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Construction of conservation laws for the Gardner equation, Landau–Ginzburg–Higgs equation, and Hirota–Satsuma equation

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Abstract
In this paper, two different methods for calculating the conservation laws are used, these are the direct construction of conservation laws and the conservation theorem proposed by Ibragimov. Using these two methods, we obtain the conservation laws of the Gardner equation, Landau–Ginzburg–Higgs equation and Hirota–Satsuma equation, respectively.

Keywords: conservation laws, Landau–Ginzburg–Higgs equation, Hirota–Satsuma equation, Gardner equation

1. Introduction

Nonlinear science runs through physics, mathematics, astronomy, and other scientific fields and is a bridge connecting mathematics and other disciplines [1–4]. It is another major development in natural science after quantum mechanics and relativity. Nonlinear differential equations transform physical phenomena into mathematical language, so it is of practical significance to study nonlinear equations. The concept of conservation law plays an important role in the study of differential equations and many applications. The mathematical concept of conservation law is derived from familiar representations of physical laws such as energy [5], momentum [6], and angular momentum [7]. In the jet problem [8], the conserved quantity plays an important role in the solution process, which is used to determine the unknown index in the similar solution that cannot be obtained under homogeneous boundary conditions. In addition, we can see that many results, such as the existence, uniqueness, and stability of rational solutions, are determined by conservation law [9–11].

The well-known method of constructing conservation laws is the Noether theorem [12], which will produce all the conservation laws of the partial differential equation system with a Lagrangian. Specifically, every nontrivial conservation law is generated by some variational symmetries, but not necessarily variational point symmetries. The application of the Noether theorem depends on the appropriate Lagrangian formulation. In fact, many equations do not have it, so we look for additional methods to find the conservation laws. The first method is that Anco and Bluman proposed an algorithm to obtain the field equations for any system of local conservation laws in [13, 14], regardless of whether the system has a Lagrangian formulation. The algorithm is based on the conservation law multiplier to obtain the conservation law of the equation system by using the formula. The second method is that Ibragimov introduced the concept of adjoint equations of differential equations and took the new differential equations composed of given equations and their adjoint equations as the research object [15]. Using the adjoint variational principle [16] proposed by Atherton *et al.* , the Lagrangian formulation of the new differential equations was obtained, and the theorem of finding conservation laws was given. Furthermore, there are many ways to construct conservation laws, such as the variational approach [17], the partial Noether approach [18], and so on [19–21].

The outline of this paper is as follows: In section 2, the method of directly constructing conservation laws is reviewed. Section 3 applies the method of directly constructing conservation laws to the Gardner equation, the Landau–Ginzburg–Higgs (LGH) equation, and the Hirota–Satsuma equation; the conservation laws are obtained. In section 4, we discuss the nonlinear self-adjointness of the above three equations and obtain new conserved quantities by using the conservation theorem proposed by Ibragimov.

2. Direct construction method for conservation laws of partial differential equations

2.1. Gardner equation

We first consider the Gardner equation

$$G \equiv u_t + 6uu_x - 6u^2u_x + u_{xxx} = 0. \tag{2.1}$$

The equation is widely used in various branches of physics, such as plasma physics, fluid physics, and quantum field theory [22–24]. It also describes various wave phenomena in plasma and solid [33, 34]. Its symmetries with infinitesimal generator $Xu = \eta$ [25, 26] satisfy the determining equation

$$\begin{aligned} 0 &= D_t\eta + 6u_x\eta + 6uD_x\eta - 12uu_x\eta \\ &-6u^2D_x\eta + D_x^3\eta \quad \text{when} \quad G = 0, \end{aligned} \tag{2.2}$$

where $D_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + u_{tt}\partial_{u_t} + \cdots$ and $D_x = \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{tx}\partial_{u_t} + \cdots$ are total derivative operators with respect to t and x. The adjoint of the symmetry equation (2.2) is the determining equation for adjoint symmetries $G'^*(\omega)|_\epsilon = 0$, where ϵ will denote the solution space of the equation (2.1). Therefore, the adjoint of the equation (2.2) is given by

$$\begin{aligned} 0 &= -D_t\omega - 6uD_x\omega + 6u^2D_x\omega - D_x^3\omega \\ &\text{when} \quad G = 0, \end{aligned} \tag{2.3}$$

which is the determining equation for the adjoint symmetries ω of the Gardner equation. Next, we consider local conservation laws $D_t\Phi^t + D_x\Phi^x = 0$ on all solutions $u(t, x)$ of the equation (2.1). All nontrivial conservation laws' conserved densities in this form can be constructed from multipliers Λ on the Gardner equation, analogous to integrating factors, where Λ depends only on t, x, u, u_x and u_{xx}. In particular, by moving off the Gardner equation solution space, we have

$$\begin{aligned} D_t\Phi^t + D_x\Phi^x &= (u_t + 6uu_x - 6u^2u_x + u_{xxx})\Lambda_0 \\ &+D_x(u_t + 6uu_x - 6u^2u_x + u_{xxx})\Lambda_1 + \cdots, \end{aligned} \tag{2.4}$$

for some expression Λ_0, Λ_1, … with no dependence on u_t or differential consequence. This yields the multiplier

$$\begin{aligned} &D_t\Phi^t + D_x(\Phi^x - \Gamma) \\ &=(u_t + 6uu_x - 6u^2u_x + u_{xxx})\Lambda, \\ &\Lambda = \Lambda_0 - D_x\Lambda_1 + \cdots, \end{aligned} \tag{2.5}$$

where $\Gamma = 0$ when u is restricted to be a Gardner equation solution.

The definition for multipliers $\Lambda(t, x, u, u_x, u_{xx})$ is that $(u_t + 6uu_x - 6u^2u_x + u_{xxx})\Lambda$ must be a divergence expression for all functions $u(t, x)$. The determining condition is expressed by

$$\begin{aligned} 0 &= E_u(G\Lambda) = E_u((u_t + 6uu_x - 6u^2u_x + u_{xxx})\Lambda) \\ &= -D_t\Lambda - 6uD_x\Lambda + 6u^2D_x\Lambda - D_x^3\Lambda \\ &+ G\Lambda_u - D_x(G\Lambda_{u_x}) + D_x^2(G\Lambda_{u_{xx}}), \end{aligned} \tag{2.6}$$

where $E_u = \partial_u - D_t\partial_{u_t} - D_x\partial_{u_x} + D_tD_x\partial_{u_{tx}} + D_x^2\partial_{u_{xx}} + \cdots$ is the standard Euler operator which annihilates divergence expressions.

By calculation, we can get the general form of the multiplier of the Gardner equation as follows (see appendix A):

$$\begin{aligned} \Lambda &= c_1\left[(2u^3 - 3u^2 + u - u_{xx})t - \frac{x}{6} + \frac{1}{3}xu\right] \\ &+ c_2[-2u^3 + 3u^2 + u_{xx}] + c_3u + c_4, \end{aligned} \tag{2.7}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. So for the Gardner equation, here are some multipliers

$$\begin{aligned} &\Lambda_1 = (2u^3 - 3u^2 + u - u_{xx})t - \frac{x}{6} + \frac{1}{3}xu, \\ &\Lambda_2 = -2u^3 + 3u^2 + u_{xx}, \\ &\Lambda_3 = u, \quad \Lambda_4 = 1. \end{aligned} \tag{2.8}$$

For the multiplier $\Lambda_1 = (2u^3 - 3u^2 + u - u_{xx})t - \frac{x}{6} + \frac{1}{3}xu$, according to the definition of multiplier $\Lambda_1G = D_t(\Phi_1^t) + D_x(\Phi_1^x)$, we have the following conserved vector:

$$\begin{aligned} \Phi_1^t &= t\left(\frac{1}{2}u^4 - u^3 + \frac{1}{2}u^2 - u_tu_x + \frac{1}{2}u_x^2\right) \\ &- \frac{x}{6}u + \frac{x}{6}u^2, \\ \Phi_1^x &= t(-2u^6 + 6u^5 - 6u^4 + 2u^3 + 2u^3u_{xx} \\ &- 3u^2u_{xx} + uu_{xx} - \frac{1}{2}u_x^2) \\ &- \frac{x}{2}u^4 + xu^3 - \frac{x}{2}u^2 - \frac{x}{6}u_{xx} \\ &+ \frac{1}{6}u_x + \frac{x}{3}xuu_{xx} - \frac{x}{2}u_x^2 - uu_x. \end{aligned} \tag{2.9}$$

Similarly, for the multiplier $\Lambda_2 = -2u^3 + 3u^2 + u_{xx}$, $\Lambda_3 = u$ and $\Lambda_4 = 1$, we have

$$\begin{aligned} \Phi_2^t &= -\frac{1}{2}u^4 + u^3 - \frac{1}{2}u_x^2, \\ \Phi_2^x &= 2u^6 - 6u^5 + \frac{9}{2}u^4 + \frac{1}{2}u_{xx}^2 \\ &+ 3u^2u_{xx} - 2u^3u_{xx} + u_xu_t, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \Phi_3^t &= \frac{1}{2}u^2, \\ \Phi_3^x &= -\frac{3}{2}u^4 + 2u^3 - \frac{1}{2}u_x^2 + uu_{xx}, \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \Phi_4^t &= u, \\ \Phi_4^x &= 3u^2 - 2u^3 + u_{xx}. \end{aligned} \tag{2.12}$$

2.2. Landau–Ginzburg–Higgs equation

Consider the LGH equation

$$G = u_{tt} - u_{xx} + g^2u^N = 0, \tag{2.13}$$

where N is an integer and g denotes an arbitrary constant. The LGH equation was introduced by Lev Devidovich Landau and Vitaly Lazarevich Ginzburg in [27], and has a very wide range of applications in radially inhomogeneous plasmas with a constant phase relation of ion-cyclotron waves [28]. It proves the superconductivity and unidirectional wave propagation in nonlinear media [29].

Symmetries of the LGH equation with infinitesimal generator $Xu = \eta$ [25, 26] satisfy the determining equation

$$D_t^2\eta - D_x^2\eta + Ng^2u^{N-1}\eta = 0 \quad \text{when} \quad G = 0. \tag{2.14}$$

Obviously, equation (2.14) is self-adjointed. For any local conservation laws

$$D_t\Phi^t + D_x\Phi^x = 0, \tag{2.15}$$

by moving off the LGH equation solution space, we have

$$D_t\Phi^t + D_x\Phi^x = (u_{tt} - u_{xx} + g^2u^N)\Lambda_0 + D_x(u_{tt} - u_{xx} + g^2u^N)\Lambda_1 + \cdots, \tag{2.16}$$

for some expressions Λ_0, Λ_1, … with no dependence on u_{tt} and differential consequences. This yields the multiplier

$$D_t\Phi^t + D_x(\Phi^x - \Gamma) = (u_{tt} - u_{xx} + g^2u^N), \quad \Lambda = \Lambda_0 - D_x\Lambda_1 + \cdots, \tag{2.17}$$

where $\Gamma = 0$ when u is restricted to be a LGH equation solution. Multipliers Λ are defined by the condition that $(u_{tt} - u_{xx} + g^2u^N)\Lambda$ is a divergence expression for all functions $u(t, x)$. We restrict attention to Λ of first-order, depending on t, x, u, u_t, u_x. This leads to the necessary and sufficient determining condition

$$\begin{aligned} 0 &= E_u((u_{tt} - u_{xx} - m^2u + g^2u^3)\Lambda) \\ &= D_t^2\Lambda - D_x^2\Lambda + Ng^2u^{N-1}\Lambda \\ &\quad + (u_{tt} - u_{xx} + g^2u^N)\Lambda_u \\ &\quad - D_x(\Lambda_{u_x}(u_{tt} - u_{xx} + g^2u^N)) \\ &\quad - D_t(\Lambda_{u_t}(u_{tt} - u_{xx} + g^2u^N)). \end{aligned} \tag{2.18}$$

By solving equations (2.18), we can obtain the general form of the multiplier of the Hirota–Satsuma equation as follows (see appendix B) :

$$\Lambda = a_1(u_x - u_t) + a_2(u_t + u_x) + c_1u_t, \tag{2.19}$$

where a_1, a_2 and c_1 are arbitrary constants. For the LGH equation, here are some multipliers

$$\Lambda_1 = u_x - u_t, \quad \Lambda_2 = u_x + u_t, \quad \Lambda_3 = u_t. \tag{2.20}$$

For the multiplier $\Lambda_1 = u_t + u_x$, according to the definition of multiplier $\Lambda_1 G = D_t(\Phi_1^t) + D_x(\Phi_1^x)$, we have the following conserved vector:

$$\begin{aligned} \Phi_1^t &= u_xu_t - \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - \frac{g^2}{N+1}u^{N+1}, \\ \Phi_1^x &= u_xu_t - \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{g^2}{N+1}u^{N+1}. \end{aligned} \tag{2.21}$$

Similarly, for the multiplier $\Lambda_2 = u_x$ and $\Lambda_3 = u_t$, we have

$$\begin{aligned} \Phi_2^t &= u_xu_t + \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{g^2}{N+1}u^{N+1}, \\ \Phi_2^x &= -u_xu_t - \frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 + \frac{g^2}{N+1}u^{N+1}, \end{aligned} \tag{2.22}$$

and

$$\begin{aligned} \Phi_3^t &= \frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{g^2}{N+1}u^{N+1}, \\ \Phi_3^x &= -u_tu_x. \end{aligned} \tag{2.23}$$

2.3. Hirota–Satsuma equation

Consider the Hirota–Satsuma (HS) equation

$$\begin{cases} G^1 = u_t - 6puu_x + rvv_x - pu_{xxx} = 0, \\ G^2 = v_t + 3puv_x + pv_{xxx} = 0, \end{cases} \tag{2.24}$$

where p and r are arbitrary constants. The HS equation describes the interaction of two long waves with different dispersion relations. It is related to most types of weak dispersive long waves, internal waves, acoustic waves, and planetary waves in geophysical hydrodynamics [30–32].

Next, we consider the linearized equation of equations (2.24)

$$\begin{aligned} (\mathscr{L}_g)_1^1V^1 &= -6pu_xV^1 - 6puD_xV^1 - pD_x^3V^1, \\ (\mathscr{L}_g)_1^2V^1 &= 3pv_xV^1, \\ (\mathscr{L}_g)_2^1V^2 &= 2rv_xV^2 + rvD_xV^2, \\ (\mathscr{L}_g)_2^2V^2 &= 3puD_xV^2 + pD_x^3V^2. \end{aligned} \tag{2.25}$$

And the adjoint system of system (2.25) is given by

$$\begin{aligned} (\mathscr{L}_g^*)_1^1W_1 &= 6puD_xW_1 + pD_x^3W_1, \\ (\mathscr{L}_g^*)_1^2W_2 &= 3pv_xW_2, \\ (\mathscr{L}_g^*)_2^1W_1 &= -rvD_xW_1, \\ (\mathscr{L}_g^*)_2^2W_2 &= -3pu_xW_2 - 3puD_xW_2 - pD_x^3W_2, \end{aligned} \tag{2.26}$$

here $\mathcal{L}_g$ and $\mathcal{L}_g^*$ denote a linearization operator defined as follows:

$$(\mathscr{L}_\Lambda)_{\sigma\rho}V^\rho = \frac{\partial\Lambda_\sigma}{\partial u^\rho}V^\rho + \frac{\partial\Lambda_\sigma}{\partial u_i^\rho}D_iV^\rho + \cdots + \frac{\partial\Lambda_\sigma}{\partial u_{i_1\cdots i_p}^\rho}D_{i_1}\cdots D_{i_p}V^\rho, \tag{2.27}$$

and

$$(\mathscr{L}^*_{\Lambda})_{\sigma\rho} W^{\rho} = \frac{\partial \Lambda_{\rho}}{\partial u^{\sigma}} W^{\rho} - D_i\left(\frac{\partial \Lambda_{\rho}}{\partial u_i^{\sigma}} W^{\rho}\right) + \cdots + (-1)^p D_{i_1} \cdots D_{i_p}\left(\frac{\partial \Lambda_{\rho}}{\partial u^{\sigma}_{i_1\cdots i_p}} W^{\rho}\right). \quad (2.28)$$

Let $\mathscr{D}_t = \partial_t - (g^1\partial_u + g^2\partial_v + \cdots)$, where $g^1 = -6puu_x + rvv_x - pu_{xxx}$, and $g^2 = 3puv_x + pv_{xxx}$. For multipliers of the form $\Lambda(x, t, \boldsymbol{u}, \boldsymbol{u}_x, \boldsymbol{u}_{xx})$ and $\boldsymbol{u}$ means u, v we have

$$\begin{aligned} 0 &= E_{u^{\sigma}}(u_t\Lambda_1 + g^1\Lambda_1 + v_t\Lambda_2 + g^2\Lambda_2) \\ &= -D_t\Lambda_{\sigma} + (\mathscr{L}^*_g)^{\rho}_{\sigma}\Lambda_{\rho} + (\mathscr{L}^*_g)_{\sigma\rho}(u_t^{\rho} + g^{\rho}), \\ \sigma &= 1, 2. \end{aligned} \quad (2.29)$$

By solving equations (2.29), we can obtain the general form of the multiplier of the HS equation as follows (see appendix C):

$$\begin{aligned} \Lambda_1 &= a_1 u_{xx} - \frac{r}{4p} a_1 v^2 + 3a_1 u^2 + k_1, \\ \Lambda_2 &= -\frac{r}{2p} a_1 v_{xx} - \frac{r}{2p} a_1 uv, \end{aligned} \quad (2.30)$$

where a_1 and k_1 are arbitrary constants. So for the HS equation, here are some multipliers

$$\begin{aligned} &\Lambda_{11} = 1, \quad \Lambda_{12} = 0, \\ &\Lambda_{21} = u_{xx} - \frac{r}{4p} v^2 + 3u^2, \\ &\Lambda_{22} = \frac{r}{2p} v_{xx} - \frac{r}{2p} uv. \end{aligned} \quad (2.31)$$

For the multiplier $\Lambda_1 = (1, 0)$, according to the definition of multiplier $\Lambda_i G^i = D_t(\Phi_1^t) + D_x(\Phi_1^x)$, we have the following conserved vector:

$$\begin{aligned} \Phi_1^t &= u, \\ \Phi_1^x &= -3pu^2 + \frac{r}{2} v^2 - pu_{xx}. \end{aligned} \quad (2.32)$$

Similarly, for the multiplier $\Lambda_2 = \left(u_{xx} - \frac{r}{4p}v^2 + 3u^2, \frac{r}{2p}v_{xx} - \frac{r}{2p}uv\right)$, we also have the following conserved vector:

$$\begin{aligned} \Phi_2^t &= \frac{1}{2}uu_{xx} - \frac{r}{12p}uv^2 + u^3 - \frac{r}{4p}vv_{xx} - \frac{r}{6p}uv^2, \\ \Phi_2^x &= \frac{1}{24}(-216p^2u^4 + 36pru^2v^2 - 3r^2v^4 \\ &\quad - 312p^2u^2u_{xx} - 32pruvv_{xx} \\ &\quad + 34prv_x^2 u + 6pru_{xx}v^2 - 34prvv_x u_x \\ &\quad - 24p^2uu_{xxxx} - 24p^2u_{xx}^2 \\ &\quad - 12prvv_{xxxx} + 12prv_x v_{xxx} - 12prv_{xx}^2 + p^2u_x u_{xxx}). \end{aligned} \quad (2.33)$$

Anco *et al* consider the construction of conservation laws in some special cases [35], and here we will consider the conservation laws constructed by the more general case of equations (2.24).

3. Nonlinear self-adjointness and conservation laws

In this section, we consider the nonlinear self-adjointness and the conservation laws of equations (2.1) and (2.24) by the method proposed by Ibragimov [15]. We know that this method cannot produce any new conservation laws unless they are already derived from multipliers. In particular, the multiplier method is complete, that is to say, every nontrivial conservation law comes from a multiplier. This section is mainly to supplement the conservation law that may be missed when the multiplier method is used to find the conservation law.

Firstly, we introduce a general form of systems

$$F_{\alpha}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{(1)}, \ldots, \boldsymbol{u}_{(r)}) = 0, \quad \alpha = 1, 2, \cdots, m, \quad (3.1)$$

where $\boldsymbol{x} = (x^1, x^2, \ldots, x^n)$ denotes n independent variables, $\boldsymbol{u} = (u^1, u^2, \ldots, u^m)$ represents m dependent variables, and $\boldsymbol{u}_{(r)}$ is a class of the partial derivatives of r-th order of $\boldsymbol{u}$. Then an adjoint system of (3.1) is given by

$$F^*_{\alpha}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_{(1)}, \boldsymbol{v}_{(1)}, \ldots, \boldsymbol{u}_{(r)}, \boldsymbol{v}_{(r)}) = 0, \quad \alpha = 1, 2, \cdots, m, \quad (3.2)$$

with

$$F^*_{\alpha}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{u}_{(1)}, \boldsymbol{v}_{(1)}, \ldots, \boldsymbol{u}_{(r)}, \boldsymbol{v}_{(r)}) \equiv \frac{\delta \mathcal{L}}{\delta u^{\alpha}} = 0, \quad \alpha = 1, 2, \cdots, m. \quad (3.3)$$

Here $\mathcal{L} = \sum_{\beta=1}^{m} \boldsymbol{v}^{\beta} F_{\beta}$ is a Lagrangian form of (3.1), with $\boldsymbol{v} = (v^1, \ldots, v^m)$ being new dependent variables. We use $\frac{\delta}{\delta u^{\alpha}}$ for the Euler–Lagrange operator

$$\frac{\delta}{\delta u^{\alpha}} = \frac{\partial}{\partial u^{\alpha}} - D_{i_1}\frac{\partial}{\partial u^{\alpha}_{i_1}} + D_{i_1}D_{i_2}\frac{\partial}{\partial u^{\alpha}_{i_1 i_2}} + \cdots + (-1)^s D_{i_1}D_{i_2}\cdots D_{i_s}\frac{\partial}{\partial u^{\alpha}_{i_1 i_2 \cdots i_s}}. \quad (3.4)$$

Next, the definition of nonlinear self-adjointness and the theorem of conservation laws are given in [15].

Definition 1. The systems of m differential equations (3.1) are said to be nonlinearly self-adjointed if the adjoint equations (3.3) are satisfied for all solutions $\boldsymbol{u}$ of the original system (3.1) upon a substitution

$$v^{\alpha} = \psi^{\alpha}(x, u), \quad \alpha = 1, \cdots, m, \quad (3.5)$$

such that $\psi(x, u) \neq 0$.

Theorem 1. *Let the system of differential equations (3.1) be nonlinearly self-adjoint. Specifically, let the adjoint system (3.3) to (3.1) be satisfied for all solutions of equations (3.1) upon a substitution (3.5). Then any Lie point, contact or Lie-Bäcklund symmetry*

$$X = \xi^i(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{(1)}, \ldots)\frac{\partial}{\partial x^i} + \eta^j(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{u}_{(1)}, \ldots)\frac{\partial}{\partial u^j}, \quad (3.6)$$

as well as a nonlocal symmetry of equations (3.1) leads to a conservation law $D_i C^i|_{(4.1)} = 0$ *constructed by the following* *formulas:*

$$C^i = \xi^i \mathcal{L} + W^\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + D_j D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right) - \cdots \right] + D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}) + \cdots\right] + D_j D_k(W^\alpha) \times \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots\right], \tag{3.7}$$

where

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha, \tag{3.8}$$

and $\mathcal{L}$ *is the formal Lagrangian for the system (3.1).*

Proposition 1. *The Gardner equation is nonlinearly self-adjoint.*

Proof. Taking the multiplier $\mu = \frac{1}{u}$ so that for

$$\frac{1}{u}(u_t + 6uu_x - 6u^2 u_x + u_{xxx}) = 0. \tag{3.9}$$

Then, the adjoint system of equation (3.9) is obtained as follows:

$$\frac{\delta \mathcal{L}}{\delta u} = -\frac{\omega}{u^2}(u_t + 6uu_x - 6u^2u_x + u_{xxx}) - D_t\left(\frac{\omega}{u}\right) - D_x(6\omega) + 6u_x\frac{\omega}{u} - 12u_x\omega + D_x(6u\omega) - D_x^3\left(\frac{\omega}{u}\right), \tag{3.10}$$

with the formal Lagrangian

$$\mathcal{L} = \frac{\omega}{u}(u_t + 6uu_x - 6u^2u_x + u_{xxx}). \tag{3.11}$$

Substituting $\omega = u$ into (3.4), one has

$$\frac{\delta \mathcal{L}}{\delta u} = -\frac{1}{u}(u_t + 6uu_x - 6u^2u_x + u_{xxx}). \tag{3.12}$$

By comparing (3.12) and (3.9), we find that equation (2.1) is strictly self-adjoint. Thus, the Gardner equation is nonlinearly self-adjoint.The Gardner equation admits three infinitesimal generators as follows:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \left(t + \frac{x}{3}\right)\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + \left(-\frac{u}{3} + \frac{1}{6}\right)\frac{\partial}{\partial u}. \tag{3.13}$$

The conservation laws of symmetry (3.13) have forms as follows:

$$[D_t C^t + D_x C^x]|_{(3.1)} = 0, \tag{3.14}$$

where the conserved vector $C = (C^t, C^x)$ is given by (3.7).

For the generator X_1, the corresponding Lie characteristic function is given by $W = -u_t$. Thus, formulas (3.7) yield the following conserved vector:

$$\begin{aligned} C^t &= \xi^t \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_t}\right] \\ &= v(u_t + 6uu_x - 6u^2u_x + u_{xxx}) - u_t v, \\ C^x &= \xi^x \mathcal{L} + W\left[\frac{\partial \mathcal{L}}{\partial u_x} + D_x^2 \frac{\partial \mathcal{L}}{\partial U_{xxx}}\right] + D_x W\left[-D_x\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right] + D_x^2 W\left[\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right] \\ &= -u_t(6uv - 6u^2v + v_{xx}) + u_{tx}v_x - u_{txx}v. \end{aligned} \tag{3.15}$$

Similarly, for the generator X_2, we have $W = -u_x$. Thus, formulas (3.7) yield the following conserved vector

$$\begin{aligned} C^t &= -u_x v, \\ C^x &= v(u_t + 6uu_x - 6u^2u_x + u_{xxx}) \\ &\quad - u_x(6uv - 6u^2v + v_{xx}) + u_{xx}v_x - u_{xxx}v. \end{aligned} \tag{3.16}$$

For the generator X_3, we have $W = -\frac{u}{3} + \frac{1}{6} - \left(t + \frac{x}{3}\right)u_x - tu_t$. According to formula (3.7), we have the conservation vector of the Gardner equation as follows:

$$\begin{aligned} C^t &= t(u_t + 6uu_x - 6u^2u_x + u_{xxx})v \\ &\quad - \left(\frac{u}{3} - \frac{1}{6} + \left(t + \frac{x}{3}\right)u_x + tu_t\right)v, \\ C^x &= \left(t + \frac{x}{3}\right)v(u_t + 6uu_x - 6u^2u_x + u_{xxx}) \\ &\quad + \left(-\frac{u}{3} + \frac{1}{6} - \left(t + \frac{x}{3}\right)u_x - tu_t\right)(6uv \\ &\quad - 6u^2v + v_{xx}) - \left(-\frac{2}{3}u_x - \left(t + \frac{x}{3}\right)u_{xx} - tu_{tx}\right)v_x \\ &\quad + \left(-u_{xx} - \left(t + \frac{x}{3}\right)u_{xxx} - tu_{txx}\right)v. \end{aligned} \tag{3.17}$$

For the generators X_1 and X_2, the translational symmetry yields trivial conservation laws. Compared with the conservation law obtained by the multiplier method, there is in fact no new conservation law, but it may be useful for us to study the problem later.

For the LGH equation, it is easy to prove that it is not nonlinearly self-adjoint. According to Theorem 3, it cannot be obtained by formula (3.7). For the HS equation, let $r = 3p$, and then equation (2.24) becomes the following form:

$$\begin{cases} G^1 = u_t - 6puu_x + 3pvv_x - pu_{xxx} = 0, \\ G^2 = v_t + 3puv_x + pv_{xxx} = 0. \end{cases} \tag{3.18}$$

Proposition 2. *The HS equation is nonlinearly self-adjoint.*

Proof. The formal Lagrangian of the HS equation (3.18) can be written as

$$\mathcal{L} = \omega^1(u_t - 6puu_x + 3pvv_x - pu_{xxx}) + \omega^2(v_t + 3puv_x + pv_{xxx}). \quad (3.19)$$

Then, one has

$$\frac{\delta\mathcal{L}}{\delta u} = -\omega_t^1 + 6pu\omega_x^1 + p\omega_{xxx}^1 - 3pv_x\omega^2,$$
$$\frac{\delta\mathcal{L}}{\delta v} = -3pv\omega_x^1 - \omega_t^2 + 3pu_x\omega^2 + 3pu\omega_x^2 - p\omega_{xxx}^2. \quad (3.20)$$

Substituting $\omega^1 = u$ and $\omega^2 = v$ into equation (3.20), we have

$$\frac{\delta\mathcal{L}}{\delta u} = -(u_t - 6puu_x + 3pvv_x - pu_{xxx}),$$
$$\frac{\delta\mathcal{L}}{\delta v} = -(v_t + 3puv_x + pv_{xxx}). \quad (3.21)$$

Hence, the special HS equation is nonlinearly self-adjoint.□

The special HS equation admits three infinitesimal generators as follows:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{x}{3}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{2}{3}u\frac{\partial}{\partial u} - \frac{2}{3}v\frac{\partial}{\partial v}. \quad (3.22)$$

The corresponding conservation law is limited by

$$[D_t C^t + D_x C^x]|_{(4.21)} = 0, \quad (3.23)$$

where $C = (C^t, C^x)$ denotes the conserved vector of (3.7). For the generator, the corresponding Lie characteristic function is given by $W^1 = -u_t$, $W^2 = -v_t$. One can find the conservation vector of equation (3.18) given by

$$\begin{aligned} C^t &= \xi^t\mathcal{L} + W^1\left[\frac{\partial\mathcal{L}}{\partial u_t}\right] + W^2\left[\frac{\partial\mathcal{L}}{\partial v_t}\right] \\ &= \omega^1(u_t - 6puu_x + 3pvv_x - pu_{xxx}) \\ &\quad + \omega^2(v_t + 3puv_x + pv_{xxx}) - u_t\omega_1 - v_t\omega_2, \\ C^x &= \xi^x\mathcal{L} + W^1\left[\frac{\partial\mathcal{L}}{\partial u_x} + D_x^2\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right] \\ &\quad + D_xW^1\left[-D_x\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right] + D_x^2W^1\frac{\partial\mathcal{L}}{\partial u_{xxx}} \\ &\quad + W^2\left[\frac{\partial\mathcal{L}}{\partial v_x} + D_x^2\frac{\partial\mathcal{L}}{\partial v_{xxx}}\right] \\ &\quad + D_xW^2\left[-D_x\frac{\partial\mathcal{L}}{\partial v_{xxx}}\right] + D_x^2W^2\frac{\partial\mathcal{L}}{\partial v_{xxx}} \\ &= u_t(6pu\omega^1 + pw_{xx}^1) - pu_{xt}\omega_x^1 + pu_{xxt}\omega^1 \\ &\quad - v_t(3pv\omega^1 - 3pu\omega^2 + p\omega_{xx}^2) \\ &\quad + pv_{xt}\omega_x^2 - pv_{xxt}\omega^2. \end{aligned} \quad (3.24)$$

For the generator X_2, we have $W^1 = -u_x$, $W^2 = -v_x$. Thus, formulas (3.7) yield the following conserved vector

$$\begin{aligned} C^t &= -u_x\omega^1 - v_x\omega^2, \\ C^x &= \omega^1(u_t - 6puu_x + 3pvv_x - pu_{xxx}) + \omega^2 \\ &\quad \times (v_t + 3puv_x + pv_{xxx}) + u_x(6pu\omega^1 + pw_{xx}^1) \\ &\quad - pu_{xx}\omega_x^1 + pu_{xxx}\omega^1 - v_x(3pv\omega^1 - 3pu\omega^2 + p\omega_{xx}^2) \\ &\quad + pv_{xx}\omega_x^2 - pv_{xxx}\omega^2. \end{aligned} \quad (3.25)$$

For the generator X_3, we have $W^1 = -\left(\frac{2}{3}u + \frac{x}{3}u_x + \frac{2t}{3}u_t\right)$, $W^2 = -\left(\frac{2}{3}v + \frac{x}{3}v_x + \frac{2t}{3}v_t\right)$. Thus, formulas (3.7) yield the following conserved vector

$$\begin{aligned} C^t &= t[\omega^1(u_t - 6puu_x + 3pvv_x - pu_{xxx}) \\ &\quad + \omega^2(v_t + 3puv_x + pv_{xxx})] \\ &\quad - \left(\frac{2}{3}u + \frac{x}{3}u_x + \frac{2t}{3}u_t\right)\omega^1 \\ &\quad - \left(\frac{2}{3}v + \frac{x}{3}v_x + \frac{2t}{3}v_t\right)\omega^2, \\ C^x &= \frac{x}{3}[\omega^1(u_t - 6puu_x + 3pvv_x - pu_{xxx}) \\ &\quad + \omega^2(v_t + 3puv_x + pv_{xxx})] \\ &\quad - \left(\frac{2}{3}u + \frac{x}{3}u_x + \frac{2t}{3}u_t\right)(-6pu\omega^1 + p\omega_{xx}^1) \\ &\quad - p\left(u_x + \frac{x}{3}u_x + \frac{2}{3}tu_{tx}\right)\omega_x^1 \\ &\quad - \left(\frac{4}{3}u_{xx} + \frac{x}{3}u_{xxx} + \frac{2}{3}tu_{txx}\right)\omega^1 - \left(\frac{2}{3}v + \frac{x}{3}v_x \right.\\ &\quad \left. + \frac{2t}{3}v_t\right)(3pv\omega^1 - 3pu\omega^2 + p\omega_{xx}^2) \\ &\quad + p\left(v_x + \frac{x}{3}v_{xx} + \frac{2t}{3}v_{tx}\right)\omega_x^2 - p\left(\frac{4}{3}v_{xx}\right. \\ &\quad \left. + \frac{x}{3}v_{xxx} + \frac{2t}{3}v_{txx}\right)\omega^2. \end{aligned} \quad (3.26)$$

According to the above calculation, we can find that the Gardner equation has infinite polynomial conservation laws, while the HS equation has only three conservation laws, whether using the above method or other methods.

4. Conclusion and discussion

In this paper, we use two methods to calculate the conservation laws, the direct construction of conservation laws and the conservation theorem method proposed by Ibragimov, and construct the conservation laws of the Gardner equation, the LGH equation, and the Hirota–Satsuma equation. The conservation theorem method is more widely used than the Noether theorem. It does not require the equation system to have a Lagrangian function, but it does require it to be nonlinearly self-adjoint. The direct construction conservation law method has no requirement for the equation system, but the calculation of its multiplier is more complicated. Compared with the Noether theorem, the multiplier method is complete. In fact, every conservation law is generated by multipliers. In contrast, the use of symmetric, adjoint symmetric formulas, partial Lagrangians, and other methods is usually incomplete. They either do not apply to every partial differential equations system, or do not necessarily produce all conservation laws. The multiplier method generates all conservation laws for any partial differential equations system, regardless of whether the system has a Lagrangian function.

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Appendix A

By comparing the coefficient G and D_{xG} of equation (2.6), it is found that the system consists of the adjoint symmetry determining equations on Λ

$$-\mathscr{D}_t\Lambda - 6uD_x\Lambda + 6u^2D_x\Lambda - D_x^3\Lambda = 0, \tag{A1}$$

and extra determining equations on Λ,

$$-D_x\Lambda_{u_x} + D_x^2\Lambda_{u_{xx}} = 0, \tag{A2}$$

$$\Lambda_{u_x} - D_x\Lambda_{u_{xx}} = 0, \tag{A3}$$

where $\mathscr{D}_t = \partial_t - (6uu_x - 6u^2u_x + u_{xxx})\partial_u - (6uu_x - 6u^2u_x + u_{xxx})_x \partial_{u_x} + \cdots$ is the total derivative operator, which expresses t derivatives of u through the Gardner equation. By expanding equation (A3), we find that the highest order derivative term is $u_{xxx}\Lambda_{u_{xx}u_{xx}}$, and hence $u_{xxx}\Lambda_{u_{xx}u_{xx}} = 0$, so that

$$\Lambda = a(t, x, u, u_x)u_{xx} + b(t, x, u, u_x). \tag{A4}$$

Substitute Λ in the above formula into (A3), we can get

$$b_{u_x} - a_u u_x - a_x = 0. \tag{A5}$$

By comparing the coefficients u_{xxxx}, u_{xxx} and so on in equation (A1), we can get the following equations

$$\begin{aligned}
&3\Lambda_{xu_x} + 3u_{xx}\Lambda_{u_xu_{xx}} + 3u_x\Lambda_{uu_{xx}} = 0,\\
&6u_x\Lambda_{xuu_{xx}} + 3u_x^2\Lambda_{uuu_{xx}} + 3\Lambda_{xu_x} + 3\Lambda_{xxu_x} + 3u_{xx}\Lambda_{uu_{xx}} = 0,\\
&\Lambda_t - (6uu_x - 6u^2u_x)\Lambda_u - (18u_xu_{xx} - 12u_x^3\\
&-36uu_xu_{xx})\Lambda_{u_{xx}} + (6u - 6u^2)(\Lambda_x + u_x\Lambda_u)\\
&-(6u_x^2 - 12uu_x^2)\Lambda_{u_x} + 3u_xu_{xx}\Lambda_{uu} + 3u_x\Lambda_{xxu}\\
&+3u_{xx}\Lambda_{xu} + 3u_x^2\Lambda_{xuu} + u_x^3\Lambda_{uuu} + \Lambda_{xxx} = 0.
\end{aligned}$$

By solving the above equations and equations (A4) and (A5), we can obtain the general form of Λ as follows:

$$\Lambda = c_1[(2u^3 - 3u^2 + u - u_{xx})t - \frac{x}{6} + \frac{1}{3}xu] + c_2[-2u^3 + 3u^2 + u_{xx}] + c_3u + c_4, \tag{A6}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Appendix B

Equation (2.18) leads to a split system of two determining equations for $\Lambda(t, x, u, u_t, u_x)$, consisting of

$$\mathscr{D}_t^2\Lambda - D_x^2\Lambda + Ng^2u^{N-1}\Lambda = 0, \tag{B1}$$

which is the symmetry determining equation (2.14) on Λ, and

$$2\Lambda_u + \mathscr{D}_t\Lambda_{u_t} - D_x\Lambda_{u_x} = 0, \tag{B2}$$

which is an extra determining equation on Λ. Here $D_t = \partial_t + u_t\partial_u + u_{tx}\partial_{u_x} + (u_{tt} - u_{xx} + g^2u^N)\partial_{u_t} + \cdots$ is the total derivative operator which expresses t derivatives through the LGH equation. And from equation (B2), we can get

$$\begin{aligned}&2\Lambda_u + \Lambda_{tu_t} + u_t\Lambda_{uu_t} + (u_{xx} - g^2u^N)\Lambda_{u_tu_t}\\ &-\Lambda_{xu_x} - u_x\Lambda_{uu_x} - u_{xx}\Lambda_{u_xu_x} = 0.\end{aligned} \tag{B3}$$

Since Λ does not contain u_{xx}, we have

$$\Lambda_{u_tu_t} = \Lambda_{u_xu_x}, \tag{B4}$$

$$2\Lambda_u + \Lambda_{tu_t} + u_t\Lambda_{uu_t} - g^2u^N\Lambda_{u_tu_t} - \Lambda_{xu_x} - u_x\Lambda_{uu_x} = 0. \tag{B5}$$

We next consider equation (B1), comparison coefficient u_{xx}, we can get

$$-2u_x\Lambda_{uu_x} - 2\Lambda_{xu_x} + 2\Lambda_{tu_t} + 2u_t\Lambda_{uu_t} - 2g^2u^N\Lambda_{u_tu_t} = 0. \tag{B6}$$

Combining equations (B5) and (B6), we have

$$4\Lambda_u = 0. \tag{B7}$$

Compare the coefficients of u_{xt} in equation (B1), we have

$$-2\Lambda_{xu_t} + 2\Lambda_{tu_x} - 2g^2u^N\Lambda_{u_tu_x} = 0. \tag{B8}$$

By solving equations (B4), (B5), (B7) and (B8), we can get the general form of multiplier is

$$\begin{aligned}\Lambda &= (u_x - u_t)F_1(t - x)\\ &+(u_t + u_x)F_2(t + x) + c_1u_t + F_3(x, t),\end{aligned} \tag{B9}$$

where $F_1(x + t)$ is an arbitrary function of $x + t$, $F_2(t)$ is an arbitrary function of t, F_3 is only related to x, t. Substitute equations (B9) into (B1), we can get

$$\begin{aligned}&-(u_t + u_x)F_{1tt} + (u_t + u_x)F_{2tt} + F_{3tt}\\ &+(-u_t + u_x)F_{1xx} - (u_t + u_x)F_{2xx}\\ &-F_{3xx} + ((-u_t + u_x)F_1 + (u_t + u_x)F_2\\ &+c_1u_t + F_3)Ng^2u^{N-1} = 0,\end{aligned} \tag{B10}$$

Equation (B10) separates into

$$F_2'' = 0, \quad F_1' = 0, \quad F_3 = 0, \tag{B11}$$

so we can obtain $F_1 = a_1 = \text{constant}$, $F_2 = a_2 = \text{constant}$ and $F_3 = 0$. Therefore, we can get the specific form of multipliers

$$\Lambda = a_1(u_x - u_t) + a_2(u_t + u_x) + c_1 u_t, \tag{B12}$$

where a_1, a_2 and c_1 are arbitrary constants.

Appendix C

Substitute equations (2.26) into (2.29) when σ is 1, 2, respectively. We have

$$\begin{aligned} &- D_t\Lambda_1 + 6puD_x\Lambda_1 + pD_x x^3\Lambda_1 + 3pv_x\Lambda_2 \\ &+ \left(\frac{\partial\Lambda_1}{\partial u}G^1 - D_x\left(\frac{\partial\Lambda_1}{\partial u_x}G^1\right)\right. \\ &\left. + D_x^2\left(\frac{\partial\Lambda_1}{\partial u_{xx}}G^1\right)\right) + \left(\frac{\partial\Lambda_2}{\partial u}G^2 - D_x\right. \\ &\left. \times\left(\frac{\partial\Lambda_2}{\partial u_x}G^2\right) + D_x^2\left(\frac{\partial\Lambda_2}{\partial u_{xx}}G^2\right)\right) = 0, \end{aligned} \tag{C1}$$

$$\begin{aligned} &- D_t\Lambda_2 - rvD_x\Lambda_1 - 3p\Lambda_2 - 3puD_x\Lambda_2 - pD_x^3\Lambda_2 \\ &+ \left(\frac{\partial\Lambda_1}{\partial v}G^1 - D_x\left(\frac{\partial\Lambda_1}{\partial v_x}G^1\right)\right. \\ &\left. + D_x^2\left(\frac{\partial\Lambda_1}{\partial v_{xx}}G^1\right)\right) + \left(\frac{\partial\Lambda_2}{\partial v}G^2 - D_x\left(\frac{\partial\Lambda_2}{\partial v_x}G^2\right)\right. \\ &\left. + D_x^2\left(\frac{\partial\Lambda_2}{\partial v_{xx}}G^2\right)\right) = 0. \end{aligned} \tag{C2}$$

Consequently, the non-leading terms in equation (2.29) are given by

$$-\mathscr{D}_t\Lambda_1 + 6puD_x\Lambda_1 + pD_x x^3\Lambda_1 + 3pv_x\Lambda_2 = 0, \tag{C3}$$

$$-\mathscr{D}_t\Lambda_2 - rvD_x\Lambda_1 - 3p\Lambda_2 - 3puD_x\Lambda_2 - pD_x^3\Lambda_2 = 0. \tag{C4}$$

Comparing the coefficients of derivatives of G and G, we can get the leading term in equation (2.29)

$$-\frac{\partial\Lambda_1}{\partial v_{xx}} + \frac{\partial\Lambda_2}{\partial u_{xx}} = 0, \tag{C5}$$

$$2\frac{\partial\Lambda_1}{\partial u_x} - 2D_x\frac{\partial\Lambda_1}{\partial u_{xx}} = 0, \quad 2\frac{\partial\Lambda_2}{\partial v_x} - 2D_x\frac{\partial\Lambda_2}{\partial v_{xx}} = 0, \tag{C6}$$

$$\begin{aligned} &\frac{\partial\Lambda_1}{\partial v_x} + \frac{\partial\Lambda_2}{\partial u_x} - 2D_x\frac{\partial\Lambda_2}{\partial u_{xx}} = 0, \\ &\times\frac{\partial\Lambda_2}{\partial u_x} + \frac{\partial\Lambda_1}{\partial v_x} - 2D_x\frac{\partial\Lambda_1}{\partial v_{xx}} = 0, \end{aligned} \tag{C7}$$

$$-D_x\frac{\partial\Lambda_1}{\partial u_x} + D_x^2\frac{\partial\Lambda_1}{\partial u_{xx}} = 0, \quad -D_x\frac{\partial\Lambda_2}{\partial v_x} + D_x^2\frac{\partial\Lambda_2}{\partial v_{xx}} = 0, \tag{C8}$$

$$\begin{aligned} &-\frac{\partial\Lambda_1}{\partial v} + \frac{\partial\Lambda_2}{\partial u} - D_x\frac{\partial\Lambda_2}{\partial u_x} + D_x^2\frac{\partial\Lambda_2}{\partial u_{xx}} = 0, \\ &-\frac{\partial\Lambda_2}{\partial u} + \frac{\partial\Lambda_1}{\partial v} - D_x\frac{\partial\Lambda_2}{\partial v_x} + D_x^2\frac{\partial\Lambda_2}{\partial v_{xx}} = 0. \end{aligned} \tag{C9}$$

Consider equation (C6). Its highest-order term is $u_{xxx}\Lambda_{1u_{xx}u_{xx}}$ and $v_{xxx}\Lambda_{2v_{xx}v_{xx}}$, hence $\Lambda_{1u_{xx}u_{xx}} = 0$ and $\Lambda_{2v_{xx}v_{xx}} = 0$. Similarly, we have $\Lambda_{1v_{xx}v_{xx}} = 0$ and $u_{xxx}\Lambda_{1u_{xx}u_{xx}}$ from equation (C7). This means that Λ_1 and Λ_2 are linear with respect to u_{xx} and v_{xx}, combining equation (C5), we have the general form of Λ_1 and Λ_2

$$\begin{aligned} \Lambda_1 &= a_1(t, x, \boldsymbol{u}, \boldsymbol{u}_x)u_{xx} + b_1(t, x, \boldsymbol{u}, \mathbf{u}_x)v_{xx} \\ &\quad + c_1(t, x, \boldsymbol{u}, \boldsymbol{u}_x), \\ \Lambda_2 &= b_1(t, x, \boldsymbol{u}, \boldsymbol{u}_x)u_{xx} + b_2(t, x, \boldsymbol{u}, \boldsymbol{u}_x)v_{xx} \\ &\quad + c_2(t, x, \boldsymbol{u}, \boldsymbol{u}_x), \end{aligned} \tag{C10}$$

Then the remaining terms in equation (C6), after some cancellations,

$$\begin{aligned} &b_{1u_x}v_{xx} + c_{1u_x} - (a_{1x} + a_{1u}u_x + a_{1v}v_x + a_{1v_x}v_{xx}) = 0, \\ &b_{1v_x}v_{xx} + c_{2v_x} - (b_{2x} + b_{2u}u_x + b_{2v}v_x + b_{2u_x}u_{xx}) = 0. \end{aligned} \tag{C11}$$

According to the coefficients of u_{xxxx} and v_{xxxx} of equations (C3) and (C4), we have

$$\begin{aligned} &a_{1v_x}u_{xx} + b_{1v_x}v_{xx} + c_{1v_x} = 0, \\ &b_{1u_x}u_{xx} + b_{2u_x}u_{xx} + c_{2u_x} = 0. \end{aligned} \tag{C12}$$

Clearly, equation (C12) separates into

$$a_{1v_x} = b_{1v_x} = c_{1v_x} = 0, \quad b_{1u_x} = b_{2u_x} = c_{2u_x} = 0. \tag{C13}$$

Hence $a_{1x} = a_{1u} = a_{1v} = 0$, $b_{2x} = b_{2u} = b_{2v} = 0$ from equation (C11). Similarly, we have $a_{1u_x} = b_{1x} = b_{1u} = b_{1v} = b_{2v_x} = 0$. That is to say, a_1, b_1 and b_2 are independent of x, $\boldsymbol{u}$, $\boldsymbol{u}_x$, equation (C10) becomes

$$\begin{aligned} \Lambda_1 &= a_1(t)u_{xx} + b_1(t)v_{xx} + c_1(t, x, \boldsymbol{u}), \\ \Lambda_2 &= b_1(t)u_{xx} + b_2(t)v_{xx} + c_2(t, x, \boldsymbol{u}), \end{aligned} \tag{C14}$$

Next, we solve the concrete forms of a_1, b_1, b_2, c_1, c_2. The coefficient of u_{xxxxx} from equation (C3), we have

$$2pb_1(t) = 0, \tag{C15}$$

hence $b_1(t) = 0$. Comparing the coefficient v_{xxx} in equation (C3) with the coefficient u_{xxx} in equation (C4), we have

$$2pc_{1_v} + rva_1(t) = 0, \quad 2pc_{2_u} + rva_1(t) = 0. \tag{C16}$$

It yields that

$$\begin{aligned} c_1 &= -\frac{r}{4p}v^2 a_1(t) + k_1(x, t, u), \\ c_2 &= -\frac{r}{2p}uva_1(t) + l_1(x, t, v). \end{aligned} \tag{C17}$$

Furthermore, by comparing the coefficients of $\boldsymbol{u}_x$, $\boldsymbol{u}_{xx}$, and constant terms in equations (C3) and (C4), we can obtain the following equations:

$$-a'_1(t) - 18a_1(t)pu_x + 3pu_x c_{1uu} + 3pc_{1xu} = 0, \tag{C18}$$

$$\frac{r}{2}a_1(t) + pb_2(t) = 0, \tag{C19}$$

$$b'_2(t) - 3pb_2(t) + 3pc_{2uv} = 0, \tag{C20}$$

$$9puc_{2u} + rvc_{1u} + 3pc_2 = 0, \tag{C21}$$

$$-c'_1(t) + 6puc_{1x} + pc_{1xxx} = 0, \tag{C22}$$

$$c_{2t} + rvc_{1x} + 3puc_{2x} + pc_{2xxx} = 0, \tag{C23}$$

$$c_{1xv} = c_{2xu} = c_{2vv} = 0. \tag{C24}$$

Simultaneous equations (C17)–(C24), we can get

$$c_1 = -\frac{r}{4p}a_1v^2 + 3a_1u^2 + k_1,$$
$$c_2 = -\frac{r}{2p}a_1uv, \quad b_2 = -\frac{r}{2p}a_1, \tag{C25}$$

where a_1 and k_1 are arbitrary constants. Therefore, we can get the specific form of multipliers

$$\Lambda_1 = a_1u_{xx} - \frac{r}{4p}a_1v^2 + 3a_1u^2 + k_1,$$
$$\Lambda_2 = -\frac{r}{2p}a_1v_{xx} - \frac{r}{2p}a_1uv, \tag{C26}$$

where a_1 and k_1 are arbitrary constants.

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