of BHs are fundamental in a quantum gravity framework. BHs are indeed considered as being theoretical laboratories for testing different models of quantum gravity. Bekenstein was the first physicist who observed that, in some respects, BHs play the same role in gravitation that atoms played in the nascent quantum mechanics [3]. This analogy implies that BH energy could have a discrete spectrum [3]. Therefore, BHs combine in some sense both the 'hydrogen atom' and the 'quasi-thermal' emission in quantum gravity [4]. As a consequence, BH quantization could be the key to a quantum theory of gravity and, for that reason, BH quantization became, and currently remains, one of the most important research fields in theoretical physics of the last 50 years. Various authors proposed and still propose various different approaches. Hence, the current literature is very rich, see for example [5–18] and references within.

An important longstanding problem in quantum gravity is the following: what is the BH horizon and what happens at it? One finds various reasons to take the BH horizon as a concrete place having real physical meaning. The Bekenstein-Hawking entropy of the BH is proportional to the area of the horizon [19]. This permits us to assign one bit of information to each unit of area (in Planck units) of the BH horizon, and, in principle, to quantize the BH horizon obtaining an integer number of bits [19]. In 1976 Hawking [20] found that if a BH horizon forms and if one assumes that effective field theory concretely works away from such a BH horizon, then BH radiance is obtained in a mixed state from the point of view of an observer external to the BH and static with respect to it, while a freely falling observer detects nothing unusual when crossing the BH horizon. This is the BH information loss paradox. Among the various attempts to resolve the paradox we recall the proposal that the BH horizon could be described as a quantum surface having approximately one degree of freedom per Planckian unit of area [21], the *principle of BH complementarity* [22] and the *firewalls* [23]. In 2014 Hawking [24] proposed that BH event horizons could not be the final result of the gravitational collapse. He hypothesized that the BH event horizon should be replaced by an 'apparent horizon' where infalling matter is suspended and then released. Hawking did not give a mechanism for how this can work. Hawking's conclusion was supported by Vaz, who found the cited mechanism via entire solutions of the Wheeler–DeWitt equation, obtaining a compact shell, that is, a dark star generated by matter condensing on the apparent horizon during quantum collapse [25]. In this paper the same results of Hawking and Vaz will be obtained via a different approach, that is Schrödinger theory of BHs, by showing that quantum gravity effects modify the BH semi-classical structure not at the Planck scale but at the Schwarzschild scale. This is in full agreement with the results of Vaz in [25]. On the other hand, before the works of Hawking and Vaz, Mitra [26], Schild, Leiter and Robertson [27] and other authors, see details in the recent book of Mitra [28], proposed various approaches in which the final result of the stellar gravitational collapse should be an object having neither horizons nor singularities. A breakdown of classical theory at the Schwarzschild scale is present also in the fuzzball paradigm, which is a BH quantum description proposed by string theory [29, 30]. But the final result of the analysis in this paper is different with respect to such a fuzzball paradigm where the quantum BH should be a sphere of strings with a definite volume. In the approach of this paper, it is instead shown that the final result of the gravitational collapse is a self-interacting, highly excited, spherically symmetric, quantum shell, a massive membrane generated by matter condensing on the apparent horizon, which concretely realizes the membrane paradigm. Hence, a series of remarkable consequences arise from this result: (i) BHs have neither horizons nor singularities; (ii) there is neither information loss in BH evaporation, nor BH complementarity, nor firewall paradox.

2. The quantum black hole: from Bohr to Schrödinger

In the framework of quantum BHs, the author and collaborators developed a semiclassical approach to BH quantization, see for example [31–33] and the complete review [34], which is somewhat similar to the historical semi-classical approach to the structure of a hydrogen atom introduced by Bohr in 1913 [35, 36]. Recently, the author and collaborators improved the analysis [37–39]. An approach to quantization due to the famous collaborator of Einstein, Rosen [40] has been applied to the historical Oppenheimer and Snyder gravitational collapse [41]. Such an approach shows that the BHs are the gravitational analog of hydrogen atoms because they result as being composed by a particle, the 'electron', which interacts through a quantum gravitational interaction with a central field, the 'nucleus'. This could, in principle, drive to a space-time quantization based on a quantum mechanical particle-like approach. Let us ask, what is the physical interpretation of the BH 'electron'? If one follows the analogy with the hydrogen atom, the de Broglie hypothesis [42] can be evoked, which enables the wave nature of the BH 'electron'. In other words, this 'particle' does not orbit the nucleus in an analogous way as a planet orbits the Sun. Instead, it should be a standing wave representing a large and often oddly shaped 'atmosphere' (the BH 'electron'), which results in distribution around a central field (the BH 'nucleus'). Such an 'atmosphere' is interpreted as being the BH horizon modes [37–39]. The remarkable consequence is, in turn, that quantum gravity effects modify the BH semi-classical structure not at the Planck scale but at the Schwarzschild scale. The framework where the radius of the event horizon shows quantum oscillations was semi-classically introduced about 50 years ago [43] in terms of the so-called BH quasi-normal modes (QNMs). Such horizon oscillations represent the BH back reaction due to external perturbations. The absorptions of external particles, as well as the emissions of Hawking quanta, are BH perturbations. The Bohr-like approach to BH quantum physics [31–34] has been developed by the author and collaborators starting from such an observation, by improving previous works of Hod [4] and Maggiore [44]. The QNMs framework is a semiclassical one similar to the approach that Bohr developed in 1913 [35, 36] on the structure of the hydrogen atom. But one also recalls that horizon modes have an importance in quantum gravity frameworks if one considers them in terms of the periodic motion of their particle-like analogue [45], by evoking the de Broglie hypothesis. An energy spectrum that scales as $\sim\sqrt{n}$ has been obtained in [45], and this seems in agreement with the results in [31–34] and in [37–39]. In the Bohr-like approach [31–34], by considering the absorptions of external particles, including the original BH formation, and the emissions of Hawking quanta, the BH horizon is not fixed at a constant radius from the BH 'nucleus' [33]. Due to energy conservation, the horizon contracts during the emission of a particle and expands during an absorption [33]. Those quantum contractions/expansions must not be considered as being 'one shot processes' [33]. Instead, the horizon oscillates [33].

The results in [37–39] seem consistent with the ones of Hajicek and Kiefer [11, 12], and permit us to write down, explicitly, the gravitational potential and the Schrödinger equation for the BH as (hereafter Planck units will be used, i.e. $G = c = k_B = \hbar = \frac{1}{4\pi\epsilon_0} = 1$) [37–39]

$$V(r) = -\frac{M_E^2}{r}, \tag{1}$$

$$-\frac{1}{2M_E}\left(\frac{\partial^2 X}{\partial r^2} + \frac{2}{r}\frac{\partial X}{\partial r}\right) + VX = EX, \tag{2}$$

where $E = -\frac{M_E}{2}$ the BH total *energy* and M_E the *BH effective mass* which has been introduced in BH physics by the author and collaborators [31–34]. The BH effective mass is the average of the BH initial and final masses which are involved in a quantum transition. M_E indeed represents the BH mass *during* the BH expansion (contraction), which is triggered by an absorption (emission) of a particle [31–34]. The rigorous definition of the BH effective mass is [31–34]

$$M_E \equiv M \pm \frac{\omega}{2}, \tag{3}$$

where ω the mass-energy of the absorbed (emitted) particle (the sign plus concerns absorptions, the sign minus concerns emissions). Hence, one sees that introducing the BH effective mass in the BH dynamical framework is very intuitive. But, for the sake of mathematical rigor, that introduction has been completely justified via the Hawking's periodicity argument [46], see [31–34] for details. Therefore, in a quantum gravity framework the hole can be interpreted as a particle, the 'electron', which interacts with a central field, the 'nucleus' through the quantum gravitational interaction of equation (1) [31–34]. One also observes that the Schrödinger equation for the BH of equation (2) is formally identical to the traditional Schrödinger equation of the s states ($l = 0$) of the hydrogen atom which obeys the Colombian potential [47]

$$V(r) = -\frac{e^2}{r}. \tag{4}$$

The absence of angular dependence in equation (2) makes it simpler than the corresponding Schrödinger equation for the hydrogen atom from a point of view. From another point of view, it is more complicated instead. In fact, differently from equation (4), where the charge of the electron is constant, equation (2) shows a variation of the BH mass. This is the physical reason for the introduction of the BH effective mass, which is indeed a dynamical quantity, in [31–34]. In the recent work [39], the author and collaborators re-obtained the BH Schrödinger equation (2) via Feynman's path integral approach. In this context, the attentive reader [48] observes that despite a star is made of a very large number of interacting particles (excitations of nonlinear quantum fields), the dust star is here treated as a whole system rather than a large number of dust particles, hence reducing the number of degrees of freedom to just the star radius prior to quantisation. One might thus wonder if the resulting quantum theory reduces to a simple hydrogen-like problem because of this huge simplification - essentially the reduction of many dust particle dynamics to an effective one-body problem [48]. In this case, both Birkoff's theorem and the no-hair theorem [49] come to our aid from the classical theory. Birkhoff's theorem [49] states that a spherical, nonrotating, BH must be the Schwarzschild BH, which is also the final result of the Oppenheimer and Snyder gravitational collapse above discussed. While the no-hair theorem states that the Schwarzschild BH is characterized by only its mass (radius) [49]. Therefore it is really difficult to think that the uniqueness of the 'black hole' object that is obtained in the classical theory can be lost when we pass to the quantum theory. On the other hand, we will see that further support for this conclusion will be given later, in section 4, when we see that the same Schrödinger equation is also obtained from a non-perfectly homogeneous gravitational collapse.

3. Rigorous solution of the black hole Schrödinger equation: energy spectrum and 'gravitational fine structure constant'

One recalls that $e^2 = \alpha$ is the fine structure constant, which, in standard units, combines the constants $\frac{e^2}{4\pi\epsilon_0}$ from electromagnetism, $\hbar$ from quantum mechanics, and the speed of light c from relativity, into the dimensionless irrational number $\alpha \simeq \frac{1}{137,036}$, which is one of the most important numbers in Nature. From the present approach, one argues that the gravitational analogous of the fine structure constant is not a constant. It is a dynamical quantity instead. As in natural units, the fine structure constant is exactly the squared electron charge e^2, in the current case the charge is replaced by the effective mass (the gravitational charge). Thus, calling the square of the BH effective mass the 'gravitational fine structure constant' is due to this analogy. In fact, if one labels the 'gravitational fine structure constant' as α_G, by confronting equations (1) and (4) one gets

$$\alpha_G = \left(\frac{M_E}{m_p}\right)^2, \tag{5}$$

in standard units, where m_p is the Planck mass. It will be indeed shown that α_G has a spectrum of values that coincides with the set of natural numbers $\mathbb{N}$. In an analogous way, the Rydberg constant, $R_\infty = \frac{1}{2}m_e\alpha^2$, is defined in terms of the electron mass m_e and on the fine structure constant. In the current gravitational approach the effective mass and the 'charge' are the same. Hence, the quantum gravitational analogous of the Rydberg constant is

$$(R_\infty)_G = \frac{M_E^5}{2m_p^5 l_p}, \tag{6}$$

in standard units, where l_p is the Planck length. Again, it will be shown that $(R_\infty)_G$ is not a constant but a dynamical quantity having a particular spectrum of values.

By introducing the variable y

$$y(r) \equiv rX, \tag{7}$$

equation (2) becomes

$$-\left(\frac{1}{2M_E}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{M_E^2}{r} + E\right)y = 0. \tag{8}$$

Setting $y' \equiv \frac{\mathrm{d}}{\mathrm{d}r}$ and using $E = -\frac{M_E}{2}$, equation (8) can be rewritten as

$$y'' + M_E^2\left(\frac{2M_E}{r} - 1\right)y = 0. \tag{9}$$

As it is $E < 0$, the asymptotic form of the solution, which is regular at the origin, is a linear combination of exponentials $\exp(M_E r)$, $\exp(-M_E r)$. If one wants this solution to be an acceptable eigensolution, the coefficient in front of $\exp(M_E r)$ must vanish. This happens only for certain discrete values of E. Such values will be the energies of the discrete spectrum of the BH and the corresponding wave function represents one of the possible BH bound states.

If one makes the change of variable

$$x = 2M_E r \tag{10}$$

equation (9) results equivalent to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{M_E^2}{x} - \frac{1}{4}\right]y = 0, \tag{11}$$

and y is the solution which goes as x at the origin. For x very large, it increases exponentially, except for certain particular values of the BH effective mass where it behaves as $\exp\left(-\frac{x}{2}\right)$. One wants to determine such special values and their corresponding eigenfunctions. One starts to perform the change of function

$$y = x\exp\left(-\frac{x}{2}\right)z(x), \tag{12}$$

which changes equation (11) to

$$\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2 - x)\frac{\mathrm{d}}{\mathrm{d}x} - (1 - M_E^2)\right]z = 0. \tag{13}$$

This last equation is a Laplace-like equation. Within a constant, one finds only a solution that is positive at the origin. All the other solutions have a singularity in x^{-1}. One can show that this solution is the confluent hypergeometric series

$$Z = \sum_{i=1}^{\infty}\frac{\Gamma(1 + i - M_E^2)}{\Gamma(1 - M_E^2)}\frac{1}{(1 + i)!}\frac{x^i}{i!}. \tag{14}$$

In fact, one expands the solution of equation (13) in Mc Laurin series at the origin as

$$z = 1 + \alpha_1 x + \alpha_2 x^2 + \dots \alpha_i x^i + \dots \tag{15}$$

By inserting equation (15) in equation (13) one writes the LHS in terms of a power series of x. All the coefficients of this expansion must be null. Thus,

$$\begin{aligned} 2\alpha_1 &= 1 - M_E^2 \\ 2*3\alpha_2 &= (2 - M_E^2)\alpha_1 \\ &\dots\dots \\ i(1 + i)\alpha_i &= (i - M_E^2)\alpha_{i-1}. \end{aligned} \tag{16}$$

Then, one gets

$$\alpha_i = \frac{(i - M_E^2)}{(1 + i)}\frac{(i - 1 - M_E^2)}{(1 + i - 1)}\dots\frac{1 - M_E^2}{1 + 1}\frac{1}{i!}, \tag{17}$$

which is exactly the coefficient of x^i in the expansion (15).

From a mathematical point of view, the series (14) is infinite and behaves as

$$\frac{\exp x}{x^{(1+M_E^2)}}$$

for large x. Consequently, y behaves in the asymptotic region as

$$\frac{\exp\frac{x}{2}}{x^{M_E^2}}$$

and cannot be, in general, an eigensolution. On the other hand, for particular values of M_E^2 all the coefficients will vanish from a certain order on. In that case, the series (14) reduces to a polynomial. The requested condition is

$$1 - M_E^2 \leqslant 0, \text{ with } |1 - M_E^2| \in \mathbb{N}, \tag{18}$$

which implies the quantization condition

$$M_E^2 = n = n' + 1, \text{ with } n' = 0, 1, 2, 3, \dots +\infty. \tag{19}$$

The hypergeometric series becomes a polynomial of degree n' and y behaves in the asymptotic region as

$$\frac{x^n}{\exp\frac{x}{2}}.$$

Then, the regular solution of the BH Schrödinger equation becomes an acceptable eigensolution. The quantization condition (19) permits us to obtain the spectrum of the effective mass as

$$(M_E)_n = \sqrt{n}, \textit{ with } n = 1, 2, 3, \dots +\infty. \tag{20}$$

The quantization condition (19) gives also the spectrum of the 'gravitational fine structure constant' that one writes as

$$(\alpha_G)_n = \left[\frac{(M_E)_n}{m_p}\right]^2 = n, \text{ with } n = 1, 2, 3, \dots +\infty. \tag{21}$$

Thus, the gravitational analogous of the fine structure constant is not a constant. It is a dynamical quantity instead, which has the spectrum of values (21) which coincides with the set of non-zero natural numbers $\mathbb{N} - \{0\}$. In the same way, one finds the spectrum of the 'gravitational Rydberg constant' as

$$[(R_\infty)_G]_n = \frac{(M_E)_n^5}{2m_p^5 l_p} = \frac{n^{\frac{5}{2}}}{2},$$
$$\text{with } n = 1, 2, 3, \ldots +\infty. \tag{22}$$

Now, from the quantum point of view, one wants to obtain the mass eigenvalues as being absorptions starting from the BH formation, that is from the BH having null mass, where with 'the BH having null mass' one means the situation of the gravitational collapse before the formation of the first event horizon. This implies that one must replace $M \to 0$ and $\omega \to M$ in equation (3). Thus, one gets

$$M_E \equiv \frac{M}{2}. \tag{23}$$

By combining equations (20) and (23) one immediately gets the BH mass spectrum as

$$M_n = 2\sqrt{n}, \tag{24}$$

which is the same result as [37–39]. Equation (24) is consistent with the BH mass spectrum found by Bekenstein in 1974 [1]. Bekenstein indeed obtained $M_n = \sqrt{\frac{n}{2}}$ by using the Bohr–Sommerfeld quantization condition because he argued that the BH behaves as an adiabatic invariant. It is also consistent with other BH mass spectra in the literature, see for example [44, 45]. The relationship between the BH effective mass and the BH total energy is [37–39]

$$E = -\frac{M_E}{2}, \tag{25}$$

which permits to find the BH energy spectrum as

$$E_n = -\sqrt{\frac{n}{4}}. \tag{26}$$

The (unnormalized) wave function corresponding to each energy level is

$$X(r) = \frac{1}{r}\left[\frac{2M_E r}{\exp\frac{x}{2}}\left(\sum_{i=0}^{n-1}(-)^i \times \frac{(n-1)!}{(n-1-i)!(i+1)!i!}\right)(2M_E r)^i\right]. \tag{27}$$

One observes that the number of nodes of the wave function is exactly $n-1$ and that the energy spectrum (26) contains a denumerably infinite number of levels because n can take all the infinite values $n \in \mathbb{N} - \{0\}$. One can calculate the energy jump between two neighboring levels as

$$\triangle E = E_{n+1} - E_n = \sqrt{\frac{n}{4}} - \sqrt{\frac{n+1}{4}} = -\frac{1}{4\left(\sqrt{\frac{n}{4}} + \sqrt{\frac{n+1}{4}}\right)}. \tag{28}$$

For large n one obtains

$$\triangle E \simeq -\frac{1}{4\sqrt{n}}$$

when $n \to +\infty$ one gets $\triangle E \to 0$. Thus, one finds that the energy levels become more and more closely spaced and their difference tends to $\triangle E = 0$ at the limit, at which point the continuous spectrum of the BH begins.

One also observes that a two particle Hamiltonian

$$H(\vec{p}, r) = \frac{p^2}{2M_E} - \frac{M_E^2}{r}, \tag{29}$$

which governs the BH quantum mechanics, must exist in correspondence of equations (1) and (2). Therefore, the square of the wave function (27) must be interpreted as the probability density of a single particle in a finite volume. Thus, the integral over the entire volume must be normalized to unity as

$$\int dx^3 |X|^2 = 1. \tag{30}$$

For stable particles, this normalization must remain the same at all times of the BH evolution. This issue has deep implications for the BH information paradox [20] because it guarantees the preservation of quantum information. As the wave function (27) obeys the BH Schrödinger equation (2), this is assured if and only if the Hamiltonian operator (29) is Hermitian [51]. In other words, the Hamiltonian operator (29) must satisfy for arbitrary wave functions X_1 and X_2 the equality [51]

$$\int dx^3 [HX_2]^* X_1 = \int dx^3 X_2^* HX_1. \tag{31}$$

One notes that both $\vec{p}$ and r are Hermitian operators. Thus, the Hamiltonian (29) will automatically be a Hermitian operator if it is a sum of a kinetic and a potential energy [51]

$$H = T + V. \tag{32}$$

This is always the case for non-relativistic particles in Cartesian coordinates and works also for BHs.

Following Rosen [40], one can also find the BH 'Bohr radius', the BH wave function and the BH expected radial distance. Such quantities and properties will be very useful in the next section. Considering Bohr's semi-classical model of a hydrogen atom, the Bohr radius, that is, the classical radius of the electron at the ground state, is [40]

$$\text{Bohr radius} = \frac{1}{m_e e^2}, \tag{33}$$

being m_e the electron mass. As one wants to obtain the corresponding 'Bohr radius' for the quantum BH, one must replace both m_e and e in equation (33) with the effective mass of the BH ground state, which is $\frac{M_1}{2} = 1$. Thus, the BH 'Bohr radius' reads

$$b_1 = 1, \tag{34}$$

which in standard units becomes $b_1 = l_P$, being $l_P = 1$, 61 625 $\times 10^{-35}$ m the Planck length. Hence, the 'Bohr radius' associated with the smallest possible BH is equal to the Planck length. One finds the wave function associated with the BH ground state as

$$\Psi_1 = 2b_1^{-\frac{3}{2}} r \exp-\left(\frac{r}{b_1}\right) = 2r\exp-(r), \tag{35}$$

being Ψ_1 is normalized as

$$\int_0^\infty \Psi_1^2 \mathrm{d}r = 1. \tag{36}$$

The expected radial distance of this BH ground state is of the order of

$$\bar{r}_1 = \int_0^\infty \Psi_1^2 r \mathrm{d}r = \frac{3}{2} b_1 = \frac{3}{2}. \tag{37}$$

In fact, one has to recall that in quantum mechanics the Bohr radius (34) represents the radius having the maximum radial probability density instead of its expected radial distance [52]. The latter is indeed 1.5 times the Bohr radius [52]. This is due to the long tail of the radial wave function [52] and it is, in turn, given by equation (37). One finds the expected radial distance for the BH excited at the level n as

$$\bar{r}_n = \frac{3}{2}(M_E)_n = \frac{3}{2}\sqrt{n}. \tag{38}$$

In Bohr's semi-classical model of hydrogen atom, the Bohr radius for the electron excited at the level n is

$$\text{Bohr radius}_n = \frac{n^2}{m_e e^2}. \tag{39}$$

Hence, if one wants to obtain the corresponding 'Bohr radius' for the quantum BH excited at the level n, one must replace both m_e and e in equation (39) with the effective mass of the BH excited at the level n. Then, by using equation (20), one obtains

$$b_n = \sqrt{n}, \tag{40}$$

that is half of the effective gravitational radius associated with the BH excited at the level n.

Thus, it has been shown that a BH is a well-defined quantum system, which obeys Schrödinger's theory. In a certain sense, it is a 'gravitational hydrogen atom'. One also observes that studying the BH in terms of a well-defined quantum mechanical system, having an ordered, discrete quantum spectrum, looks consistent with the unitarity of the underlying quantum gravity theory and with the idea that information should come out in BH evaporation.

4. Quantum shell, breakdown of black hole complementarity and of firewall paradox. Information recovery

In this section it is shown that equation (2) is the Schrödinger equation of a self-interacting massive quantum shell, generated by matter condensing on the apparent horizon, which concretely realizes the membrane paradigm. This important issue enables a series of nontrivial consequences, in particular: (i) BHs have neither horizons nor singularities; (ii) there is neither information loss in BH evaporation nor BH complementarity nor firewalls.

Let us start by rewriting the potential of equation (1) as

$$V = -\frac{M_E^2}{r} = -\frac{M^2}{4r} = -\frac{M^2}{2R}, \tag{41}$$

where one sets $R \equiv 2r$. The potential $V = -\frac{M^2}{2R}$ is well known as being the self-interaction gravitational potential of a spherical massive shell, where R is its radius [53]. One also recalls from previous sections of this paper and from [37–39] that the physical interpretation of the coordinate r in the Schrödinger equation (2) is in terms of the oscillating effective gravitational radius. Thus, from equation (23) R results to be the real oscillating gravitational radius. On the other hand, $V = -\frac{M^2}{2R}$ is also the potential of a two-particle system composed of two identical masses M gravitationally interacting with a relative position $2R$. Thus, the spherical shell is physically equivalent to a two-particle system of two identical masses, but, clearly, as the BH mass M does not double, one has to consider the two identical masses M as being fictitious and representing the real physical shell. Let us recall the general problem of a two-particle system where the particles have different masses [47]. This is a 6-dimensional problem that can be split into two 3-dimensional problems, that of a static or free particle, and that of a particle in a static potential if the sole interaction which is felt by the particles is their mutual interaction depending only on their relative position [47]. One denotes by m_1, m_2 the masses of the particles, by $\overrightarrow{d_1}$, $\overrightarrow{d_2}$ their positions and by $\overrightarrow{p_1}$, $\overrightarrow{p_2}$ the respective momenta. Being $\overrightarrow{d_1} - \overrightarrow{d_2}$ their relative position, the Hamiltonian of the system reads [47]

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(\overrightarrow{d_1} - \overrightarrow{d_2}). \tag{42}$$

One sets [47]:

$$\begin{aligned} m_T &= m_1 + m_2, \qquad \overrightarrow{D} = \frac{m_1 \overrightarrow{d_1} + m_2 \overrightarrow{d_2}}{m_1 + m_2}, \\ \overrightarrow{p_T} &= \overrightarrow{p_1} + \overrightarrow{p_2}, \\ m &= \frac{m_1 m_2}{m_1 + m_2} \qquad \overrightarrow{d} = \overrightarrow{d_1} - \overrightarrow{d_2} \\ \overrightarrow{p} &= \frac{m_1 \overrightarrow{p_1} + m_2 \overrightarrow{p_2}}{m_1 + m_2}. \end{aligned} \tag{43}$$

The change of variables of equation (43) is a canonical transformation because it conserves the Poisson brackets [47]. According to the change of variables of equations (43), the motion of the two particles is interpreted as being the motion of two fictitious particles: (i) the *center of mass*, having position $\overrightarrow{D}$, total mass m_T and total momentum $\overrightarrow{p_T}$ and, (ii) the *relative particle* (which is the particle associated with the relative motion), having position $\overrightarrow{d}$, mass m, called r*educed mass*, and momentum $\overrightarrow{p}$ [47]. The Hamiltonian of equation (42) considered as a function of the new variables of equation (43) becomes [47]:

$$H = \frac{p_T^2}{2m_T} + \frac{p^2}{2m} + V(\overrightarrow{d}). \tag{44}$$

The new variables obey the same commutation relations as if they should represent two particles of positions $\overrightarrow{D}$ and $\overrightarrow{d}$ and momenta $\overrightarrow{p_T}$ and $\overrightarrow{p}$ respectively [47]. The Hamiltonian of equation (44) can be considered as being the sum of two terms [47]:

$$H_T = \frac{p_T^2}{2m_T}, \tag{45}$$

and

$$H_m = \frac{p^2}{2m} + V(\overrightarrow{d}). \tag{46}$$

The term of equation (45) depends only on the variables of the center of mass, while the term of equation (46) depends only on the variables of the relative particle. Thus, the Schrödinger equation in the representation $\overrightarrow{D}$, $\overrightarrow{d}$ is [47]:

$$\left[\left(-\frac{1}{2m_T}\triangle_D\right) + \left(-\frac{1}{2m}\triangle_d + V(d)\right)\right] \Psi(D, d) = E\Psi(D, d), \tag{47}$$

being $\triangle_{\overrightarrow{D}}$ and $\triangle_{\overrightarrow{d}}$ the Laplacians relative to the coordinates $\overrightarrow{D}$ and $\overrightarrow{d}$ respectively. Now, one observes that the BH effective mass of equation (23) is also the reduced mass of the previously introduced two-particle system composed of two identical masses M:

$$M_E = \frac{M * M}{M + M} = \frac{M}{2}. \tag{48}$$

In that case, by recalling that in Schwarzschild coordinates the BH center of mass coincides with the origin of the coordinate system, and with the replacements

$$\begin{gathered} m \to M_E \\ d \to R, \end{gathered} \tag{49}$$

the Schrödinger equation (47) becomes

$$\left(-\frac{1}{2M_E}\triangle_{2R} + V(2R)\right)\Psi(2R) = E\Psi(2R), \tag{50}$$

which, by using equation (41) and $R = 2r$, in the representation $\overrightarrow{D} = 0$, $\overrightarrow{r}$ reads

$$-\frac{1}{2M_E}\left(\frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r}\right) + V\Psi = E\Psi \tag{51}$$

for the s-states, and coincides with equation (2), with the sole difference that in the analysis in this section, the wave function has been labelled with Ψ rather than X.

Thus, one argues the intriguing physical interpretation of the analysis in this section. The quantum BH described in previous sections in terms of a 'gravitational hydrogen atom' being composed by a particle, the 'electron', which interacts with a central field, the 'nucleus', is only a fictitious mathematical representation (in fact, perfect central fields having infinite mass and pointlike 'charge' do not exist in nature and one has to determine the correct mass distribution of the real quantum field [47]) of the real, concrete physical quantum BH, which results in being a highly excited quantum shell, a massive membrane generated by matter condensing on the apparent horizon, which results to be an 'accumulation surface' and having as radius the oscillating gravitational radius. This quantum massive shell self-interacts via the potential of equation (41). This result, which has been obtained in a rigorous way in [39] by quantizing the historical Oppenheimer and Snyder gravitational collapse through Feynman's path integral approach, is founded on the sole assumption that the final result of the Oppenheimer and Snyder gravitational collapse must be the Schwarzschild BH, which is a result older than 80 years. Only general relativity and quantum mechanics have been used in its derivation and it has been shown that this quantum BH obeys Schrödinger's theory.

The implications of this result are notable. Matter condenses on the apparent horizon during the collapse without undergoing further collapse and, consequently, with the formation of neither horizons nor singularities. Instead, quantum collapse generates the oscillating quantum massive membrane on the apparent horizon which seems to be the real physical state of the quantum BH. These BHs quantum shells are prevented from collapsing by quantum gravity in a similar way to how atoms are prevented from collapsing by quantum mechanics. This is consistent with the approaches of Hawking [24] and Vaz [25] concerning the BH information paradox [20]. In fact, in 2014 Hawking [24] proposed that BH event horizons could not be the final result of the gravitational collapse. He speculated that the BH event horizon should be replaced by an 'apparent horizon' where infalling matter is suspended and then released, exactly like in the current analysis. Hawking did not give a mechanism for how this can work, which was later given by Vaz [25], who supported Hawking's conclusion. Vaz indeed discussed an interesting quantum gravitational model of dust collapse by showing that continued collapse to a singularity can only be achieved by combining two independent and entire solutions of the Wheeler–DeWitt equation [25]. He argued that such a combination is forbidden leading in a natural way to matter condensing on the apparent horizon during quantum collapse [25], which is the same result in this section. In fact, in this section, the same results of Hawking and Vaz have been obtained via Schrödinger's theory of BHs. This cannot be a coincidence and it is also a suggestive issue because a general important problem in quantum gravity is that different approaches usually give different results. This is not the case in the current analysis, which retrieves the same results as Hawking and Vaz. Our results are also consistent with Einstein's idea of the localization of the particles within a thin spherical shell [54]. It is also important to stress that before the works of Hawking and Vaz, Mitra [26], Schild, Leiter and Robertson [27] and other authors (details can be found in the recent book of Mitra [28]) proposed various approaches in which the final result of the stellar gravitational collapse has to be an object having neither horizons nor singularities.

Let us clarify an important point [48]. The quantum description in this section implies that collapsing matter accumulates near the (apparent) horizon and forms a shell. One might rather suspect that this is a consequence of the previously cited fact that, as the dust star is here treated as a whole system rather than a large number of dust particles, hence reducing the number of degrees of freedom to just the star radius prior to quantisation. Thus, reducing the degrees of freedom to just the radius of the star is not enough to estimate how collapsed matter is actually distributed. In fact, a simple argument against the shell is provided by considering that many dust particles (hence a significant fraction of the BH mass) could already be inside the gravitational radius of the star when the collapse starts: will they be pushed outwards? Or will they not contribute to the quantised Bekenstein mass? Hence the question: how solid can one take the conclusion that there is no horizon? Is the probability of finding dust outside the gravitational radius completely negligible and for any value of the BH mass? This problem is solved in Vaz's approach [25]. By analysing a nonhomogenous dust collapse, Vaz has shown that Dirac quantization of the constraints leads to a Wheeler–DeWitt equation. For a smooth dust distribution, this equation can be regularized on a lattice [25]. Each point on the lattice represents a collapsing dust shell and the final solution (equation (5) in [25]) represents collapse with support everywhere in spacetime and for any value of the BH mass. This solution results in dust shells condensing to the apparent horizon on both sides. In fact, the wave functions leading to equation (5) in [25] are well-defined everywhere except at the apparent horizon, where there is an essential singularity. Interior and exterior solutions can be matched by deforming the integration path in the complex Ri -plane so as to go around the essential singularity at the apparent horizon [25].

One recalls from the previous section that the average size of the 'gravitational hydrogen atom' excited at the level n is given by the expected radial distance of equation (38). Thus, one finds an average size of the BH quantum shell given by

$$\bar{R}_n = 3(M_E)_n = 3\sqrt{n}\,, \tag{52}$$

and, using the mass spectrum of equation (24) one gets

$$\bar{R}_n = \frac{3}{2}M_n. \tag{53}$$

By recalling that the classical fixed gravitational radius is $R_g = 2M$, one gets

$$\bar{R}_n = \frac{3}{4}(R_g)_n, \tag{54}$$

which means that the size of the quantum BH is $\frac{3}{4}$ of the size of its classical counterpart. In an analogous way, the maximum radial probability density for the BH quantum shell is

$$2b_n = 2\sqrt{n}\,, \tag{55}$$

that is, exactly the effective BH gravitational radius, as one intuitively expects.

Hence, the surface of the Schwarzschild sphere is not an event horizon. Instead, it is a self-interacting quantum shell, a massive membrane generated by matter condensing on the apparent horizon, which oscillates around its average value of equation (52) and with a maximum radial probability density given by equation (55). This shell must have a physical thickness because of the generalized uncertainty principle and quantum fluctuations [55]. This is consistent with various previous proposals in BH physics. In fact, the first, important approach which attempted to understand which are the BH degrees of freedom to be quantized was the membrane paradigm [56]. In this framework, an observer external to the BH and static with respect to it sees the BH interactions with the external environment as being completely described in terms of a fictitious membrane that should be located close to the horizon and should have various physical properties like temperature, viscosity and electrical charge. In order to attempt solving the BH information paradox, in [57] Mathur proposed to realize the membrane paradigm by finding real degrees of freedom just outside the BH horizon via string theory, higher dimensional gravity theory and extra directions. In the picture in [57] the real degrees of freedom of the BH are given by the rapidly oscillating solutions of Einstein equations corresponding to the BH microstates. The result in the current paper is stronger, because it shows that the BH is physically a real, self-interacting oscillating quantum shell with a physical thickness. This BH quantum shell is generated by the matter which condenses on the apparent horizon. It is obtained not by a conjectured paradigm but directly from the gravitational collapse of dust and seems to be the real physical nature of the BH. This has also profound implications on the conjecture of BH complementarity [22], which states that the points of view of the free falling observer and of the observer external to the BH and static with respect to it should be complementary. Therefore, on the one hand the information should be reflected at the BH horizon and on the other hand it should pass through the BH horizon without escaping. The principle of BH complementarity implies that no ideal 'super observer' can confirm both stories simultaneously. For the observer external to the BH and static with respect to it, the infinite time dilation at the BH horizon corresponds to the infinite amount of time to reach the horizon. In that case, the stretched horizon results in being hot and having a physical meaning. Hence, the observer external to the BH and static with respect to it observes that infalling information heats up the stretched BH horizon, which, in turn, re-radiates it in terms of Hawking radiation. Consequently, BH evaporation should be unitary. The infalling observer sees instead nothing special happening at the BH horizon with both the observer and the information falling to the BH singularity. The authors of [22] proposed that both stories are complementary in the quantum sense. Thus, one gets no contradiction without violation of linearity in quantum theory. The conclusion of the approach in this paper has been instead that the final result of the gravitational collapse is a self-interacting, highly excited, spherically symmetric, massive quantum shell, generated by matter condensing on the apparent horizon, which concretely realizes the membrane paradigm. Thus, on the one hand, BHs have neither horizons nor singularities. On the other hand, one argues that the supposed BH complementarity arises from the use of the classical theory at the Schwarzschild scale. But in this paper, it has been shown that the classical theory breaks down at the Schwarzschild scale because the quantum approach that has been developed in this section and in the previous one gives dramatically different results with respect to the classical theory. Then, the observer external to the BH and static with respect to it the same BH will continue to see the horizon as being a membrane because all matter that ever fell towards the horizon seems as being frozen forever just outside the horizon. But now, the infalling observer must use quantum theory as he approaches the apparent horizon (at the Schwarzschild scale) and his final destiny will be to end up absorbed by the BH oscillating massive membrane which, in turn, will change its mass and its total energy by jumping to a higher (negative) energy level. Therefore, there is neither contradiction nor complementarity between what the infalling observer sees and what the observer external to the BH and static with respect to it sees. Likewise, there is no need to postulate the existence of a firewall.

Another fundamental issue is the absence of the information paradox. Let us start by recalling what the BH information paradox is [20]. The correct description of the vacuum around a BH event horizon leads to the emission of Hawking quanta in terms of vacuum polarization. Hawking radiation arises from just above the horizon. Thus, the unique BH behavior in general relativity implies the universality of Hawking radiation. In Hawking's original derivation [2], BH radiance was strictly thermal, featureless and independent of BH formation. A more rigorous formulation of it, based on the modern language of tunnelling, has instead shown that BH back reaction and energy conservation imply a slight deviation from the strict thermality [50]. In any case, a consequence of Hawking radiation will be that the BH will evaporate for very long times. This generates the BH information paradox. In Hawking words, verbatim [20] 'if there were an event horizon, the outgoing state would be mixed. If the black hole evaporated completely without leaving a remnant, as most people believe and would be required by CPT, one would have a transition from an initial pure state to a mixed final state and a loss of unitarity'. In other words, the BH interior state cannot be reconstructed (with the macroscopic exception of the BH 'hairs', which are mass, charge and angular momentum [49]) from the exterior data, and, in particular, from the final state of Hawking radiation. Consequently, there is no unitary transformation of states in a Hilbert space that can describe BH evaporation, which, in turn, seems to make quantum gravity non-unitary. But in the above analysis it has been shown that the real quantum state of the Schwarzschild BH is in terms of an object free of horizons and singularities, that is, a self-interacting, highly excited, spherically symmetric, massive quantum membrane, generated by matter condensing on the apparent horizon. Thus, on the one hand, the internal states of the oscillating quantum shell must be in causal connection with distant observers, in agreement with the strong equivalence principle which states that special relativity must hold locally for all of the laws of physics in all of spacetime as seen by time-like observers (see section 2.1 of [58]). This guarantees the preservation of the physical information. On the other hand, the absence of horizons implies that the BHs oscillating quantum shells must radiate from their surface like all other normal bodies, because radiation cannot arise from pair creation from the vacuum. Therefore, there is no information paradox.

It is also important to estimate the maximum value of the density of the quantum membrane. If one returns to the BH mathematical description in terms of a quantum system composed by a particle, the 'electron', which interacts through a quantum gravitational interaction with a central field, the 'nucleus', then the Born rule [59] and the Copenhagen interpretation of quantum mechanics [60] imply that one cannot exactly localize the position of the 'electron', but one can only find the probability density of finding the 'electron' at a given point which is, in turn, proportional to the square of the magnitude of the wave function of the 'electron' at that point. Being the system 'electron-nucleus' only a fictitious representation of the physical quantum shell, this implies that one cannot exactly localize the position of the quantum shell via the oscillating gravitational radius, and must, in turn, use an average radius. The average radius of the BH quantum shell is given by equation (52). Thus, by evoking again the generalized uncertainty principle, which guarantees that the BH quantum shell must have a physical thickness, at least of the order of the Planck length, one can compute the minimum volume of the BH quantum shell (in Planck units) as the difference between the volume of the sphere having radius $3\sqrt{n}+\frac{1}{2}$ and the volume of the sphere having radius $3\sqrt{n}-\frac{1}{2}$. Thus, one gets:

$$V_{\min}=\frac{4}{3}\pi\left[\left(3\sqrt{n}+\frac{1}{2}\right)^3-\left(3\sqrt{n}-\frac{1}{2}\right)^3\right]=\frac{4}{3}\pi\left(27n+\frac{1}{4}\right)=36\pi n+\frac{\pi}{3}. \tag{56}$$

On the other hand, the mass spectrum of the BH quantum shell is given by equation (24). Hence, one obtains the maximum value of the density of the BH quantum shell as

$$\rho_{\max}=\frac{2\sqrt{n}}{36\pi n+\frac{\pi}{3}}. \tag{57}$$

The maximum density decreases with increasing n, as one intuitively expects. Thus, the maximum density corresponds to the ground state of the BH quantum membrane, that, for $n=1$, is a density of

$$\rho_{\max}(n=1)=\frac{2}{36\pi+\frac{\pi}{3}}\simeq 0.0175, \tag{58}$$

in Planck units. By recalling that the Planck density is roughly 10^{93} grams per cubic centimetre in standard units, one gets a value of

$$\rho_{\max}(n=1)\simeq 1.752*10^{91}\ \text{grams per cubic centimetre} \tag{59}$$

for the density of the ground state of the BH quantum shell in standard units, which is very high but about two order of magnitude less than the Planck density. For large n equation (57) is well approximated by

$$\rho_{\max}\simeq\frac{1}{18\pi\sqrt{n}}. \tag{60}$$

For a BH having mass of the order of 10 solar masses equation (24) gives

$$\sqrt{n} = \frac{10M_{\odot}}{2} = 5M_{\odot} \sim \frac{10^{34}\ \text{grams}}{M_p} \sim 5 * 10^{38}. \tag{61}$$

being $M_{\odot} \sim 2 * 10^{33}$ grams the solar mass and $M_p \sim 2 * 10^{-5}$ grams the Planck mass. By inserting the result of equation (61) in equation (60) one gets

$$\rho_{\max}(10M_{\odot}) \sim \frac{1}{5{*}18\pi * 10^{38}} \sim 3.5 * 10^{-41} \tag{62}$$

in Planck units and, being the Planck density roughly 10^{93} grams per cubic centimetre in standard units, one finds a value of

$$\rho_{\max}(10M_{\odot}) \sim 3.5 * 10^{52} \text{ grams per cubic centimetre.} \tag{63}$$

5. From Schrödinger to Klein–Gordon

The attentive reader [48] could ask if, given the analogy with Newtonian gravity, there is any essential difference between the BH quantum description in general relativity given here and what one would obtain in Newtonian physics. For example, one could ask if the spectra are the same. In our opinion, this point, despite being interesting, is outside the goal of this paper. The cited analogy is due to the fact that the Lagrangian of equation (60) in [39] is written in the form $T-U$ where both T and U are Newtonian quantities. On the other hand, both the quantities T and U have been obtained by using the Einstein field equation of general relativity, see [39] for details. Thus, the above question can be reformulated as 'can the quantities T and U be obtained via a pure Newtonian formulation of the gravitational collapse or will one find some deviations from their obtaining via the Einstein field equation of general relativity?' This could be an interesting point to be developed in a future paper.

In this section, the non-relativistic Schröedinger equation (2) is promoted to a relativistic Klein–Gordon equation. This step needs to be explained better also in light of the previous discussion regarding the fact that up to now only non-relativistic quantities have been quantized [48]. Equations (1) and (2) are indeed a consequence of general relativity. This must be compatible with special relativity (to which general relativity reduces locally in the freely falling frame of the dust because of the Equivalence Principle) [48]. This will allow to insert the formalism of special relativity within the quantum analysis of this paper, writing an equation in covariant form. In this way, time and space will be treated in the same way and the d'Alembertian operator turns out to be relativistically invariant. This treatment seems to work also for small mass BH which could, in principle, be viewed as sorts of fundamental particles (they are represented via particle-like mathematical equations). Thus, in this section, the special relativistic corrections to the BH Schrödinger theory, which has been developed in the previous section, will be discussed. This will permit us to obtain the BH Klein–Gordon equation and the corresponding eigenvalues. By considering again the particle-like approach of sections 3 and 4, the goal of this section is to find the Klein–Gordon generalization of the BH Schrödinger equation (2). To make this, following [61] one recalls that the derivation of a wave equation for a particle of mass m can start with the relativistic dispersion relation for the free particle [61]:

$$p^{\mu}p_{\mu} = g_{\mu\nu}p^{\mu}p^{\nu} = m^2. \tag{64}$$

Then, in terms of the energy E and the three-momentum $\vec{p}$ one gets the well-known equation on the relativistic energy [61]

$$E^2 - (\vec{p})^2 = m^2. \tag{65}$$

The Principle of Minimal Electromagnetic Coupling, that is [61]

$$p_{\mu} \rightarrow \pi_{\mu} = p_{\mu} - qA_{\mu}, \tag{66}$$

describes the interaction of a particle of charge q with the electromagnetic field. In equation (66) the four-vector potential A consists of the scalar potential Φ and the vector potential $\vec{A}$ [61]. These obey $\vec{B} = \nabla \times \vec{A}$ and $\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$ [61]. For an electron $q=-e$, being e the charge on the proton, which is assumed as being positive by convention. The Coulomb potential due to a proton is $\Phi = \frac{e}{r}$ and $\vec{A} = 0$. Then, $E \rightarrow E + \frac{e^2}{r}$, where r is the proton-electron distance [61]. Now, one replaces $\vec{p} \rightarrow -i\nabla$ and allows the resulting Klein–Gordon equation to act on a spacial function $\psi(\vec{x})$ [61]

$$\left[E^2 - m^2 + 2E\left(\frac{e^2}{r}\right) + \left(\frac{e^2}{r}\right)^2 - (-i\nabla)^2\right]\psi(\vec{x}) = 0. \tag{67}$$

From the historical point of view equation (67), which is now known as the Klein–Gordon equation in a Coulomb potential, was originally derived by Schrödinger in his search for an equation describing de Broglie waves. This approach has been indeed found in Schrödinger's notebooks from late 1925, and he appears to have prepared a manuscript applying it to the hydrogen atom.

On the other hand, for solving the Klein–Gordon equation momentum space is more convenient than coordinate space [62, 63]. In momentum space equation (67) becomes [62]

$$(E\widehat{R^2} + e^2E\widehat{R} + e^2)\psi(p) = \widehat{R^2}(m^2 + p^2)\psi(p). \tag{68}$$

For the s-states the distance squared and distance operators act in momentum space as [62]

$$R^2\psi(p) = -\left(\frac{\mathrm{d}^2}{\mathrm{d}p^2} + \frac{2}{p}\frac{\mathrm{d}}{\mathrm{d}p}\right)\psi(p)$$
$$R\psi(p) = i\left(\frac{\mathrm{d}}{\mathrm{d}p} + \frac{1}{p}\right)\psi(p). \tag{69}$$

By inserting equation (69) in equation (68) one finds [62]

$$(\epsilon^2 + p^2)\frac{d^2\psi(p)}{dp^2} + \left(\frac{2\epsilon}{p} + 2iEe^2 + 6p\right)\frac{d\psi(p)}{dp} + \\ +\left(e^4 + \frac{2iEe^2}{p} + 6\right)\psi(p) = 0, \tag{70}$$

where $\epsilon^2 \equiv m^2 - E^2$. Now, following again the analogy between the s-states of the hydrogen atom and the BH particle-like representation in sections 3 and 4, in order to obtain the Klein–Gordon equation of the Schwarzschild BH one makes the following replacements in equation (70)

$$\begin{aligned} m &\to M_E \\ e &\to M_E \\ \epsilon^2 &\to \epsilon_{BH}^2 \equiv M_E^2 - E^2. \end{aligned} \tag{71}$$

Thus, the BH Klein–Gordon equation reads

$$(\epsilon_{BH}^2 + p^2)\frac{d^2\psi(p)}{dp^2} + \left(\frac{2\epsilon_{BH}}{p} + 2iEM_E^2 + 6p\right)\frac{d\psi(p)}{dp} \\ +\left(M_E^4 + \frac{2iEM_E^2}{p} + 6\right)\psi(p) = 0. \tag{72}$$

Following [62], the solution of equation (72) is obtained in terms of the hypergeometric function as

$$\psi(p) = \frac{C}{p}\left(1 + \frac{ip}{\epsilon_{BH}}\right)^{-\left(\frac{3}{2}+\mu\right)} F \\ \times\left[\frac{3}{2} + \mu, \frac{1}{2} - w + \mu, \mu + 1, 2\left(1 + \frac{ip}{\epsilon_{BH}}\right)^{-1}\right], \tag{73}$$

where F is the hypergeometric function C is a normalization constant and

$$\mu \equiv \sqrt{\frac{1}{4} - M_E^4}, \qquad w \equiv \frac{EM_E^2}{\epsilon_{BH}}. \tag{74}$$

and the quantization condition of the energy is

$$n = w - \mu - \frac{1}{2}, \tag{75}$$

which is obtained in order to ensure that the wave function (73) is square integrable. Then, the hypergeometric series reduces to a polynomial. By inserting the two quantities of equation (74) in equation (75) and by using some algebra one gets the energy spectrum

$$E_n = \frac{M_E}{\sqrt{1 + \frac{M_E^4}{n - \frac{1}{2} + \sqrt{\frac{1}{4} - M_E^4}}}}. \tag{76}$$

One notes that the BH effective mass M_E is not a constant, but it is given by the spectrum of equation (20). This implies that, as it is always $M_E \geqslant 1$, then the eigenvalues of equation (76) are all imaginary. In order to solve the problem one needs to go beyond the assumption of minimal coupling of equation (66) which works in the electromagnetic case. In fact, in the gravitational potential of equation (1) one finds the square of the mass. Thus, one needs to couple such a scalar interaction to the square of the mass in a relativistic sense in the equation of motion [63]. In general, for a particle of mass m this is obtained via the replacement [63]

$$m^2 \to m^2 + U^2(r), \tag{77}$$

where $U(r)$ is the potential of an arbitrary scalar interaction acting on the particle of mass m. The corresponding Klein–Gordon equation for the s-states with this non-minimal coupling is [63]

$$\left[\frac{d^2}{dr^2} + E^2 - m^2 - U^2(r)\right]\psi(r) = 0. \tag{78}$$

By dividing this last equation by m^2 with the replacement [63]

$$r' = rm, \tag{79}$$

one finds

$$\left[\frac{d^2}{dr'^2} + \frac{E^2}{m^2} - 1 - \frac{U^2(r')}{m^2}\right]\psi(r') = 0. \tag{80}$$

One specifies the interaction via the first of equations (71) and setting [63]

$$\begin{aligned} \frac{U^2(r')}{M_E^2} &= W(r') = -\frac{M_E^2}{r'} \\ b^2 &= 1 - \frac{E^2}{M_E^2}. \end{aligned} \tag{81}$$

Following the analysis in [63] one gets

$$\left[\frac{d^2}{dr'^2} - b^2 + \frac{M_E^2}{r'}\right]\psi(r') = 0. \tag{82}$$

Again, one replaces

$$\begin{aligned} \xi &= 2br' \\ c &= \frac{M_E^2}{2b}, \end{aligned} \tag{83}$$

obtaining

$$\left[\frac{d^2}{d\xi^2} - \frac{1}{4} + \frac{c}{\xi}\right]\psi(\xi) = 0. \tag{84}$$

Now, one analyses the asymptotics $\xi \to \infty$ and $\xi \to 0$. For $\xi \to \infty$ equation (84) becomes

$$\left[\frac{d^2}{d\xi^2} - \frac{1}{4}\right]\psi(\xi) = 0. \tag{85}$$

Thus, one immediately obtains

$$\psi(\xi) \propto \exp\left(-\frac{\xi}{2}\right). \tag{86}$$

In similar way, for $\xi \to 0$ equation (84) reduces to

$$\left[\frac{d^2}{d\xi^2} + \frac{c}{\xi}\right]\psi(\xi) = 0, \tag{87}$$

and one obtains

$$\psi(\xi) \propto \xi, \tag{88}$$

which can be normalized.

Hence, for the solution of equation (84) one chooses

$$\psi(\xi) = N\xi F(\xi)\exp\left(-\frac{\xi}{2}\right), \tag{89}$$

where one has still to determine $F(\xi)$. Inserting equation (89) in equation (84) one obtains an equation analogous to equation (13)

$$\xi\frac{d^2F}{d\xi^2} + (2-\xi)\frac{dF}{d\xi} - (1-c)F = 0, \tag{90}$$

which solution is again the confluent hypergeometric series

$$F(\xi) = Z(1-c, 2, \xi). \tag{91}$$

The confluent hypergeometric series can be normalized only if its first argument equals negative integer or zero [63]. This gives the quantization condition

$$1 - c = -n_r \tag{92}$$

with

$$n_r = 0, 1, 2, 3, \ldots +\infty. \tag{93}$$

By recalling the definition of principal quantum number [63]

$$n = 1 + n_r, \tag{94}$$

one finds from equation (92)

$$E_n = \sqrt{1 - \frac{M_E^4}{4n^2}}M_E, \tag{95}$$

which, using equation (21) for the 'gravitational fine structure constant' gives

$$E_n = \frac{\sqrt{3}}{2}(M_E)_n = \frac{\sqrt{3}}{2}\sqrt{n}, \tag{96}$$

where for the last passage equation (20) has been used. Equation (96) represents the total relativistic energy of the 'electron' in the particle-like mathematical representation which considers the BH as being an 'hydrogen atom' composed by the 'electron' interacting with a central field, the 'nucleus'. It is again a Bekenstein-like energy spectrum $\alpha\sqrt{n}$ with $\alpha = \frac{\sqrt{3}}{2}$. By using equations (26) and (24) equation (96) can be rewritten as

$$E_n = \sqrt{n} - \frac{1}{2}\sqrt{n} + \frac{\sqrt{3}-1}{2}\sqrt{n}, \tag{97}$$

where $\sqrt{n}$ is the BH effective rest mass, $-\frac{1}{2}\sqrt{n}$ is the eigenvalue of the Schrödinger equation (2) and $\frac{\sqrt{3}-1}{2}\sqrt{n}$ are the relativistic corrections.

One notes that the energy of equation (96) is not the total BH relativistic energy, but only the relativistic energy of the 'electron' in the cited fictitious particle-like mathematical representation. In order to find the total BH relativistic energy one has to add the rest mass of the center of mass in the decomposition of equation (43). In that case, by using equation (24) one has

$$(m_T)_n = M_n + M_n = 2M_n = 4\sqrt{n}. \tag{98}$$

Thus, the total BH relativistic energy, that is, the total relativistic energy of the self-interacting massive quantum shell that has been decomposed via equation (43), is

$$(E_{tot})_n = E_n + (m_T)_n = \left(4 + \frac{\sqrt{3}}{2}\right)\sqrt{n} = \left(2 + \frac{\sqrt{3}}{4}\right)M_n. \tag{99}$$

Thus, the relativistic energy of BH ground state, that is, $n = 1$, is $4 + \frac{\sqrt{3}}{2}$ in Planck units. By recalling that, in standard units the Planck energy is

$$E_P \simeq 1.96*10^9\ J, \tag{100}$$

one gets the relativistic energy of BH ground state in standard units as

$$(E_{tot})_{n=1} \simeq 9.54*10^9\ J. \tag{101}$$

Putting $\gamma \equiv 2 + \frac{\sqrt{3}}{4}$ and recalling that in special relativity the gamma factor is

$$\gamma = (1 - v^2)^{-\frac{1}{2}}, \tag{102}$$

one can rewrite equation (99) in the intriguing Einsteinian form

$$(E_{tot})_n = \gamma M_n,$$

that means that the relativistic energy of the BH self-interacting massive quantum shell is exactly the relativistic energy of a free particle having the BH rest mass and travelling with a velocity is given by the solution of the equation

$$(1 - v^2)^{-\frac{1}{2}} = 2 + \frac{\sqrt{3}}{4}, \tag{103}$$

which is

$$v \simeq 0.911. \tag{104}$$

6. Conclusion remarks

In this work, it has been shown that the final result of the quantized historical Oppenheimer and Snyder gravitational collapse is a self-interacting massive quantum shell, generated by matter condensing on the apparent horizon, which obeys a Schrödinger equation in the non-relativistic case and a Klein–Gordon equation when relativistic corrections are taken into due account. In fact, the author and collaborators recently found the Schwarzschild BH Schrödinger equation [37–39]. In that approach, the traditional classical singularity in the BH core has been replaced by a nonsingular two-particle system where the two components, the 'nucleus' and the 'electron', strongly interact with each other through a quantum gravitational interaction. Thus, in this picture a BH is nothing else than the gravitational analog of the hydrogen atom. In this paper, by following with caution the analogy between this BH Schrödinger equation and the traditional Schrödinger equation of the s states ($l = 0$) of the hydrogen atom, the BH Schrödinger equation has been solved and discussed. The approach also permitted us to find the quantum gravitational quantities which are the gravitational analogous of the fine structure constant and of the Rydberg constant. Remarkably, it has been shown that such quantities are not constants. Instead, they are dynamical quantities having well-defined discrete spectra. In particular, the spectrum of the 'gravitational fine structure constant' is exactly the set of non-zero natural numbers $\mathbb{N} - \{0\}$. Therefore, the first, interesting consequence of the results in this paper is that the BH results in a well-defined quantum gravitational system, which obeys Schrödinger's theory: the 'gravitational hydrogen atom'. This should lead to space-time quantization based on a quantum mechanical particle approach.

On the other hand, the potential energy in the BH Schrödinger equation has been identified as being the gravitational energy of a spherically symmetric shell. This permitted us to show that the Schwarzschild BH results in a self-interacting, highly excited, spherically symmetric, massive quantum shell (a membrane), generated by matter condensing on the apparent horizon, which concretely realizes the membrane paradigm. As a consequence, the quantum BH described in the above terms of a 'gravitational hydrogen atom' is only a fictitious mathematical representation of the real, concrete physical quantum BH, which results in being a self-interacting, highly excited quantum massive shell having a radius equal to the oscillating gravitational radius. Thus, a series of nontrivial consequences emerge from this interesting result. In particular: i) BHs have neither horizons nor singularities; ii) there is neither information loss in BH evaporation, nor BH complementarity, nor firewall paradox. These results are consistent with previous ones by Hawking [24], Vaz [25], Mitra [26], Schild, Leiter and Robertson [27] and other authors [28].

Finally, the special relativistic corrections to the BH Schrödinger equation and to the energy spectrum are obtained by finding the BH Klein–Gordon equation and the corresponding eigenvalues.

For the sake of completeness, we stress the phenomenological implications of the approach in this paper [48]. The mass quantisation as conjectured by Bekenstein [3] (and later by Mukhanov [5]) has been recovered. In addition, the approach discussed here seems in agreement with the corpuscular picture of Dvali and Gomez [64] and similar to the novel approach to quantum gravity (the Nexus Paradigm) of Marongwe [65], which models the quantum BH in terms of another kind of hydrogen-like solutions. An important point seems to be the fact that astronomical observations of EHT [66–71] indicate that astrophysical BHs should have dark surfaces. This seems indeed consistent with the results in this paper. In particular, the result of equation (54), that the size of the quantum BH is $\frac{3}{4}$ of the size of its classical counterpart, might shed some light on the physical origin of the dark spot in the image of supermassive BH SgrA* [72]. This dark spot is indeed noticeably smaller than the classical BH shadow. Further studies in this direction, together with increasingly precise astronomical observations of astrophysical BHs, could confirm or rule out this intriguing hypothesis.

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