Communications in Theoretical Physics

Commun. Theor. Phys. **76** (2024) 025002 (15pp)

https://doi.org/10.1088/1572-9494/ad1588

Finite dimensional irreducible representations of Lie superalgebra $D(2, 1; \alpha)$

Xi Chen (陈曦)[1,*], Wen-Li Yang (杨文力)[2,*], Xiang-Mao Ding (丁祥茂)[3] and Yao-Zhong Zhang (张耀中)[4]

[1] College of Intelligent Systems Science and Engineering, Hubei Minzu University 445000, China
[2] Institute of Modern Physics, Northwest University, Xian 710069, China
[3] Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100080, China
[4] School of Mathematics and Physics, The University of Queensland, Brisbane, QLD 4072, Australia

E-mail: chenxi@hbmzu.edu.cn and wlyang@nwu.edu.cn

Received 10 October 2023, revised 7 December 2023
Accepted for publication 14 December 2023
Published 1 February 2024

Abstract

This paper focuses on the finite dimensional irreducible representations of Lie superalgebra $D(2, 1; \alpha)$. We explicitly construct the finite dimensional representations of the superalgebra $D(2, 1; \alpha)$ by using the shift operator and differential operator representations. Unlike ordinary Lie algebra representation, there are typical and atypical representations for most superalgebras. Therefore, its typical and atypical representation conditions are also given. Our results are expected to be useful for the construction of primary fields of the corresponding current superalgebra of $D(2, 1; \alpha)$.

Keywords: superalgebra, representations, shift operator, conformal field theory

1. Introduction

Affine Lie algebras and their corresponding conformal field theories (CFTs) have essential applications in many subfields of physics [1]. Supersymmetry is the superalgebra associated with the symmetry generator. The concepts of supersymmetry relate to bosonic and fermionic degrees of freedom [2]. Supersymmetry theory is a uniform framework for the systems of bosons and fermions. The conformal field theories are based on current algebras. Current superalgebras and their corresponding two-dimensional conformal field theory have played a fundamental role in the high-energy physics and statistical physics at critical point, such as logarithmic CFTs [3], topological field theory [4], disordered systems and integer quantum Hall effects [5–11]. In most applications of conformal field theories, one needs to construct the finite-dimensional representations of a superalgebra explicitly.

Unlike ordinary bosonic algebra representation, there are typical and atypical representations for most superalgebras. The typical representation is similar to the representation that appeared in bosonic algebra. The atypical representation can be irreducible or not fully reducible. There is no atypical representation's counterpart in ordinary bosonic algebra representation [12, 13]. This makes the study of the representations of superalegbras extremely difficult. The superalgebras $psl(n|n)$ and $osp(2n+2|2n)$ stand out as a most interesting class due to the fact that the corresponding sigma models with their supergroups have a vanishing superdimension or vanishing dual Coexter number. The nonlinear sigma models based on the supergroups have a vanishing one-loop β function, which are expected to be conformal invariant without adding the Wess–Zumino terms [14]. Finite-dimensional typical and atypical representations of $osp(2|2)$ and $gl(2|2)$ have been studied in several papers [15, 16].

The sigma model associated with $psl(4|4)$ (or $su(2, 2|4)$) is related to the string theory on the $AdS_5 \times S^5$ background. Recent studies show that the superalgebra $D(2, 1; \alpha)$ is the one-parameter deformation of Lie superalgebra $D(2, 1) = osp(4|2)$ and has a vanishing dual Coexter number. It has played an important role in describing the origin of the Yangian symmetry of AdS/CFT [17, 18] and the symmetry of string theory on $AdS_3 \times S^3 \times S^3 \times S^1$. There are two types of AdS_3 geometries

* Authors to whom any correspondence should be addressed.

which preserve superconformal symmetry; the finite-dimensional subalgebras of these superconformal algebras are $psu(1,1|2)$ and $D(2,1;\alpha)$ [19]. The parameter α is related to the relative size of the radius of geometry [20]. Thus, the study of the $D(2,1;\alpha)$ model would provide essential insight into the quantization of the string theory on the $AdS_3\times S^3\times S^3\times S^1$ background.

This paper is organized as follows. In section 2, we review the definition of finite-dimensional exceptional superalgebra $D(2,1;\alpha)$ and its commutation relations. In section 3, we explicitly give the differential operator representations of all the generators. In section 4, we give the shift operators. In section 5, we construct the finite-dimensional representation of superalgebra $D(2,1;\alpha)$. In section 6, we give four atypical conditions. If none of the four atypical conditions are satisfied, then the representation is a typical representation. Section 7 is devoted to our conclusions.

2. $D(2,1;\alpha)$ superalgebra

The exceptional Lie superalgebra $D(2,1;\alpha)$ with α forms a continuous one-parameter family of superalgebras of rank 3 and dimension 17 [2]. It is a deformation of the Lie superalgebra $osp(4|2)$ with the parameter $\alpha\neq 0,-1,\infty$. The bosonic (or even) part is a $su(2)\oplus su(2)\oplus su(2)$ of dimension 9, and the fermionic (or odd) part is a spinor representation (2, 2, 2) of the bosonic part of dimension 8. In terms of the orthogonal basis vector $\epsilon_1,\epsilon_2,\epsilon_3$ with the inner product

$$(\epsilon_1,\epsilon_1)=-\frac{1+\alpha}{2},\quad (\epsilon_2,\epsilon_2)=\frac{1}{2},\quad (\epsilon_3,\epsilon_3)=\frac{\alpha}{2},\quad (\epsilon_i,\epsilon_j)=0\ \ for\ i\neq j. \tag{2.1}$$

The even roots Δ_0 and the odd roots Δ_1 of $D(2,1;\alpha)$ are given by

$$\Delta_0=\{\pm 2\epsilon_i\},\quad \Delta_1=\{\pm\epsilon_1\pm\epsilon_2\pm\epsilon_3\}. \tag{2.2}$$

Let $\Pi=\{\alpha_1=\epsilon_1-\epsilon_2-\epsilon_3,\ \alpha_2=2\epsilon_2,\ \alpha_3=\epsilon_3\}$ be the simple root system, with α_1 being fermionic and α_2,α_3 being bosonic. The positive roots system $\Delta^+=\Delta_0^+\cup\Delta_1^+$ is a union of the positive even and odd roots. The positive even roots set Δ_0^+ and positive odd roots set Δ_1^+ are given by $\Delta_0^+=\{\alpha_2,\alpha_3,2\alpha_1+\alpha_2+\alpha_3\}$, $\Delta_1^+=\{\alpha_1,\alpha_1+\alpha_2,\alpha_1+\alpha_3,\alpha_1+\alpha_2+\alpha_3\}$. The Cartan matrix a_{ij} is given by

$$a=\begin{pmatrix}0&1&\alpha\\-1&2&0\\-1&0&2\end{pmatrix}, \tag{2.3}$$

and with each positive root δ, there are generators E_δ (raising operator), $F_\delta\equiv E_{-\delta}$ (lowering operator) and H_δ (Cartan generator). These operators have definite Z_2-gradings:

$$[H_\delta]=0,\qquad [E_\delta]=[F_\delta]=\begin{cases}0, & \delta\in\Delta_0^+,\\ 1, & \delta\in\Delta_1^+.\end{cases} \tag{2.4}$$

For any two generators $a,b\in D(2,1;\alpha)$, the (anti)commutator is defined by

$$[a,b]=ab-(-)^{[a][b]}ba, \tag{2.5}$$

the commutation relations of $D(2,1;\alpha)$ are

$$[E_{\alpha_i},F_{\alpha_j}]=\delta_{ij}H_{\alpha_i},\quad [H_{\alpha_i},H_{\alpha_j}]=0,\quad i,j=1,2,3, \tag{2.6}$$

$$\begin{aligned}
&[H_{\alpha_i},E_{\alpha_j}]=a_{ij}E_{\alpha_j},\quad [H_{\alpha_i},F_{\alpha_j}]=-a_{ij}F_{\alpha_j},\quad i,j=1,2,3,\\
&[E_{\alpha_1},E_{\alpha_2}]=-E_{\alpha_1+\alpha_2},\quad [E_{\alpha_1},E_{\alpha_3}]=-E_{\alpha_1+\alpha_3},\\
&[E_{\alpha_1},E_{\alpha_1+\alpha_2+\alpha_3}]=-(1+\alpha)E_{2\alpha_1+\alpha_2+\alpha_3},\\
&[E_{\alpha_1},H_{\alpha_2}]=[E_{\alpha_1},H_{\alpha_3}]=E_{\alpha_1},\\
&[E_{\alpha_1},F_{\alpha_1+\alpha_2}]=F_{\alpha_2},\quad [E_{\alpha_1},F_{\alpha_1+\alpha_3}]=\alpha F_{\alpha_3},\\
&[E_{\alpha_1},F_{2\alpha_1+\alpha_2+\alpha_3}]=-F_{\alpha_1+\alpha_2+\alpha_3},\quad [E_{\alpha_2},E_{\alpha_1+\alpha_3}]\\
&=E_{\alpha_1+\alpha_2+\alpha_3},\\
&[E_{\alpha_2},H_{\alpha_1}]=-E_{\alpha_2},\quad [E_{\alpha_2},H_{\alpha_2}]=-2E_{\alpha_2},\\
&[E_{\alpha_2},F_{\alpha_1+\alpha_2}]=F_{\alpha_1},\quad [E_{\alpha_2},F_{\alpha_1+\alpha_2+\alpha_3}]=F_{\alpha_1+\alpha_3},\\
&[E_{\alpha_3},E_{\alpha_1+\alpha_2}]=E_{\alpha_1+\alpha_2+\alpha_3},\quad [E_{\alpha_3},H_{\alpha_1}]=-\alpha E_{\alpha_3},\\
&[E_{\alpha_3},H_{\alpha_3}]=-2E_{\alpha_3},\quad [E_{\alpha_3},F_{\alpha_1+\alpha_3}]=F_{\alpha_1},\\
&[E_{\alpha_1},F_{\alpha_1+\alpha_2+\alpha_3}]=F_{\alpha_1+\alpha_2},\\
&[E_{\alpha_1+\alpha_2},H_{\alpha_1}]=[E_{\alpha_1+\alpha_2},H_{\alpha_2}]\\
&=-E_{\alpha_1+\alpha_2},\ [E_{\alpha_1+\alpha_2},H_{\alpha_3}]=E_{\alpha_1+\alpha_2},\\
&[E_{\alpha_1+\alpha_2},E_{\alpha_1+\alpha_3}]=(1+\alpha)E_{2\alpha_1+\alpha_2+\alpha_3},\\
&[E_{\alpha_1+\alpha_2},F_{\alpha_1}]=-E_{\alpha_2},\quad [E_{\alpha_1+\alpha_2},F_{\alpha_1+\alpha_2}]=-H_{\alpha_1}+H_{\alpha_2},\\
&[E_{\alpha_1+\alpha_2},F_{\alpha_1+\alpha_2+\alpha_3}]=-\alpha F_{\alpha_3},\quad [E_{\alpha_1+\alpha_2},F_{2\alpha_1+\alpha_2+\alpha_3}]\\
&=-F_{\alpha_1+\alpha_3},\ [E_{\alpha_1+\alpha_3},H_{\alpha_1}]=-\alpha E_{\alpha_1+\alpha_3},\quad [E_{\alpha_1+\alpha_3},H_{\alpha_2}]\\
&=E_{\alpha_1+\alpha_3},\\
&[E_{\alpha_1+\alpha_3},H_{\alpha_3}]=-E_{\alpha_1+\alpha_3},\quad [E_{\alpha_1+\alpha_3},F_{\alpha_1}]\\
&=-\alpha E_{\alpha_3},\\
&[E_{\alpha_1+\alpha_3},F_{\alpha_3}]=-E_{\alpha_1},\quad [E_{\alpha_1+\alpha_3},F_{\alpha_1+\alpha_3}]\\
&=-H_{\alpha_1}+\alpha H_{\alpha_3},\\
&[E_{\alpha_1+\alpha_3},F_{\alpha_1+\alpha_2+\alpha_3}]=-F_{\alpha_2},\quad [E_{\alpha_1+\alpha_3},F_{2\alpha_1+\alpha_2+\alpha_3}]\\
&=-F_{\alpha_1+\alpha_2},\\
&[E_{\alpha_1+\alpha_2+\alpha_3},H_{\alpha_1}]=-(1+\alpha)E_{\alpha_1+\alpha_2+\alpha_3},\quad [E_{\alpha_1+\alpha_2+\alpha_3},H_{\alpha_2}]\\
&=E_{\alpha_1+\alpha_2+\alpha_3},\\
&[E_{\alpha_1+\alpha_2+\alpha_3},H_{\alpha_3}]=-E_{\alpha_1+\alpha_2+\alpha_3},\quad [E_{\alpha_1+\alpha_2+\alpha_3},F_{\alpha_2}]\\
&=-E_{\alpha_1+\alpha_3},\\
&[E_{\alpha_1+\alpha_2+\alpha_3},F_{\alpha_3}]=-E_{\alpha_1+\alpha_2},\quad [E_{\alpha_1+\alpha_2+\alpha_3},F_{\alpha_1+\alpha_2}]=\alpha E_{\alpha_3},\\
&[E_{\alpha_1+\alpha_2+\alpha_3},F_{\alpha_1+\alpha_2}]=E_{\alpha_2},\quad [E_{\alpha_1+\alpha_2+\alpha_3},F_{\alpha_1+\alpha_2+\alpha_3}]\\
&=H_{\alpha_1}-H_{\alpha_2}-\alpha H_{\alpha_3},\\
&[E_{\alpha_1+\alpha_2+\alpha_3},F_{2\alpha_1+\alpha_2+\alpha_3}]=-F_{\alpha_1},\quad [E_{2\alpha_1+\alpha_2+\alpha_3},H_{\alpha_1}]\\
&=-(1+\alpha)E_{2\alpha_1+\alpha_2+\alpha_3},\\
&[E_{2\alpha_1+\alpha_2+\alpha_3},F_{\alpha_1}]=E_{\alpha_1+\alpha_2+\alpha_3},\quad [E_{2\alpha_1+\alpha_2+\alpha_3},F_{\alpha_1+\alpha_2}]\\
&=E_{\alpha_1+\alpha_3},\\
&[E_{2\alpha_1+\alpha_2+\alpha_3},F_{\alpha_1+\alpha_3}]=E_{\alpha_1+\alpha_2},\quad [E_{2\alpha_1+\alpha_2+\alpha_3},F_{\alpha_1+\alpha_2+\alpha_3}]\\
&=E_{\alpha_1},\ [E_{2\alpha_1+\alpha_2+\alpha_3},F_{2\alpha_1+\alpha_2+\alpha_3}]=\frac{2}{1+\alpha}H_{\alpha_1}-\frac{1}{1+\alpha}H_{\alpha_2}\\
&+\frac{\alpha}{1+\alpha}H_{\alpha_3},\ [H_{\alpha_1},F_{\alpha_2}]=-F_{\alpha_2},\quad [H_{\alpha_1},F_{\alpha_3}]=-\alpha F_{\alpha_3},\\
&[H_{\alpha_1},F_{\alpha_1+\alpha_2}]=-F_{\alpha_1+\alpha_2},\quad [H_{\alpha_1},F_{\alpha_1+\alpha_3}]=-\alpha F_{\alpha_1+\alpha_2},\\
&[H_{\alpha_1},F_{\alpha_1+\alpha_2+\alpha_3}]=-(1+\alpha)F_{\alpha_1+\alpha_2+\alpha_3},\quad [H_{\alpha_1},F_{2\alpha_1+\alpha_2+\alpha_3}]\\
&=-(1+\alpha)F_{2\alpha_1+\alpha_2+\alpha_3},\\
&[H_{\alpha_2},F_{\alpha_1}]=F_{\alpha_1},\quad [H_{\alpha_2},F_{\alpha_2}]=-2F_{\alpha_2},\\
&[H_{\alpha_2},F_{\alpha_1+\alpha_2}]=-F_{\alpha_1+\alpha_2},\quad [H_{\alpha_2},F_{\alpha_1+\alpha_3}]=F_{\alpha_1+\alpha_3},\\
&[H_{\alpha_2},F_{\alpha_1+\alpha_2+\alpha_3}]=-F_{\alpha_1+\alpha_2+\alpha_3},\quad [H_{\alpha_3},F_{\alpha_1}]=F_{\alpha_1},\\
&[H_{\alpha_3},F_{\alpha_2}]=-2F_{\alpha_3},\quad [H_{\alpha_3},F_{\alpha_1+\alpha_2}]=F_{\alpha_1+\alpha_2},\\
&[H_{\alpha_3},F_{\alpha_1+\alpha_3}]=-F_{\alpha_1+\alpha_3},\quad [H_{\alpha_3},F_{\alpha_1+\alpha_2+\alpha_3}]=-F_{\alpha_1+\alpha_2+\alpha_3},\\
&[F_{\alpha_1},F_{\alpha_2}]=-F_{\alpha_1+\alpha_2},\quad [F_{\alpha_1},F_{\alpha_3}]=-F_{\alpha_1+\alpha_3},\\
&[F_{\alpha_1},F_{\alpha_1+\alpha_2+\alpha_3}]=(1+\alpha)F_{2\alpha_1+\alpha_2+\alpha_3},\\
&[F_{\alpha_2},F_{\alpha_1+\alpha_3}]=F_{\alpha_1+\alpha_2+\alpha_3},\quad [F_{\alpha_3},F_{\alpha_1+\alpha_2}]
\end{aligned}$$

$$=F_{\alpha_1+\alpha_2+\alpha_3},\quad [F_{\alpha_1+\alpha_2},E_{\alpha_1+\alpha_3}]=-(1+\alpha)F_{2\alpha_1+\alpha_2+\alpha_3}, \tag{2.7}$$

and all the other commutators are zero.

3. Differential operator representation of $D(2, 1; \alpha)$

To obtain a shift operator [22] of $D(2, 1; \alpha)$, one needs to construct the differential operator representations [23–31] of the Lie superalgebra $D(2, 1; \alpha)$. Let $\langle\Lambda|$ be the highest weight vector in the representation of $D(2, 1; \alpha)$ with the highest weights λ_i, satisfying the following conditions:

$$\langle\Lambda|F_{\alpha_i} = 0, \tag{3.1}$$

$$\langle\Lambda|H_{\alpha_i} = \lambda_i\langle\Lambda|. \tag{3.2}$$

An arbitrary vector in the representation space is parametrized by the bosonic coordinate variables x_{α_i} and fermionic coordinate variables θ_{α_i},

$$\langle\Lambda, x, \theta| = \langle\Lambda|G_+(x, \theta). \tag{3.3}$$

We constructed the corresponding $G_+(x, \theta)$ as follows:

$$G_+(x, \theta) = G_{\alpha_3}G_{\alpha_2}G_{\alpha_1+\alpha_2+\alpha_3}G_{\alpha_1+\alpha_3}G_{2\alpha_1+\alpha_2+\alpha_3}G_{\alpha_1+\alpha_2}G_{\alpha_1}, \tag{3.4}$$

and the associated G_δ are given by (e is Euler's number)

$$G_\delta = \begin{cases} e^{x_{\alpha_i}E_{\alpha_i}}, & \text{if} \quad [E_{\alpha_i}] = 0, \\ e^{\theta_{\alpha_i}E_{\alpha_i}}, & \text{if} \quad [E_{\alpha_i}] = 1. \end{cases} \tag{3.5}$$

One can define a differential operator realization $\rho^{(d)}$ of the generators of Lie superalgebra $D(2, 1; \alpha)$ by the following relation

$$\rho^{(d)}(g)\langle\Lambda, x, \theta| \equiv \langle\Lambda, x, \theta|g, \quad \forall g \in D(2, 1; \alpha). \tag{3.6}$$

Here, $\rho^{(d)}(g)$ is a differential operator of the bosonic and fermionic coordinate variables $\{x_{\alpha_i}, \theta_{\alpha_i}\}$ associated with the generator g. After some manipulations, we obtain the following differential operator representations of all generators of Lie superalgebra $D(2, 1; \alpha)$:

$$\rho^{(d)}(E_{\alpha_1}) = \partial_{\theta_{\alpha_1}}, \tag{3.7}$$

$$\rho^{(d)}(E_{\alpha_2}) = -\theta_{\alpha_1}\partial_{\theta_{\alpha_1+\alpha_2}} - \theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} + \partial_{x_{\alpha_2}}, \tag{3.8}$$

$$\rho^{(d)}(E_{\alpha_3}) = -\theta_{\alpha_1}\partial_{\theta_{\alpha_1+\alpha_3}} - (1+\alpha)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}} - \theta_{\alpha_1+\alpha_2}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} + \partial_{x_{\alpha_3}}, \tag{3.9}$$

$$\rho^{(d)}(E_{\alpha_1+\alpha_2}) = \partial_{\theta_{\alpha_1+\alpha_2}}, \tag{3.10}$$

$$\rho^{(d)}(E_{\alpha_1+\alpha_3}) = \partial_{\theta_{\alpha_1+\alpha_3}} + (1+\alpha)\theta_{\alpha_1+\alpha_2}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}}, \tag{3.11}$$

$$\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3}) = \partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} - (1+\alpha)\theta_{\alpha_1}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}}, \tag{3.12}$$

$$\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3}) = \partial_{x_{2\alpha_1+\alpha_2+\alpha_3}}, \tag{3.13}$$

$$\begin{aligned}\rho^{(d)}(H_{\alpha_1}) = &-\theta_{\alpha_1+\alpha_2}\partial_{\theta_{\alpha_1+\alpha_2}} \\ &-(1+\alpha)x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}} - \alpha\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}} \\ &-(1+\alpha)\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} \\ &-x_{\alpha_2}\partial_{x_{\alpha_2}} - \alpha x_{\alpha_3}\partial_{x_{\alpha_3}} + \lambda_{\alpha_1},\end{aligned} \tag{3.14}$$

$$\begin{aligned}\rho^{(d)}(H_{\alpha_2}) = &\theta_{\alpha_1}\partial_{\theta_{\alpha_1}} - \theta_{\alpha_1+\alpha_2}\partial_{\theta_{\alpha_1+\alpha_2}} + \theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}} \\ &- \theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} - 2x_{\alpha_2}\partial x_{\alpha_2} + \lambda_{\alpha_2},\end{aligned} \tag{3.15}$$

$$\begin{aligned}\rho^{(d)}(H_{\alpha_3}) = &\theta_{\alpha_1}\partial_{\theta_{\alpha_1}} + \theta_{\alpha_1+\alpha_2}\partial_{\theta_{\alpha_1+\alpha_2}} \\ &-\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}} \\ &-\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} \\ &-2x_{\alpha_3}\partial_{\alpha_3} + \lambda_{\alpha_3},\end{aligned} \tag{3.16}$$

$$\begin{aligned}\rho^{(d)}(F_{\alpha_1}) = &\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} - \theta_{\alpha_1+\alpha_2}\partial_{x_{\alpha_2}} \\ &+x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} \\ &-\alpha\theta_{\alpha_1+\alpha_3}\partial_{x_{\alpha_3}} - \theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\partial_{\theta_{\alpha_1+\alpha_2}} \\ &-(1+\alpha)\theta_{\alpha_1}x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}} \\ &-\alpha\theta_{\alpha_1}\theta_{\alpha_1+a_3}\partial_{\theta_{\alpha_1+\alpha_3}} - (1+\alpha)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} \\ &-\theta_{\alpha_1}x_{\alpha_2}\partial_{x_{\alpha_2}} \\ &-\alpha\theta_{\alpha_1}x_{\alpha_3}\partial_{x_{\alpha_3}} \\ &+\theta_{\alpha_1}\lambda_{\alpha_1},\end{aligned} \tag{3.17}$$

$$\begin{aligned}\rho^{(d)}(F_{\alpha_2}) = &-\theta_{\alpha_1+\alpha_2}\partial_{\theta_{\alpha_1}} - \theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}} \\ &-x_{\alpha_2}^2\partial_{x_{\alpha_2}} + x_{\alpha_2}\lambda_{\alpha_2},\end{aligned} \tag{3.18}$$

$$\begin{aligned}\rho^{(d)}(F_{\alpha_3}) = &-\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1}} + (1+\alpha)\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3} \\ &\times\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}} - \theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2}} - x_{\alpha_3}^2\partial_{x_{\alpha_3}} + x_{\alpha_3}\lambda_{\alpha_3},\end{aligned} \tag{3.19}$$

$$\begin{aligned}\rho^{(d)}(F_{\alpha_1+\alpha_2}) = &(1+\alpha)\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}} \\ &+\alpha\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} \\ &-\theta_{\alpha_1+\alpha_2}x_{\alpha_2}\partial_{x_{\alpha_2}} + x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}} \\ &+\alpha\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{x_{\alpha_3}} - \theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\partial_{\theta_{\alpha_1}} \\ &+(1+\alpha)\theta_{\alpha_1+\alpha_2}x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}} \\ &-\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}} \\ &+\alpha\theta_{\alpha_1+\alpha_2}x_{\alpha_3}\partial_{x_{\alpha_3}} \\ &+\theta_{\alpha_1+\alpha_2}\lambda_{\alpha_2} - \theta_{\alpha 1+\alpha_2}\lambda_{\alpha_1} \\ &+\theta_{\alpha_1}x_{\alpha_2}\lambda_{\alpha_2} \\ &-\theta_{\alpha_1}x_{\alpha_2}^2\partial_{x_{\alpha_2}},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\rho^{(d)}(F_{\alpha_1+\alpha_3}) = &-\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}} \\ &+\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{x_{\alpha_2}} \\ &-\alpha\theta_{\alpha_1+\alpha_3}x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}} \\ &+x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2}} \\ &-\alpha\theta_{\alpha_1+\alpha_3}x_{\alpha_3}\partial_{x_{\alpha_3}} \\ &-\alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2}} \\ &+\alpha(1+\alpha)\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}} \\ &-\alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1}} \\ &+\theta_{\alpha_1+\alpha_3}x_{\alpha_2}\partial_{x_{\alpha_2}} - \alpha\theta_{\alpha_1}x_{\alpha_3}^2\partial_{x_{\alpha_3}} \\ &+\alpha\theta_{\alpha_1}x_{\alpha_3}\lambda_{\alpha_3} - \theta_{\alpha_1+\alpha_3}\lambda_{\alpha_1} \\ &+\alpha\theta_{\alpha_1+\alpha_3}\lambda_{\alpha_3},\end{aligned} \tag{3.21}$$

$$\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3}) = \alpha\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1}} + x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1}}$$
$$+\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}}$$
$$+\theta_{\alpha_1+\alpha_3}x_{\alpha_2}^2\partial_{x_{\alpha_2}}$$
$$-\alpha(1+\alpha)\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}}$$
$$+\alpha\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2}}$$
$$+\theta_{\alpha_1+\alpha_2+\alpha_3}x_{\alpha_2}\partial_{x_{\alpha_2}}$$
$$+\alpha\theta_{\alpha_1+\alpha_2+\alpha_3}x_{\alpha_3}\partial_{x_{\alpha_3}} + \alpha\theta_{\alpha_1+\alpha_2}x_{\alpha_3}^2\partial_{x_{\alpha_3}}$$
$$-\alpha\theta_{\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_3} - \alpha\theta_{\alpha_1+\alpha_2}x_{\alpha_3}\lambda_{\alpha_3}$$
$$-\theta_{\alpha_1+\alpha_3}x_{\alpha_2}\lambda_{\alpha_2} - \theta_{\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_2} + \theta_{\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_1}, \tag{3.22}$$

$$\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3}) = \theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}}$$
$$-\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}x_{\alpha_2}\partial_{x_{\alpha_2}}$$
$$+\alpha\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}x_{\alpha_3}\partial_{x_{\alpha_3}}$$
$$-\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{x_{\alpha_2}}$$
$$-x_{2\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}}$$
$$-x_{2\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_2}\partial_{\alpha_1+\alpha_2}$$
$$-\alpha\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{x_{\alpha_3}}$$
$$-\alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2}\partial_{\alpha_1}$$
$$+\alpha(1+\alpha)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3}\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}}$$
$$+\alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_2}\partial_{\alpha_1+\alpha_2}$$
$$-\alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}x_{\alpha_3}^2\partial_{x_{\alpha_3}}$$
$$-\theta_{\alpha_1}x_{2\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1}}$$
$$-x_{2\alpha_1+\alpha_2+\alpha_3}^2\partial_{x_{2\alpha_1+\alpha_2+\alpha_3}}$$
$$+\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_3}}$$
$$-\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}x_{\alpha_2}^2\partial_{x_{\alpha_2}}$$
$$-x_{2\alpha_1+\alpha_2+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\partial_{\theta_{\alpha_1+\alpha_2+\alpha_3}}$$
$$-\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}x_{\alpha_2}\partial_{x_{\alpha_2}}$$
$$-\alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}x_{\alpha_3}\partial_{x_{\alpha_3}} + \theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\lambda_{\alpha_1}$$
$$+\frac{2}{1+\alpha}x_{2\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_1} - \theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_1}$$
$$+\alpha\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2}\lambda_{\alpha_3} - \frac{\alpha}{1+\alpha}x_{2\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_3}$$
$$+\alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}x_{\alpha_3}\lambda_{\alpha_3} + \alpha\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_3}$$
$$-\frac{1}{1+\alpha}x_{2\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_2}$$
$$+\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}x_{\alpha_2}\lambda_{\alpha_2} + \theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\lambda_{\alpha_2}. \tag{3.23}$$

One can directly check that the differential operator realizations satisfy the commutation relations of Lie superalgebra $D(2,1;\alpha)$ [21].

4. Shift operator of *D*(2, 1; *a*)

The even part of Lie superalgebra $D(2,1;\alpha)$ is $su(2)\oplus su(2)\oplus su(2)$, with the basis s_i, t_i, $u_i (i=0,\pm)$, satisfying the relations

$$[s_0, s_\pm] = \pm s_\pm, \quad [t_0, t_\pm] = \pm t_\pm,$$
$$[u_0, u_\pm] = \pm u_\pm, \quad [s_+, s_-] = 2s_0,$$
$$[t_+, t_-] = 2t_0, \quad [u_+, u_-] = 2u_0. \tag{4.1}$$

The odd part of Lie superalgebras $D(2,1;\alpha)$ is a spinor representation (2, 2, 2) of the even part, with components $R_{ijk}(i,j,k=\pm\frac{1}{2})$ [22]. In our assumption, the elements of $D(2,1;\alpha)$ are given by

$$s_0 = \frac{1}{2(1+\alpha)}(2H_{\alpha_1} - H_{\alpha_2} - \alpha H_{\alpha_3}),$$
$$s_+ = -iE_{\alpha_1+\alpha_2+\alpha_3}, \quad s_- = iF_{\alpha_1+\alpha_2+\alpha_3},$$
$$t_0 = \frac{1}{2}H_{\alpha_2}, \quad t_+ = E_{\alpha_2},$$
$$t_- = F_{\alpha_2}, \quad u_0 = \frac{1}{2}H_{\alpha_3},$$
$$u_+ = E_{\alpha_3}, \quad u_- = F_{\alpha_3},$$
$$R_{\frac{1}{2},\frac{1}{2},\frac{1}{2}} = E_{\alpha_1+\alpha_2+\alpha_3}, \quad R_{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} = iF_{\alpha_1+\alpha_2+\alpha_3},$$
$$R_{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} = E_{\alpha_1+\alpha_2}, \quad R_{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} = iF_{\alpha_1+\alpha_2},$$
$$R_{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} = E_{\alpha_1+\alpha_3}, \quad R_{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} = F_{\alpha_1+\alpha_3},$$
$$R_{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} = E_{\alpha_1}, \quad R_{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} = iF_{\alpha_1}. \tag{4.2}$$

The invariant scalars of the Lie subalgebra of $D(2,1;\alpha)$ are given by

$$S^2 = s_+s_- + s_0^2 - s_0, \quad T^2 = t_+t_- + t_0^2 - t_0,$$
$$U^2 = u_+u_- + u_0^2 - u_0. \tag{4.3}$$

Irreducible representations of Lie superalgebra can be reduced into the direct sum of a set of irreducible representations of subalgebra. The representation of $su(2)\oplus su(2)\oplus su(2)$ can be labeled by (s,t,u), where $s(s+1)$, $t(t+1)$, $u(u+1)$ are the eigenvalues of the subalgebra invariants S^2, T^2, U^2. And the representations of $D(2,1;\alpha)$ are labeled by $|s, m_s; t, m_t; u, m_u; \lambda\rangle$, where m_s, m_t, m_u are eigenvalues of the s_0, t_0, u_0. The degeneracy representations can be labeled by λ. The operator $\hat{s}$ is defined by

$$\hat{s}|s, m_s; t, m_t; u, m_u; \lambda\rangle = s|s, m_s; t, m_t; u, m_u; \lambda\rangle. \tag{4.4}$$

The operators $\hat{t}$ and $\hat{u}$ are defined in the same way. Let (p,q,r) be the corresponding (s,t,u) values, and p be the maximum s value in the reduction of a $D(2,1;\alpha)$ representation. Therefore, the decomposition into $su(2)\oplus su(2)\oplus su(2)$ is given by

$$\mathbb{F} = \{(p,q,r), \left(p-\frac{1}{2}, q\pm\frac{1}{2}, r\pm\frac{1}{2}\right), \left(p-\frac{1}{2}, q\pm\frac{1}{2}, r\mp\frac{1}{2}\right),$$
$$(p-1, q\pm1, r), (p-1, q, r\pm1), (p-1, q, r; 1), (p-1, q, r; 2),$$
$$\left(p-\frac{3}{2}, q\pm\frac{1}{2}, r\pm\frac{1}{2}\right), \left(p-\frac{3}{2}, q\pm\frac{1}{2}, r\mp\frac{1}{2}\right), (p-2, q, r)\}. \tag{4.5}$$

The $(s, t, u) = (p-1, q, r)$ is a twofold degeneracy. Therefore, the multiplicity of the (s, t, u) representation is denoted as $|p-1, m_p; q, m_q; r, m_r; \lambda\rangle (\lambda = 1, 2)$.

The shift operators $O^{i,j,k}\left(i, j, k = \frac{1}{2}\right)$ have been studied by Hughes and Yadegar [32]. The shift operators for $D(2, 1; \alpha)$ are given by Van der Jeugt [22]

$$O^{\frac{1}{2},\frac{1}{2},\frac{1}{2}} = -\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3})(\hat{s}+s_0+1)(\hat{t}+t_0+1)\times(\hat{u}+u_0+1) - \rho^{(d)}(F_{\alpha_1})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0+1)(\hat{u}+u_0+1) -\rho^{(d)}(E_{\alpha_1+\alpha_3})(\hat{s}+s_0+1)\rho^{(d)}(E_{\alpha_2})(\hat{u}+u_0+1) -\rho^{(d)}(E_{\alpha_1+\alpha_2})(\hat{s}+s_0+1)(\hat{t}+t_0+1)\rho^{(d)}(E_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_2})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(E_{\alpha_2})(\hat{u}+u_0+1) -\rho^{(d)}(E_{\alpha_1})(\hat{s}+s_0+1)\rho^{(d)}(E_{\alpha_2})\rho^{(d)}(E_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_3})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})(\hat{t}+t_0+1)\rho^{(d)}(E_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}\times(E_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(E_{\alpha_2})\rho^{(d)}(E_{\alpha_3}), \tag{4.6}$$

$$O^{\frac{1}{2},-\frac{1}{2},\frac{1}{2}} = -\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3})\times(\hat{s}+s_0+1)\rho^{(d)}(F_{\alpha_2})(\hat{u}+u_0+1) -\rho^{(d)}(F_{\alpha_1})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(F_{\alpha_2})(\hat{u}+u_0+1) +\rho^{(d)}(E_{\alpha_1+\alpha_3})(\hat{s}+s_0+1)(\hat{t}+t_0)(\hat{u}+u_0+1) -\rho^{(d)}(E_{\alpha_1+\alpha_2})(\hat{s}+s_0+1)\rho^{(d)}(F_{\alpha_2})\rho^{(d)}(E_{\alpha_3}) +\rho^{(d)}(F_{\alpha_1+\alpha_2})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0)(\hat{u}+u_0+1) +\rho^{(d)}(E_{\alpha_1})(\hat{s}+s_0+1)(\hat{t}+t_0)\rho^{(d)}(E_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_3})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}\times(F_{\alpha_2})\rho^{(d)}(E_{\alpha_3}) +\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0)\rho^{(d)}(E_{\alpha_3}), \tag{4.7}$$

$$O^{\frac{1}{2},\frac{1}{2},-\frac{1}{2}} = -\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3})\times(\hat{s}+s_0+1)(\hat{t}+t_0+1)\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0+1)\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(E_{\alpha_1+\alpha_3})(\hat{s}+s_0+1)\rho^{(d)}\times(E_{\alpha_2})\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1+\alpha_2})(\hat{s}+s_0+1)\times(\hat{t}+t_0+1)(\hat{u}+u_0) -\rho^{(d)}(F_{\alpha_1+\alpha_2})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}\times(E_{\alpha_2})\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1})(\hat{s}+s_0+1)\rho^{(d)}\times(E_{\alpha_2})(\hat{u}+u_0) +\rho^{(d)}(F_{\alpha_1+\alpha_3})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0+1)(\hat{u}+u_0) +\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times\rho^{(d)}(E_{\alpha_2})(\hat{u}+u_0), \tag{4.8}$$

$$O^{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} = -\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3})\times(\hat{s}+s_0+1)\rho^{(d)}(F_{\alpha_2})\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(F_{\alpha_2})\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1+\alpha_3})(\hat{s}+s_0+1)(\hat{t}+t_0)\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1+\alpha_2})(\hat{s}+s_0+1)\rho^{(d)}\times(F_{\alpha_2})(\hat{u}+u_0) +\rho^{(d)}(F_{\alpha_1+\alpha_2})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0)\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(E_{\alpha_1})(\hat{s}+s_0+1)(\hat{t}+t_0)(\hat{u}+u_0) +\rho^{(d)}(F_{\alpha_1+\alpha_3})\rho^{(d)}(E_{2\alpha_1+\alpha_2+\alpha_3})\times\rho^{(d)}(F_{\alpha_2})(\hat{u}+u_0) -\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}\times(E_{2\alpha_1+\alpha_2+\alpha_3})(\hat{t}+t_0)(\hat{u}+u_0), \tag{4.9}$$

$$O^{-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}} = -\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\times\rho^{(d)}(F_{\alpha_2})\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1+\alpha_2})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\times\rho^{(d)}(F_{\alpha_2})(\hat{u}+u_0) +\rho^{(d)}(E_{\alpha_1+\alpha_3})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0)\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0)(\hat{u}+u_0) +\rho^{(d)}(F_{\alpha_1})(\hat{s}+s_0)\rho^{(d)}\times(F_{\alpha_2}\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_3})(\hat{s}+s_0)\times\rho^{(d)}(F_{\alpha_2})(\hat{u}+u_0) -\rho^{(d)}(F_{\alpha_1+\alpha_2})(\hat{s}+s_0)\times(\hat{t}+t_0)\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})(\hat{s}+s_0)\times(\hat{t}+t_0)(\hat{u}+u_0), \tag{4.10}$$

$$O^{-\frac{1}{2},\frac{1}{2},\frac{1}{2}} = -\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3})\times\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})(\hat{t}+t_0+1)(\hat{u}+u_0+1) -\rho^{(d)}(E_{\alpha_1+\alpha_2})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\times(\hat{t}+t_0+1)\rho^{(d)}(E_{\alpha_3}) -\rho^{(d)}(E_{\alpha_1+\alpha_3})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\times\rho^{(d)}(E_{\alpha_2})(\hat{u}+u_0+1) -\rho^{(d)}(E_{\alpha_1})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}\times(E_{\alpha_2})\rho^{(d)}(E_{\alpha_3}) +\rho^{(d)}(F_{\alpha_1})(\hat{s}+s_0)(\hat{t}+t_0+1)(\hat{u}+u_0+1) +\rho^{(d)}(F_{\alpha_1+\alpha_3})(\hat{s}+s_0)\times(\hat{t}+t_0+1)\rho^{(d)}(E_{\alpha_3}) +\rho^{(d)}(F_{\alpha_1+\alpha_2})(\hat{s}+s_0)\rho^{(d)}\times(E_{\alpha_2})(\hat{u}+u_0+1) +\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})(\hat{s}+s_0)\times\rho^{(d)}(E_{\alpha_2})\rho^{(d)}(E_{\alpha_3}), \tag{4.11}$$

$$O^{-\frac{1}{2},-\frac{1}{2},\frac{1}{2}} = -\rho^{(d)}(E_{\alpha_1+\alpha_2+\alpha_3}) \times \rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(F_{\alpha_2})(\hat{u}+u_0+1) -\rho^{(d)}(E_{\alpha_1+\alpha_2})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(F_{\alpha_2})\rho^{(d)}(E_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1+\alpha_3})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})(\hat{t}+t_0)(\hat{u}+u_0+1) +\rho^{(d)}(E_{\alpha_1})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})(\hat{t}+t_0)\rho^{(d)}(E_{\alpha_3}) +\rho^{(d)}(F_{\alpha_1})(\hat{s}+s_0)\rho^{(d)}(F_{\alpha_2})(\hat{u}+u_0+1) +\rho^{(d)}(F_{\alpha_1+\alpha_3})(\hat{s}+s_0)\rho^{(d)}(F_{\alpha_2})\rho^{(d)}(E_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_2})(\hat{s}+s_0)(\hat{t}+t_0)(\hat{u}+u_0+1) -\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3})(\hat{s}+s_0)(\hat{t}+t_0)\rho^{(d)}(E_{\alpha_3}), \tag{4.12}$$

$$O^{-\frac{1}{2},\frac{1}{2},-\frac{1}{2}} = -\rho^{(d)} \times (E_{\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})(\hat{t}+t_0+1)\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1+\alpha_2})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})(\hat{t}+t_0+1)(\hat{u}+u_0) -\rho^{(d)}(E_{\alpha_1+\alpha_3})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(E_{\alpha_2})\rho^{(d)}(F_{\alpha_3}) +\rho^{(d)}(E_{\alpha_1})\rho^{(d)}(F_{2\alpha_1+\alpha_2+\alpha_3})\rho^{(d)}(E_{\alpha_2})(\hat{u}+u_0) +\rho^{(d)}(F_{\alpha_1})(\hat{s}+s_0) \times (\hat{t}+t_0+1)\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_3})(\hat{s}+s_0) \times (\hat{t}+t_0+1)(\hat{u}+u_0) +\rho^{(d)}(F_{\alpha_1+\alpha_2})(\hat{s}+s_0)\rho^{(d)} \times (E_{\alpha_2})\rho^{(d)}(F_{\alpha_3}) -\rho^{(d)}(F_{\alpha_1+\alpha_2+\alpha_3}) \times (\hat{s}+s_0)\rho^{(d)}(E_{\alpha_2})(\hat{u}+u_0). \tag{4.13}$$

The shift operators $O^{i,j,k}$ shift an eigenstate into one or two eigenstates (for the twofold degenerate case),

$$O^{i,j,k}|s, m_s; t, m_t; u, m_u; \lambda\rangle \propto \sum_{\lambda'} |s+i, m_s+i; t+j, m_t+j; u+k, m_u+k; \lambda'\rangle. \tag{4.14}$$

The normalized shift operator $A^{i,j,k}$ is

$$A^{i,j,k} = O^{i,j,k}\left[(\hat{s}+s_0+\frac{1}{2}+i)\left(\hat{t}+t_0+\frac{1}{2}+j\right)\left(\hat{u}+u_0+\frac{1}{2}+k\right)\right]^{-\frac{1}{2}}. \tag{4.15}$$

5. Representations of $D(2, 1; \alpha)$

The exceptional Lie superalgebra $D(2, 1; \alpha)$ $(\alpha \neq 0, -1)$ forms

$$|p-2, m_p; q, m_q; r, m_r\rangle = \theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3} \times\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}, \tag{5.1}$$

$$\left|p-\frac{3}{2}, m_p; q+\frac{1}{2}, m_q; r+\frac{1}{2}, m_r\right\rangle =\left(q+m_q+\frac{1}{2}\right)\left(r+m_r+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}} \times \chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} -(q-m_q+\frac{1}{2})\left(r+m_r+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} +\left(q+m_q+\frac{1}{2}\right)\left(r-m_r+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}} -\left(q-m_q+\frac{1}{2}\right)\left(r-m_r+\frac{1}{2}\right) \times \theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}, \tag{5.2}$$

$$\left|p-\frac{3}{2}, m_p; q-\frac{1}{2}, m_q; r+\frac{1}{2}, m_r\right\rangle =(r+m_r+\frac{1}{2})\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} +\left(r+m_r+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} +\left(r-m_r+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}} +\left(r-m_r+\frac{1}{2}\right)\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}, \tag{5.3}$$

$$\left|p-\frac{3}{2}, m_p; q+\frac{1}{2}, m_q; r-\frac{1}{2}, m_r\right\rangle =\left(q+m_q+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}} \times \chi_{\alpha_3}^{r+\frac{1}{2}-m_r} - \left(q-m_q+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}} \times \chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} - \left(q+m_q+\frac{1}{2}\right)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}} +(q-m_q+\frac{1}{2})\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3} \times \chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}} \chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}, \tag{5.4}$$

$$\left|p-\frac{3}{2}, m_p; q-\frac{1}{2}, m_q; r-\frac{1}{2}, m_r\right\rangle$$
$$=\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\,\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\,\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}$$
$$+\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\,\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\,\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\times\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}$$
$$-\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}$$
$$\times\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}-\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\,\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}$$
$$\times\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}, \tag{5.5}$$

$$|p-1, m_p; q, m_q; r-1, m_r\rangle$$
$$=-4(p-1)q\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$-4(p-1)q\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r+1}$$
$$+4(p-1)q\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$-4(p-1)q\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r-1}-(p-m_p-1)$$
$$[4(1+\alpha)(p-1)q+\alpha(p-m_p-2)(q+m_q)]$$
$$\theta_{\alpha_1}\theta_{\alpha_2}\theta_{\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\,\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}\,\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}, \tag{5.6}$$

$$|p-1, m_p; q-1, m_q; r, m_r\rangle$$
$$=4(p-1)r\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$+4(p-1)r\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q+1}\chi_{\alpha_3}^{r-m_r}$$
$$+4(p-1)r\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$-4(p-1)r\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q-1}\chi_{\alpha_3}^{r-m_r}$$
$$+(p-m_p-1)\alpha(p-m_p-2)(r-m_r)$$
$$\times\theta_{\alpha_1}\theta_{\alpha_2}\theta_{\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}, \tag{5.7}$$

$$|p-1, m_p; q+1, m_q; r, m_r\rangle$$
$$=4(p-1)r(q+m_q+1)(q+m_q)r\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q+1}\chi_{\alpha_3}^{r-m_r}$$
$$-4(p-1)r(q+m_q+1)(q-m_q+1)r\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\times\chi_{\alpha_3}^{r-m_r}$$
$$-4(p-1)r(q+m_q+1)(q-m_q+1)r\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$+4(p-1)r(q-m_q+1)(q-m_q)r\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q-1}\chi_{\alpha_3}^{r-m_r}-(p-m_p-1)$$
$$(q+1-m_q)(q+m_q+1)\alpha(p-m_p-2)(r-m_r)$$
$$\theta_{\alpha_1}\theta_{\alpha_2}\theta_{\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}, \tag{5.8}$$

$$|p-1, m_p; q, m_q; r+1, m_r\rangle$$
$$=-2(r+m_r+1)(r+m_r)q\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r+1}$$
$$-2(r-m_r+1)(r+m_r+1)q\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$+2(r-m_r+1)(r+m_r+1)q\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$-2(r-m_r+1)(r-m_r)q\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r-1}+(r+m_r+1)$$
$$[(q+m_q)\alpha(p-m_p-2)+4(1+\alpha)(p-1)q]$$
$$\theta_{\alpha_1}\theta_{\alpha_2}\theta_{\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}, \tag{5.9}$$

$$|p-1, m_p; q, m_q; r, m_r; 1\rangle$$
$$=4(p-1)(q+1)(r+m_r)(q+m_q+1)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r+1}+2(p-1)(q+m_q+1)$$
$$[(q+m_q+1)(r+m_r)-(q-m_q+1)(r-m_r)]$$
$$\times\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$+4(p-1)r(q+m_q+1)(q+m_q)\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q+1}\chi_{\alpha_3}^{r-m_r}+2(p-1)(q+m_q+1)$$
$$[(q+m_q+1)(r-m_r)-(r+m_r)(q-m_q+1)]$$
$$\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}$$
$$-4(p-1)r(q+m_q+1)(q-m_q)\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q-1}\chi_{\alpha_3}^{r-m_r}$$
$$-4(p-1)(q+1)(q+m_q+1)(r-m_r)\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r-1}+(p-m_p-1)(q+m_q+1)$$
$$[-4\alpha r(q+1)(r+1)+4(1+\alpha)(p-1)(q+1)m-r$$
$$-\alpha(p-m_p-2)(r-m_r)(q+m_q+2)+4qr(q+1)]$$
$$\theta_{\alpha_1}\theta_{\alpha_2}\theta_{\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}$$
$$\times\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}$$
$$\times\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}, \tag{5.10}$$

$$
\begin{aligned}
&|p-1, m_p; q, m_q; r, m_r; 2\rangle = -4(p-1)(r+m_r)q\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r+1}\\
&+4(p-1)(q+m_q)r\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\ \chi_{\alpha_2}^{q-m_q+1}\chi_{\alpha_3}^{r-m_r}\\
&+2(p-1)[(q+m_q)(r+m_r)-(q-m_q)(r-m_r)]\\
&\theta_{\alpha_1+\alpha_3}\theta_{\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}\\
&+2(p-1)[-(q-m_q)(r+m_r)+(q+m_q)(r-m_r)]\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}\\
&-4(p-1)(q-m_q)r\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q-1}\chi_{\alpha_3}^{r-m_r}\\
&-4(p-1)(r-m_r)q\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r-1}\\
&+(p-m_p-1)\left\{2(1+\alpha)(1-p)\left[m_r(2q+1)-\frac{1}{2}\right]+4qr(q+\alpha r+1+\alpha)\right.\\
&+m_q(2m_r-1)+\alpha m_r(2m_q-1)+\alpha(p-2-m_p)(q+m_q)(r-m_r)\}\\
&\theta_{\alpha_1}\theta_{\alpha_2}\theta_{\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r},
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
&\left|p-\frac{1}{2}, m_p; q-\frac{1}{2}, m_q; r-\frac{1}{2}, m_r\right\rangle = -4(2p-1)(p-1)q(2r-1)\theta_{\alpha_1+\alpha_2}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\\
&\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}-4(2p-1)(p-1)q(2r-1)\theta_{\alpha_1}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&-4(2p-1)(p-1)q(2r-1)\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\\
&\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}-4(2p-1)(p-1)q(2r-1)\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&+\left(p-m_p-\frac{1}{2}\right)\left\{4(p-1)q\left[-\left(r+m_r-\frac{1}{2}\right)\left(-(1+\alpha)(p-2)-m_q-\frac{1}{2}+\alpha r\right)\right.\right.\\
&-\left(r+m_r+\frac{1}{2}\right)\left(\alpha\left(r-m_r-\frac{1}{2}\right)+(1+\alpha)(2p-3)\right)+\left(r-m_r-\frac{5}{2}\right)\left(q-m_q+\frac{1}{2}\right)\\
&\left.\left.+\left(r-m_r+\frac{3}{2}\right)\left(q-m_q+\frac{3}{2}\right)\right]-\alpha\left(p-m_p-\frac{3}{2}\right)\left(q+m_q+\frac{1}{2}\right)(2p-3)\left(r+m_r+\frac{1}{2}\right)\right\}\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&+(p-m_p-\frac{1}{2})\left\{4(p-1)q\left[-\left(r+m_r-\frac{1}{2}\right)(-(1+\alpha)(p-1)-q+\alpha(r+1))\right.\right.\\
&\left.+\left(r+m_r+\frac{1}{2}\right)\left(-3(1+\alpha)(p-1)-\alpha\left(p-m_p-\frac{3}{2}\right)-\alpha r+1-(1+\alpha)p+q\right)\right]\\
&\left.-\alpha\left(p-m_p-\frac{1}{2}\right)\left(q+m_q+\frac{1}{2}\right)(2p-3)\left(r+m_r+\frac{1}{2}\right)\right\}\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&+\left(p-m_p-\frac{1}{2}\right)\left\{4(p-1)q\left[-(2r-1)\left(q-m_q+\frac{1}{2}\right)+\left(r-m_r-\frac{1}{2}\right)\right.\right.\\
&\left(\left(1+\alpha)(p-1)-m_q-\frac{1}{2}+\alpha(r+1\right)\right)+\left(r-m_r+\frac{1}{2}\right)\left(\alpha\left(r+m_r-\frac{3}{2}\right)-(1+\alpha)(2p-3)\right)\\
&\left.-\left(r+m_r-\frac{1}{2}\right)\left((1+\alpha)(p-1)+m_q+\frac{1}{2}-\alpha m_r+\frac{1}{2}\alpha\right)+2(p-1)(1+\alpha)+\alpha\right]\\
&\left.-\alpha\left(p-m_p-\frac{3}{2}\right)\left(q+m_q+\frac{1}{2}\right)(2p-3)\left(r-m_r+\frac{1}{2}\right)\right\}\\
&\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&+\left(p-m_p-\frac{1}{2}\right)\{4(p-1)q\left[\left(r+m_r-\frac{1}{2}\right)(\alpha r-(1+\alpha)(p-1)-(q+1))+\left(r-m_r-\frac{1}{2}\right)\right.\\
&\left.\left(\left(1+\alpha)(p-1)+\alpha(p-m_p-\frac{3}{2})+\alpha r-(q+1\right)\right)-(1+\alpha)(2p-3)\left(r-m_r+\frac{1}{2}\right)\right]\\
&\left.-\alpha\left(p-m_p-\frac{3}{2}\right)\left(q+m_q+\frac{1}{2}\right)(2p-3)\left(r-m_r+\frac{1}{2}\right)\right\}\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}},
\end{aligned}
\tag{5.12}
$$

$$
\left| p-\frac{1}{2}, m_p; q+\frac{1}{2}, m_q; r-\frac{1}{2}, m_r \right\rangle
$$

$$
\begin{aligned}
&= -4(2p-1)(p-1)q(2r-1)\left(q+m_q+\frac{1}{2}\right)\theta_{\alpha_1}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&-4(2p-1)(p-1)q(2r-1)\left(q-m_q+\frac{1}{2}\right)\theta_{\alpha_1+\alpha_2}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&-4(2p-1)(p-1)q(2r-1)\left(q+m_q+\frac{1}{2}\right)\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&-4(2p-1)(p-1)q(2r-1)\left(q-m_q+\frac{1}{2}\right)\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&+\left(q+m_q+\frac{1}{2}\right)\left(p-m_p-\frac{1}{2}\right)\left\{4(p-1)q\left[\left(r-m_r-\frac{1}{2}\right)\left(-\left(p+m_p+\frac{3}{2}\right)-2\alpha r-2q-2\alpha p\right)\right.\right.\\
&\left.+1+(1+\alpha)\left(p+m_p-\frac{1}{2}\right)\left(r+m_r-\frac{1}{2}\right)-\left(r+m_r+\frac{1}{2}\right)+\left(q-m_q+\frac{1}{2}\right)\right]\\
&\left.-\alpha\left(p-m_p-\frac{3}{2}\right)\left(q+m_q-\frac{1}{2}\right)\left(r+m_r+\frac{1}{2}\right)\right\}\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}+\left(q+m_q+\frac{1}{2}\right)\\
&\left(p-m_p-\frac{1}{2}\right)\left\{4(p-1)q\left[\left(r-m_r-\frac{1}{2}\right)\left((1+\alpha)(p-1)+\alpha\left(p-m_p-\frac{3}{2}\right)+\alpha r+q\right)+\left(r+m_r-\frac{1}{2}\right)\right.\right.\\
&(-(1+\alpha)(p-1)+q+\alpha r)+q+\alpha r)+(1+\alpha)(2p-1)\\
&\left.\left.-(1+\alpha)\left(r-m_r+\frac{1}{2}\right)\right]-\alpha\left(p-m_p-\frac{3}{2}\right)\left(q+m_q-\frac{1}{2}\right)\left(r-m_r+\frac{1}{2}\right)\right\}\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&+\left(q-m_q+\frac{1}{2}\right)\left(p-m_p-\frac{1}{2}\right)\left\{4(p-1)q\left[\left(r+m_r-\frac{1}{2}\right)(-(1+\alpha)(p-2)+q+\alpha r)\right.\right.\\
&\left.+\left(r-m_r-\frac{1}{2}\right)\left(\alpha\left(r-m_r-\frac{1}{2}\right)+\alpha r+3q-(1+\alpha)p\right)+\left(r+m_r+\frac{1}{2}\right)\right]\\
&\left.+\alpha\left(p-m_p-\frac{3}{2}\right)\left(q-m_q-\frac{1}{2}\right)\left(r+m_r+\frac{1}{2}\right)\right\}\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&+\left(p-m_p-\frac{1}{2}\right)\left(q-m_q+\frac{1}{2}\right)\left\{4(p-1)q\left[-\left(r-m_r-\frac{1}{2}\right)((1+\alpha)(p-1)+q+\alpha(r+1))\right.\right.\\
&+\left(r+m_r-\frac{1}{2}\right)((1+\alpha)(p-1)-q-\alpha r)+\left(r-m_r+\frac{1}{2}\right)(1+2\alpha)\\
&\left.\left.-2(1+\alpha)\left(p+m_p-\frac{3}{2}\right)\right]+\alpha\left(p-m_p-\frac{3}{2}(q+m_q-\frac{1}{2})(r-m_r+\frac{1}{2})\right)\right\}\\
&\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}},
\end{aligned}
\tag{5.13}
$$

$$
\begin{aligned}
&\left| p-\frac{1}{2}, m_p; q-\frac{1}{2}, m_q; r+\frac{1}{2}, m_r \right\rangle \\
&=4(2p-1)(p-1)r(2q-1)\left(r+m_r+\frac{1}{2}\right)\theta_{\alpha_1}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} \\
&+4(2p-1)(p-1)r(2q-1)\left(r+m_r+\frac{1}{2}\right)\theta_{\alpha_1+\alpha_2}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} \\
&-4(2p-1)(p-1)r(2q-1)\left(r-m_r+\frac{1}{2}\right)\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}} \\
&-4(2p-1)(p-1)r(2q-1)\left(r-m_r+\frac{1}{2}\right)\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}} \\
&+\left\{4(p-1)r\left(p-m_p-\frac{1}{2}\right)\left(r+m_r+\frac{1}{2}\right)\left[-\left(q+m_q-\frac{1}{2}\right)(-(1+\alpha)(p-1)+q+\alpha r)\right.\right. \\
&\left.-\left(q-m_q-\frac{1}{2}\right)\left((1+\alpha)\left(m_p+\frac{1}{2}\right)+q+\alpha r\right)\right]+\left(p-m_p-\frac{1}{2}\right)\alpha\left(p-m_p-\frac{3}{2}\right) \\
&\left.\left(r-m_r+\frac{1}{2}\right)\left(q+m_q+\frac{1}{2}\right)\left(r+m_r+\frac{1}{2}\right)\right\}\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} \\
&+\{4(p-1)r\left(p-m_p-\frac{1}{2}\right)\left(r+m_r+\frac{1}{2}\right)(2q-1)((1+\alpha)(p-1)-\alpha r-q) \\
&-\left(p-m_p-\frac{1}{2}\right)\alpha(p-m_p-\frac{3}{2})\left(r-m_r+\frac{1}{2}\right)\left(q-m_q+\frac{1}{2}\right)\left(r+m_r+\frac{1}{2}\right)\Big\} \\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} \\
&+\left\{4(p-1)r\left(p-m_p-\frac{1}{2}\right)\left(r-m_r+\frac{1}{2}\right)\left(\left(q+m_q-\frac{1}{2}\right)\alpha\left(p-m_p-\frac{3}{2}\right)\right.\right. \\
&+(2q-1)[(1+\alpha)p+q+\alpha r])+\left(p-m_p-\frac{1}{2}\right)\alpha\left(p-m_p-\frac{3}{2}\right) \\
&\left.\left(r-m_r+\frac{1}{2}\right)^2\left(q+m_q+\frac{1}{2}\right)\right\}\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}} \\
&+\left\{4(p-1)r\left(p-m_p-\frac{1}{2}\right)\left(r-m_r+\frac{1}{2}\right)\left(-\left(q-m_q-\frac{1}{2}\right)\left[-\alpha\left(m_r+\frac{1}{2}\right)-q+(1+\alpha)p\right]\right.\right. \\
&\left.-\alpha\left(r+m_r+\frac{1}{2}\right)(2q-1)\right]-\left(q+m_q-\frac{1}{2}\right)\left(q-m_q+\frac{1}{2}\right)\Bigg)-\left(p-m_p-\frac{1}{2}\right) \\
&\left.\alpha\left(p-m_p-\frac{3}{2}\right)\left(r-m_r+\frac{1}{2}\right)^2\left(q-m_q+\frac{1}{2}\right)\right\} \\
&\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}},
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
&\left| p-\frac{1}{2}, m_p; q+\frac{1}{2}, m_q; r+\frac{1}{2}, m_r \right\rangle = -4(2p-1)(p-1)r(2q+3)\Big(q+m_q+\frac{3}{2}\Big)\Big(q+m_q+\frac{1}{2}\Big)\Big(r+m_r+\frac{1}{2}\Big)\\
&\theta_{\alpha_1}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}} + 4(2p-1)(p-1)r(2q+3)\Big(q+m_q+\frac{3}{2}\Big)\Big(q+m_q+\frac{1}{2}\Big)\Big(r-m_r+\frac{1}{2}\Big)\\
&\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&+4(2p-1)(p-1)r(2q+3)\Big(q+m_q+\frac{3}{2}\Big)\Big(q-m_q+\frac{1}{2}\Big)\Big(r+m_r+\frac{1}{2}\Big)\\
&\theta_{\alpha_1+\alpha_2}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&-4(2p-1)(p-1)r(2q+3)\Big(q+m_q+\frac{3}{2}\Big)\Big(q-m_q+\frac{1}{2}\Big)\Big(r-m_r+\frac{1}{2}\Big)\\
&\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{1}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&+\Big[4(p-1)r\Big(q+m_q+\frac{3}{2}\Big)\Big(q+m_q+\frac{1}{2}\Big)\Big(p-m_p-\frac{1}{2}\Big)\Big\{\Big(r+m_r+\frac{1}{2}\Big)\\
&\Big[(2q+3)(-(1+\alpha)(p-1)-(1+q))+\alpha r\Big(q-m_q+\frac{3}{2}\Big)+\Big(q+m_q+\frac{3}{2}\Big)\\
&\Big(\alpha m_r-\frac{1}{2}\alpha-2\Big(q-m_q+\frac{1}{2}\Big)\Big)\Big]+\Big(q-m_q+\frac{1}{2}\Big)\Big(q+m_q+\frac{3}{2}\Big)\\
&\alpha\Big(p-m_p-\frac{3}{2}\Big)\Big(r-m_r+\frac{1}{2}\Big)\Big(r+m_r+\frac{1}{2}\Big)\Big]\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&+\Big[4(p-1)r\Big(p-m_p-\frac{1}{2}\Big)\Big(q+m_q+\frac{3}{2}\Big)\Big(q+m_q+\frac{1}{2}\Big)\Big(r-m_r+\frac{1}{2}\Big)\\
&\Big\{\Big(q+m_q+\frac{3}{2}\Big)(-1-q+(1+\alpha)p+\alpha(r+1))+\alpha\Big(r+m_r+\frac{3}{2}\Big)\Big(q-m_q+\frac{3}{2}\Big)\\
&+\Big((1+\alpha)(p-1)+\alpha\Big(p-m_p-\frac{3}{2}\Big)-\Big(-m_q+\frac{1}{2}\Big)-\alpha\Big(m_r+\frac{1}{2}\Big)\Big)\\
&-\Big(q+m_q-\frac{1}{2}\Big)\Big(q-m_q+\frac{3}{2}\Big)\Big\}-\Big(p-m_p-\frac{1}{2}\Big)\Big(q-m_q+\frac{1}{2}\Big)(q+m_q+\frac{3}{2})\\
&\alpha(p-m_p-\frac{3}{2})\Big(r-m_r+\frac{1}{2}\Big)^2\Big]\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q+\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}}\\
&+\Big[4(p-1)r\Big(p-m_p-\frac{1}{2}\Big)\Big(q+m_q+\frac{3}{2}\Big)\Big(q-m_q+\frac{1}{2}\Big)\Big\{\Big(r+m_r+\frac{1}{2}\Big)(2q+3)\\
&((1+\alpha)(p-1)+1+q-\alpha r)+\Big(r-m_r+\frac{1}{2}\Big)\Big(q+m_q+\frac{1}{2}\Big)\Big(\alpha\Big(p-m_p-\frac{3}{2}\Big)\\
&-\alpha\Big(m_r+\frac{1}{2}\Big)+\Big(m_q-\frac{1}{2}\Big)+(1+\alpha)(p-1)\Big)\Big\}-\Big(p-m_p-\frac{1}{2}\Big)\Big(q-m_q+\frac{1}{2}\Big)\\
&\Big(q+m_q+\frac{3}{2}\Big)\alpha\Big(p-m_p-\frac{3}{2}\Big)\Big(r-m_r+\frac{1}{2}\Big)\Big(r+m_r+\frac{1}{2}\Big)\Big]\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r+\frac{1}{2}}\\
&+\Big[4(p-1)r\Big(p-m_p-\frac{1}{2}\Big)\Big(q+m_q+\frac{3}{2}\Big)\Big(r-m_r+\frac{1}{2}\Big)\Big(q-m_q+\frac{1}{2}\Big)\Big\{-(q-m_q+\frac{1}{2})\\
&\Big\{-\Big(q-m_q+\frac{3}{2}\Big)\Big(-1+\alpha\Big(p-m_p-\frac{3}{2}\Big)-\alpha\Big(m_r+\frac{1}{2}\Big)-q+(1+\alpha)(p-1)\\
&-\alpha\Big(q-m_q+\frac{1}{2}\Big)\Big)\Big\}-\Big(p-m_p-\frac{1}{2}\Big)\Big(q-m_q+\frac{1}{2}\Big)\Big(q+m_q+\frac{3}{2}\Big)\\
&\alpha(p-m_p-\frac{3}{2})\Big(r-m_r+\frac{1}{2}\Big)^2\Big]\\
&\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-\frac{3}{2}}\chi_{\alpha_2}^{q-m_q-\frac{1}{2}}\chi_{\alpha_3}^{r-m_r-\frac{1}{2}},
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
&|p, m_p; q, m_q; r, m_r\rangle\\
&=32pq^2r(p-1)(2p-1)(2r-1)\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}\\
&+(2p-1)(r+m_r)[-(q+m_q-1)V-(q-m_q+1)W+4(p-m_p)\\
&(p-1)q(2r-1)(q-m_q)(-(1+\alpha)(p-1)+\alpha(r-1)+q)]\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r+1}\\
&+(2p-1)[(q+m_q)(r-m_r+1)V+(-(r+m_r)+(q+m_q))Y\\
&+(r+m_r)(q-m_q+1)Z+4(p-m_p)(p-1)q(2r-1)\{(q-m_q)\\
&[(r-m_r)(m_q+\alpha m_r-(1+\alpha)p)-(r+m_r)(\alpha(r-m_r+1)-(q+m_q))]\\
&+(q+m_q)[(1+\alpha)(2p-1)-(q-m_q+1)(r-m_r)+(r+m_r)\\
&(-(1+\alpha)p+m_q+\alpha r)\}]\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}\\
&+(2p-1)(q+m_q)[(r-m_r+1)W-(r-m_r+1)Z+4(p-m_p)(p-1)\\
&q(2r-1)\{(r-m_r+1)(\alpha(p-m_p)+\alpha r+(1+\alpha)(p-1)+q)\\
&+(r+m_r)(-(1+\alpha)p+\alpha r+q)+(1+\alpha)p-\alpha(p-m_p)-q+\alpha r)\}]\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q+1}\chi_{\alpha_3}^{r-m_r}\\
&+(2p-1)[(r-m_r+1)V-(r-m_r+1)(q-m_q+1)W-(q-m_q+1)Z\\
&-((q+m_q)(r+m_r)-1)Y+4(p-m_p)(p-1)q(2r-1)\\
&\{(q+m_q)[(r+m_r)\alpha(r-m_r+1)+(r-m_r)(\alpha(p-m_p-1)\\
&-\alpha m_r+m_q+(1+\alpha)p)]+(q-m_q)[\alpha(r-m_r+1)-(q+m_q)(r+m_r)\\
&+(r+m_r)(m_q+(1+\alpha)p-\alpha m_r)-(1+\alpha)(2p-1)-\alpha-(q+m_q+1)\\
&-(r-m_r)\alpha(r+m_r+1)+(r-m_r+1)(r+m_r)\}]\\
&\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r}\\
&+(2p-1)(q-m_q)[(r-m_r+1)V+(r+m_r-1)Y+4(p-1)q(2r-1)\\
&(p-m_p)\{2r(-(q+1)+(1+\alpha(p-\alpha r)-(2p-1)(1+\alpha)\}]\\
&\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q-1}\chi_{\alpha_3}^{r-m_r}\\
&+(2p-1)(r-m_r)[-(q+m_q-1)Y-(q-m_q+1)Z+4(p-1)q(2r-1)\\
&(p-m_p)\{(q-m_q)(-\alpha(p-m_p-1)+\alpha-(1+\alpha)p-q-\alpha r)\\
&-(r+m-r)\alpha(q-m_q)+(q-m_q)(\alpha(m_r+1)-(1+\alpha)p-q)\}\\
&\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-1}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r-1}\\
&+(p-m_p-1)[4(p-m_p)(p-1)(2p-1)q(2r-1)\{-2q(r-m_r+1)\\
&\alpha(1+\alpha)-(q+m-q)(1+\alpha)(-\alpha(p-m_p-2)+\alpha(r-m_r)\\
&-(1+\alpha)p-q+\alpha r)-(1+\alpha)(q-m_q)(\alpha m_r-(1+\alpha)p-q)\}\\
&+(r-m_r+1)V\{-(q+m_q)(-\alpha(p-m_p-2)-(1+\alpha)(p-1)\\
&-m_q-\alpha r)+(m_q-\alpha r-(1+\alpha)(p-1))-(q-m_q)(q+m_q+1)\}\\
&+(q-m_q+1)W(r-m_r+1)(q+(1+\alpha)(p-1)+\alpha r)\\
&+Y\{((1+\alpha)p-m_q-\alpha m_r)((r+m_r)(q+m_q-1)-1)\\
&-(r+m_r+1)((q-m_q)(q+m_q+1)+(q+m_q-1)\alpha(r-m_r))\\
&-(q+m_q)(-(1+\alpha)(p-1)-\alpha(p-m_p-2)+\alpha m_r-m_q)\}\\
&+Z(q-m_q+1)\{-(r+m_r)(-(1+\alpha)p+q+\alpha m_r)\\
&+(r-m_r)((q-m_q+1)-\alpha(r+m_r+1))+((1+\alpha)(p-1)+m_q-\alpha m_r)\\
&\theta_{\alpha_1}\theta_{\alpha_1+\alpha_2}\theta_{\alpha_1+\alpha_3}\theta_{\alpha_1+\alpha_2+\alpha_3}\chi_{2\alpha_1+\alpha_2+\alpha_3}^{p-m_p-2}\chi_{\alpha_2}^{q-m_q}\chi_{\alpha_3}^{r-m_r},
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
V=\;&4(p-1)q(p-m_p)[-(r+m_r-1)(-(1+\alpha)(p-2)-m_q+\alpha r)\\
&-(r+m_r)(\alpha(r-m_r)+(1+\alpha)(2p-3))+(r+m_r-3)\\
&(q-m_q+1)(r-m_r+2)(q+m_q+1)]-\alpha(p-m_p-1)\\
&(q+m_q)(2p-3)(r+m_r),
\end{aligned}
\tag{5.17}
$$

$$W = 4(p-1)q(p-m_p)[-(r+m_r-1)(-(1+\alpha)(p-1)-q+\alpha(r+1)) \\ +(r+m_r)(-3(1+\alpha)(p-1)-\alpha(p-m_p-1)-\alpha r+1+q)] \\ -\alpha(p-m_p-1)(q+m_q)(2p-3)(r+m_r), \tag{5.18}$$

$$Y = 4(p-1)q(p-m_p)[-(2r-1)(q-m_q+1)+(r-m_r)((1+\alpha)(p-1) \\ -m_q+\alpha(r+1))+(r-m_r+1)(\alpha(r+m_r-2)-(1+\alpha)(2p-3)) \\ -(r+m_r-1)((1+\alpha)(p-1)+m_q-\alpha m_r+\alpha)+2(p-1)(1+\alpha) \\ +\alpha]-\alpha(p-m_p-1)(q+m_q)(2p-3)(r-m_r-1), \tag{5.19}$$

$$Z = 4(p-1)q(p-m_p)[(r+m_r+1)(\alpha r-(1+\alpha)(p-1)-(q+1)) \\ +(r-m_r)((1+\alpha)9p-1)-\alpha(p-m_p-1)+\alpha r(q+1))-(1+\alpha)(2p-3) \\ (r-m_r+1)]-\alpha(p-m_p-1)(q+m_q)(2p-3)(r-m_r-1). \tag{5.20}$$

6. The typical and atypical representation of $D(2, 1; \alpha)$

The (s, t, u) components must satisfy

$$s, t, u \in \frac{1}{2}N = \left\{0, \frac{1}{2}, 1, \ldots\right\}, \tag{6.1}$$

and the (p, q, r) also belongs to this set. If $p \geqslant 2$, $q \geqslant 1$, $r \geqslant 1$, there are four atypical conditions [22] given by

$$(1+\alpha)p+q+\alpha r=0, \tag{6.2}$$

$$(1+\alpha)p-(q+1)-\alpha r=0, \tag{6.3}$$

$$(1+\alpha)p+q-\alpha(r+1)=0, \tag{6.4}$$

$$(1+\alpha)p-(q+1)-\alpha(r+1)=0. \tag{6.5}$$

If none of the four atypical conditions are satisfied, then the representation is a typical representation, which decomposes into 16 subalgebra irreducible representations. If one of the conditions is satisfied, the representation is reducible but indecomposable generally. The shift operator will separate the 16-dimensional lattice into two 8-dimensional lattices. Since

$$O^{\frac{1}{2},-j,-k}O^{-\frac{1}{2},j,k}|s, t, u> = 0. \tag{6.6}$$

For (6.2), $j=\frac{1}{2}$, $k=\frac{1}{2}$, the two 8-dimensional lattices are given by:

$$\mathbb{F}_1 = \{(p, q, r), \left(p-\frac{1}{2}, q+\frac{1}{2}, r-\frac{1}{2}\right), \left(p-\frac{1}{2}, q-\frac{1}{2}, r+\frac{1}{2}\right), \\ \left(p-\frac{1}{2}, q-\frac{1}{2}, r-\frac{1}{2}\right), (p-1, q, r; 1), (p-1, q, r-1), \\ (p-1, q-1, r), \left(p-\frac{3}{2}, q-\frac{1}{2}, r-\frac{1}{2}\right)\}, \tag{6.7}$$

$$\mathbb{F}_2 = \{\left(p-\tfrac{1}{2}, q+\tfrac{1}{2}, r+\tfrac{1}{2}\right), (p-1, q+1, r), (p-1, q, r+1) \\ (p-1, q, r; 2), \left(p-\tfrac{3}{2}, q+\tfrac{1}{2}, r+\tfrac{1}{2}\right), \left(p-\tfrac{3}{2}, q+\tfrac{1}{2}, r-\tfrac{1}{2}\right), \\ \left(p-\tfrac{3}{2}, q-\tfrac{1}{2}, r+\tfrac{1}{2}\right), (p-2, q, r)\}. \tag{6.8}$$

For (6.3), $j=-\frac{1}{2}$, $k=\frac{1}{2}$, the two 8-dimensional lattices are given by:

$$\begin{aligned}&\mathbb{F}_1=\{(p,q,r),\left(p-\frac{1}{2},q+\frac{1}{2},r-\frac{1}{2}\right),\left(p-\frac{1}{2},q+\frac{1}{2},r+\frac{1}{2}\right),\\&\left(p-\frac{1}{2},q-\frac{1}{2},r-\frac{1}{2}\right),(p-1,q,r;1),(p-1,q,r-1),\\&(p-1,q,r-1);\left(p-\frac{3}{2},q-\frac{1}{2},r-\frac{1}{2}\right)\},\end{aligned}\tag{6.9}$$

$$\begin{aligned}\mathbb{F}_2=\Big\{&\left(p-\tfrac{1}{2},q-\tfrac{1}{2},r+\tfrac{1}{2}\right),(p-1,q-1,r),(p-1,q,r+1)\\&(p-1,q,r;2),\left(p-\tfrac{3}{2},q-\tfrac{1}{2},r+\tfrac{1}{2}\right),\left(p-\tfrac{3}{2},q-\tfrac{1}{2},r-\tfrac{1}{2}\right),\\&\left(p-\tfrac{3}{2},q+\tfrac{1}{2},r+\tfrac{1}{2}\right),(p-2,q,r)\Big\}.\end{aligned}\tag{6.10}$$

For (6.4), $j=\frac{1}{2}$, $k=-\frac{1}{2}$, the two 8-dimensional lattices are given by:

$$\begin{aligned}&\mathbb{F}_1=\{(p,q,r),\left(p-\frac{1}{2},q+\frac{1}{2},r+\frac{1}{2}\right),\left(p-\frac{1}{2},q-\frac{1}{2},r-\frac{1}{2}\right),\\&\left(p-\frac{1}{2},q-\frac{1}{2},r+\frac{1}{2}\right),(p-1,q,r;1),(p-1,q,r-1),\\&(p-1,q,r-1);\left(p-\frac{3}{2},q-\frac{1}{2},r+\frac{1}{2}\right)\},\end{aligned}\tag{6.11}$$

$$\begin{aligned}\mathbb{F}_2=\{&\left(p-\tfrac{1}{2},q+\tfrac{1}{2},r-\tfrac{1}{2}\right),(p-1,q,r-1),(p-1,q+1,r)\\&(p-1,q,r;2),\left(p-\tfrac{3}{2},q-\tfrac{1}{2},r-\tfrac{1}{2}\right),\left(p-\tfrac{3}{2},q+\tfrac{1}{2},r-\tfrac{1}{2}\right),\\&\left(p-\tfrac{3}{2},q+\tfrac{1}{2},r+\tfrac{1}{2}\right),(p-2,q,r)\}.\end{aligned}\tag{6.12}$$

For (6.5), $j=-\frac{1}{2}$, $k=-\frac{1}{2}$, the two 8-dimensional lattices are given by:

$$\begin{aligned}&\mathbb{F}_1=\{(p,q,r),\left(p-\frac{1}{2},q-\frac{1}{2},r+\frac{1}{2}\right),\left(p-\frac{1}{2},q+\frac{1}{2},r+\frac{1}{2}\right),\\&\left(p-\frac{1}{2},q+\frac{1}{2},r-\frac{1}{2}\right),(p-1,q,r;1),(p-1,q,r+1),\\&(p-1,q+1,r);\left(p-\frac{3}{2},q+\frac{1}{2},r+\frac{1}{2}\right)\},\end{aligned}\tag{6.13}$$

$$\begin{aligned}&\mathbb{F}_2=\{\left(p-\frac{1}{2},q-\frac{1}{2},r-\frac{1}{2}\right),(p-1,q-1,r),(p-1,q,r-1)\\&(p-1,q,r;2),\left(p-\frac{3}{2},q-\frac{1}{2},r+\frac{1}{2}\right),\left(p-\frac{3}{2},q-\frac{1}{2},r-\frac{1}{2}\right),\\&\left(p-\frac{3}{2},q+\frac{1}{2},r-\frac{1}{2}\right),(p-2,q,r)\}.\end{aligned}\tag{6.14}$$

If $p<2$, $q<1$, $r<1$, only none-negative value elements appear in the decomposition of the (s,t,u) lattice.

7. Conclusions

First, we have reviewed the explicit differential operator representations for Lie superalgebra $D(2,1;\alpha)$. Based on the shift operator and differential operator representations, we have constructed the explicitly finite-dimensional representations of superalgebra $D(2,1;\alpha)$ by using bosonic and fermionic coordinates. Our results are expected to be useful for the construction of primary fields of the corresponding current superalgebra of $D(2,1;\alpha)$, which play an important role in the computation of quantization of the string theory on the $AdS_3\times S^3\times S^3\times S^1$ background.

Acknowledgments

This work received financial support from the National Natural Science Foundation of China (Grant No. 11 405 051). Yao-Zhong Zhang was supported by the Australian Research Council Discovery Project DP190101529. Xiang-Mao Ding was supported by NSFC Grant 11 775 299.

References

[1] Di Francesco P, Mathieu P and Senehal D 1997 *Conformal Field Theory* (Berlin: Springer)
[2] Frappat L, Sorba P and Scarrino A 2000 *Dictionary on Lie Algebras and Superalgebras* (New York: Academic)
[3] Flohr M 2003 Bits and pieces in logarithmic conformal field theory *Int. J. Mod. Phys.* A **18** 4497–592
[4] Isidro M and Ramallo A V 1994 gl(N,N) current algebras and topological field theories *Nucl. Phys.* B **414** 715–62
[5] Efetov K 1983 Supersymmetry and theory of disordered metals *Adv. Phys.* **32** 53–127
[6] Bernard S *Conformal Field Theory Applied To 2D Disordered Systems: An Introduction* hep-th/9509137
[7] Mudry C, Chamon C and Wen X G 1996 Two-dimensional conformal field theory for disordered systems at criticality *Nucl. Phys.* B **466** 383–443
[8] Maassarani Z and Serban D 1997 Non-unitary conformal field theory and logarithmic operators for disordered systems *Nucl. Phys.* B **489** 603–25
[9] Zirnbauer M R *Conformal field theory of the integer quantum Hall plateau transition* hep-th/9905054
[10] Bassi Z S and LeClair A 2000 The exact s-matrix for an $osp(2|2)$ disordered system *Nucl. Phys.* B **578** 577–627
[11] Guruswamy S, LeClair A and Ludwig A W W 2000 *Nucl. Phys.* B **583** 475
[12] Kac V G 1977 Characters of typical representations of classical Lie superalgebras *Commun. Algebra.* **5** 889–97
[13] Kac V G 1978 *Lecture Notes in Mathematics* vol 676 (Berlin: Springer) 597
[14] Bershadsky M, Zhukov S and Vaintrob A 1999 $PSL(n|n)$ sigma model as a conformal field theory *Nucl. Phys.* B **559** 205–34
[15] Zhang Y Z 2004 Coherent state construction of representations of $osp(2|2)$ and primary fields of $osp(2|2)$ conformal field theory *Phys. Lett.* A **327** 442–51
[16] Zhang Y Z and Gould M D 2005 A unified and complete construction of all finite dimensional irreducible representations of $gl(2|2)$ *J. Math. Phys.* **46** 013505
[17] Matsumoto T and Moriyama S 2008 An exceptional algebraic origin of the AdS/CFT Yangian symmetry *J. High Energy Phys.* JHEP04(2008)022
[18] Matsumoto T and Moriyama S 2009 Serre relation and higher grade generators of the AdS/CFT Yangian symmetry *J. High Energy Phys.* JHEP09(2009)097
[19] Ohlsson Sax O and Stefanski B 2011 Integrability, spin-chains and the AdS3/CFT2 correspondence *J. High Energy Phys.* JHEP08(2011)029
[20] Gauntlett J P, Myers R C and Townsend P K 1999 Supersymmetry of rotating branes *Phys. Rev.* D **59** 025001
[21] Chen X, Yang W-L, Ding X-M, Feng J, Ke S-M, Wu K and Zhang Y-Z 2012 Free-field realization of the exceptional current superalgebra $D(2, 1; \alpha)_k$ *J. Phys.* A **45** 405204
[22] Van der Jeugt J 1985 Irreducible representations of the exceptional Lie superalgebras $D(2, 1; \alpha)$ *J. Math. Phys.* **26** 913–24
[23] Feigin B and Frenkel E 1990 Affine Kac-Moody algebras and semi-infinite flag manifolds *Commun. Math. Phys.* **128** 161–89
[24] Bouwknegt P, McCarthy J and Pilch K 1990 Free field approach to 2-dimensional conformal field theories *Prog. Phys.* **102** 67–135
[25] Ito K 1990 Feigin-Fuchs representation of generalized parafermions *Phys. Lett.* B **252** 69–73
[26] Rasmussen J 1998 Free field realizations of affine current superalgebras, screening currents and primary fields *Nucl. Phys.* B **510** 688
[27] Ding X-M, Gould M and Zhang Y-Z 2003 $gl(2|2)$ current superalgebra and non-unitary conformal field theory *Phys. Lett.* A **318** 354–63
[28] Zhang Y-Z, Liu X and Yang W-L 2005 Primary fields and screening currents of $gl(2|2)$ non-unitary conformal field theory *Nucl. Phys.* B **704** 510–26
[29] Yang W-L, Zhang Y-Z and Liu X 2006 $gl(4|4)$ current superalgebra: free field realization and screening currents *Phys. Lett.* B **641** 329–34
[30] Yang W-L and Zhang Y-Z 2008 Free field realization of the current algebra *Phys. Rev.* D **78** 106004
[31] Yang W-L, Zhang Y-Z and Kault S 2009 Differential operator realizations of super-algebras and free field representations of corresponding current algebras *Nucl. Phys.* B **823** 372–402
[32] Hughes J W B and Yadergar J 1978 o(3) shift operators: the general analysis *J. Math. Phys.* **19** 2068