 Communications in Theoretical Physics

Commun. Theor. Phys. 76 (2024) 065004 (11pp) https://doi.org/10.1088/1572-9494/ad3b8b

Decomposition solutions and Bäcklund transformations of the B-type and C-type Kadomtsev–Petviashvili equations

Xiazhi Hao[1] and **S Y Lou**[2,*]

[1] College of Science, Zhejiang University of Technology, Hangzhou 310014, China
[2] School of Physical Science and Technology, Ningbo University, Ningbo 315211, China

E-mail: lousenyue@nbu.edu.cn

Received 8 January 2024, revised 8 March 2024
Accepted for publication 7 April 2024
Published 17 May 2024

Abstract

This paper introduces a modified formal variable separation approach, showcasing a systematic and notably straightforward methodology for analyzing the B-type Kadomtsev–Petviashvili (BKP) equation. Through the application of this approach, we successfully ascertain decomposition solutions, Bäcklund transformations, the Lax pair, and the linear superposition solution associated with the aforementioned equation. Furthermore, we expand the utilization of this technique to the C-type Kadomtsev–Petviashvili (CKP) equation, leading to the derivation of decomposition solutions, Bäcklund transformations, and the Lax pair specific to this equation. The results obtained not only underscore the efficacy of the proposed approach, but also highlight its potential in offering a profound comprehension of integrable behaviors in nonlinear systems. Moreover, this approach demonstrates an efficient framework for establishing interrelations between diverse systems.

Keywords: decomposition solution, Bäcklund transformation, Lax pair, linear superposition solution

1. Introduction

The pursuit of exact solutions for nonlinear partial differential equations (PDEs) has spurred the development of a multitude of methodologies, each offering unique insights and approaches. From the inverse scattering method [1–3] to Hirota's method [4–8], and the group theoretical method [9] to direct methods [10–20], researchers have explored diverse avenues to unravel the complexities of nonlinear PDEs. Central to these methodologies is the Bäcklund transformation (BT), a powerful analytical tool that establishes crucial connections between solutions of specific nonlinear PDEs, thereby advancing the field of integrability [21, 22]. The rich tapestry of research surrounding BTs underscores their enduring appeal to mathematicians and physicists alike. With active developments spanning differential geometry, algebra, and nonlinear science, BTs have emerged as a cornerstone in addressing integrability challenges across various disciplines. The wealth of literature dedicated to BTs reflects the profound interest and significance attributed to these transformative tools [23–26].

Within this landscape of methodologies, the decomposition method based on the formal variable separation approach (FVSA) emerges as a notable approach for constructing BTs [27–29]. In this paper, we introduce a novel decomposition technique in section 2, that not only enables the construction of BTs, but also demonstrates remarkable efficacy in tackling two prominent nonlinear PDEs: the B-type and C-type Kadomtsev–Petviashvili equations (BKP and CKP). Our assertion of the straightforward and natural approach to constructing BTs heralds novel discoveries with potential applications across a spectrum of equations. Sections 3 and 4 of this paper delve into a detailed exploration of the utilization of the decomposition method in analyzing the potential BKP equation and potential CKP equation, respectively. Through a systematic investigation, we unveil BTs and Lax pairs for both equations in a highly intuitive manner, underscoring the

[*] Author to whom any correspondence should be addressed.

coherence and significance of the results. The paper culminates in a comprehensive summary of the findings in section 5, with intricate calculations thoughtfully relocated to the appendix to enhance readability and focus on the key insights derived from the application of the decomposition method.

2. Modified formal variable separation approach

The modified FVSA outlined in this section focuses on deriving the general decomposition solution

$$w_y = F(x,\, y,\, t,\, w,\, w_x,\, w_{xx},\ldots,w_{xm}), \tag{1}$$

$$w_t = G(x,\, y,\, t,\, w,\, w_x,\, w_{xx},\ldots,w_{xn}) \tag{2}$$

for a (2+1)-dimensional PDE. The key objective is to maintain the integrability condition

$$w_{yt} - w_{ty} = [F,\, G] = 0, \tag{3}$$

while seeking a solution in the form of equations (1) and (2). Here, w is a function of x, y, t, with F and G depending on these variables and derivatives of w with respect to x.

The method involves a systematic procedure:

(1) Determination of admissible values: by analyzing the terms in the (2+1)-dimensional PDE and comparing their magnitudes, suitable values of integers m and n in equations (1) and (2) are identified to prioritize specific terms over others.
(2) Substitution analysis: substituting the decomposed forms into the PDE and the integrability condition (3), expressions for w and its derivatives are derived, leading to an overdetermined system of equations.
(3) Solutions for F and G: by solving the system of equations from step (2), the functions F and G are determined, resulting in the desired decomposition solutions, the corresponding BT and Lax pair.

The systematic methodology not only guarantees the maintenance of integrability, but also establishes a well-defined framework for constructing BTs and Lax pairs. The upcoming application of this technique is primed to shed light on the dynamics of (2+1)-dimensional PDEs, demonstrating the effectiveness and potential of the modified FVSA.

3. The decomposition solutions, Bäcklund transformation, Lax pair and linear superposition solutions of the potential BKP equation

An illustrative starting point often resides in examples. Let us consider the BKP equation

$$\begin{aligned} &u_{xt} + (u_{x4} + 15uu_{x2} + 15u^3 - 15uv \\ &- 5u_{xy})_{xx} - 5u_{yy} = 0,\ v_x = u_y, \end{aligned} \tag{4}$$

where $u_x = \partial_x u$, $u_{x2} = \partial_x^2 u$, $u_{x3} = \partial_x^3 u,\ldots$, a (2+1)-dimensional extension of the Sawada–Kotera (SK) equation, renowned for its manifestation of elegant integrable properties within the realm of academic discourse [30, 31]. Analytical treatment of a potential function w is advantageous, achieved by defining $u = w_x$ within the context of equation (4). Consequently, the equation governing w can be expressed as follows:

$$\begin{aligned} w_{xt} = 5w_{yy} - (&w_{x5} + 15w_x w_{x3} \\ &+ 15w_x^3 - 15w_x w_y - 5w_{xxy})_x. \end{aligned} \tag{5}$$

The analysis of the potential BKP equation (5) reveals that for the chosen leading orders $m = 3$ and $n = 5$, the expressions of w_y and w_t can be determined in terms of

$$w_y = F(x,\, y,\, t,\, w,\, w_x,\, w_{xx},\, w_{xxx}), \tag{6}$$

$$w_t = G(x,\, y,\, t,\, w,\, w_x,\, w_{xx},\, w_{xxx},\, w_{xxxx},\, w_{xxxxx}). \tag{7}$$

Consequently, it becomes necessary to impose the integrability condition

$$F_t = G_y \tag{8}$$

on equations (6) and (7) while ensuring the satisfaction of equation (5). The derivation of the functions F and G is straightforward but involves a detailed procedure, which is provided in the appendix for reference. The outcome of the calculations leads directly to the formulation of the following theorems.

Theorem 1. *If w satisfies the decomposition system of the consistent variable coefficient potential Korteweg–de Vries (KdV) equation*

$$v_y = v_{xxx} - \frac{a}{2}v_x^2 + \frac{m_0}{a}x + n_y, \tag{9}$$

$$\begin{aligned} v_t &= 9v_{xxxxx} - 15av_x v_{xxx} - \frac{15a}{2}v_{xx}^2 \\ &+\frac{5a^2}{2}v_x^3 + \left[15m_2 + 15m_1 y - \left(5m_0^2 + \frac{1}{2}m_{0t}\right)y^2\right. \\ &\left.-5m_0 x\right]v_x - 5m_0 v + [(10m_0^2 + m_{0t})y \\ &-15m_1]\frac{x}{a} + \frac{y^3}{6a}m_0(10m_0^2 + m_{0t}) \\ &-\frac{15y^2}{2a}m_0 m_1 - \frac{15y}{a}m_0 m_2 \\ &+5m_0 n + n_t + m_3, \end{aligned} \tag{10}$$

$$\begin{aligned} w_y &= w_{xxx} + 3w_x^2 + av_x w_x \\ &+\frac{a^2}{6}v_x^2 - \frac{1}{3}m_0(x + m_0 y^2) \\ &-\frac{m_{0t}y^2}{30} + m_1 y + m_2, \end{aligned} \tag{11}$$

$$
\begin{aligned}
w_t =\ & 9w_{xxxxx} + 15(av + 6w)_x w_{xxx} \\
& + 5a(av + 3w)_x v_{xxx} + 15(av + 3w)_{xx} w_{xx} \\
& + \frac{15}{2} w_x[(av + 3w)_x^2 + 3w_x^2] \\
& + \frac{5a^2}{2} v_{xx}^2 + w_x(15m_2 + 15m_1 y \\
& - 5m_0^2 y^2 - 5m_0 x - \frac{1}{2} m_{0t} y^2) - 5m_0 w \\
& - \frac{xy}{3}(m_{0t} + 10m_0^2) + 5m_1 x \\
& - \frac{y^3}{90}(m_{0tt} + 30m_0 m_{0t} + 100m_0^3) \\
& + \frac{y^2}{2}(m_{1t} + 10m_0 m_1) \\
& + (m_{2t} + 10m_0 m_2)y + m_4,
\end{aligned} \tag{12}
$$

where m_i, $i = 0, 1, ..., 4$ are arbitrary functions of t, n is an arbitrary function of y and t, and a is an arbitrary constant, then w also satisfies the potential BKP equation (5).

The arbitrary parameter a plays a crucial role in determining whether the function v, defined by equations (9)-(10), can serve as a decomposition solution of the potential BKP equation (5). It is of interest to demonstrate that v can indeed be transformed into a solution of the potential BKP equation (5) for specific values of a, namely $a = -3$ and $a = -6$. In a more general sense, we can establish the following corollaries.

Corollary 1. *Considering the decomposition presented in theorem 1, assuming specific values for the constant a and the function n*

$$
\begin{aligned}
a = -3,\ n =\ & m_2 y + \frac{1}{2} m_1 y^2 \\
& - \frac{1}{90}(10m_0^2 + m_{0t})y^3 + n_0(t),
\end{aligned} \tag{13}
$$

it can be established that both v and w, as defined by equations (9)-(12), satisfy the potential BKP equation (5).

We hereby present an additional corollary stemming from theorem *1*.

Corollary 2. *By considering the functions v and w determined through equations (9)-(12) as prescribed in the statement of theorem 1, subject to the supplementary condition*

$$
a = -6,\ n = m_2 y + n_0(t),\quad m_0 = m_1 = 0, \tag{14}
$$

it can be concluded that these functions satisfy the potential BKP equation (5).

The implications derived from Corollaries 1 and 2 provide compelling evidence for the existence of variable coefficient KdV decomposition solutions (9)-(10) satisfying the prescribed conditions (13) and/or (14) within the framework of the potential BKP equation (5) [32]. In parallel, the variable coefficient KdV decompositions (11)-(12) subject to (13) and/or (14) engender BTs that establish connections between two distinct solutions, denoted as v and w, of the potential BKP equation (5). An additional significant implication arising from the decomposition analysis is the establishment of a comprehensive BT of the potential BKP equation (5). This consequential result is succinctly formulated as the following theorem.

Theorem 2. (***Bäcklund Transformation***) *The decomposition relation*

$$
\begin{aligned}
p_y =\ & p_{xxx} + \frac{3}{2}(w + v)_{xx} p \\
& + \frac{3}{2}(w_x^2 - v_x^2) + \frac{1}{4}(p^3)_x,\ p \equiv w - v,
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
p_t =\ & 9p_{xxxxx} + \frac{9}{16}(p^5)_x \\
& + \frac{45}{8}(w + v)_{xx} p^3 \\
& - \frac{45}{8}(3v_x^2 - 3w_x^2 - 2p_{xxx})p^2 \\
& + \frac{15}{4}(9v_x^2 + 9w_x^2 - 6v_x w_x \\
& + 4v_{xxx} + 4w_{xxx} + 2v_y + 2w_y)_x p \\
& - 15w_{xxx}(v - 4w)_x - 15v_{xxx}(4v - w)_x \\
& + 45w_{xx}^2 - 45v_{xx}^2 \\
& + \frac{45}{2} p_x(v_x^2 + w_x^2) + \frac{15}{2}(w + v)_y p_x,
\end{aligned} \tag{16}
$$

serves as a BT connecting two solutions, denoted as w and v, of the potential BKP equation (5).

The BT exhibits a distinctive characteristic in the form of a conservation law

$$
\begin{aligned}
p_y =\ & \left[p_{xx} + \frac{3}{2} p(w_x + v_x) + \frac{p^3}{4}\right]_x, \\
p_t =\ & \left[9p_{xxxx} + \frac{9p^5}{16}\right. \\
& + \frac{45}{8}(w + v)_x p^3 + \frac{45}{4} p^2 p_{xx} \\
& + \left(15(w + v)_{xxx} + \frac{45}{2}(w_x^2 + v_x^2)\right. \\
& \left. + \frac{15}{2}(w + v)_y\right) p \\
& \left. + \frac{45}{2}(p_x(w + v)_x)_x\right]_x.
\end{aligned}
$$

The identification of the BT for a PDE is equivalent to the discovery of its corresponding Lax pair, as indicated in the references [33, 34]. Consequently, it is reasonable to explore the BT further to facilitate the construction of the associated Lax pair. Building upon the insights provided by theorem 2, we can derive the widely recognized Lax pair formulation

[35]

$$\psi_y = \psi_{xxx} + 3w_x\psi_x, \tag{17}$$

$$\psi_t = 9\psi_{xxxxx} + 45w_x\psi_{xxx} + 45w_{xx}\psi_{xx} + 15(2w_{xxx} - 3w_x^2 + w_y)\psi_x \tag{18}$$

of the potential BKP equation (5) by substitution of the transformation $v = w + \ln(\psi^2)_x$ *into the BT (15)-(16) as may be verified by direct calculation. The elimination of w is made between the Lax pair, and results in the Schwarzian form of the potential BKP equation (5)*

$$S_{xxx} + C_x + 4S_x^2 - 5K_x^2 + 5(S_xK - S_y - K_y) = 0 \tag{19}$$

with Schwarzian derivatives $S = \frac{\psi_{xxx}}{\psi_x} - \frac{3\psi_{xx}^2}{2\psi_x^2}$, $C = \frac{\psi_t}{\psi_x}$ and $K = \frac{\psi_y}{\psi_x}$.

It is notable that each of the two corollaries and theorem 2 exhibit a system of coupled differential equations involving two unknown functions. Moreover, these results collectively imply that if one of the two functions satisfies the potential BKP equation (5), then the second function also serves as a solution to the same equation. Additionally, the BT (15)-(16) not only implies a particular form of the Sharma–Tasso–Olver (STO) decomposition solution for the potential BKP equation (5) when v = 0 [32], but also provides a methodology for identifying the Lax pair and the associated Schwarzian equation.

Theorem 3. *Let the function w be a solution to the consistent variable coefficient Svinolupov–Sokolov (SS) system*

$$w_y = w_{xxx} + \frac{3w_{xx}^2}{2W} + \frac{3}{2}w_x^2 - \frac{27}{4}M^2 + \frac{3}{2}m_1x + \frac{3y}{20}(m_1y + 2m_2)_t + m_3, \tag{20}$$

$$\begin{aligned} w_t = {} & 9w_{xxxxx} + 45w_xw_{xxx} + \frac{45}{2}w_x^3 + \frac{3}{4}[30m_1x - 135M^2 \\ & + 3y(m_1y + 2m_2)_t + 20m_3]w_x + \frac{45}{2}m_1w - \frac{3}{2}(45m_1M - m_{1t}y \\ & - m_{2t})x + \frac{y^2}{20}(m_1y + 3m_2)_{tt} - \frac{27}{4}y(m_1y + 2m_2)_tM \\ & + y(m_{3t} - 45m_1m_3) + m_4 + \frac{45}{2W}[2w_{xx}w_{xxxx} + w_{xxx}^2 \\ & + (3M + w_x)w_{xx}^2] + \frac{180}{W^2}w_{xx}^2w_{xxx} + \frac{315}{2W^3}w_{xx}^4 + \frac{405}{4}M^3, \end{aligned} \tag{21}$$

where $M \equiv m_1y + m_2$, $W \equiv 3M - 2w_x$ *and* m_i, $i = 1, 2, 3, 4$ *are arbitrary functions of t, then w is a solution of the potential BKP equation (5).*

Theorem 4. *The potential BKP equation (5) possesses the following variable coefficient KdV decomposition solution*

$$w_y = -\frac{1}{2}w_{xxx} - \frac{3}{2}w_x^2 + 6Mw_x + \frac{y}{10}(m_1y + 2m_2)_t + m_1x - 6M^2 + m_3,\ M \equiv m_1y + m_2, \tag{22}$$

$$\begin{aligned} w_t = {} & -\frac{9}{4}w_{xxxxx} - \frac{45}{4}(2w_xw_{xxx} + w_{xx}^2 + 2w_x^3) + 15m_1w + \frac{3}{2}y(m_1y \\ & + 2m_2)_tw_x + 15(6M^2 + m_1x + m_3)w_x + \frac{y^2}{30}(m_1y + 3m_2)_{tt} - 3(y^2m_{1t} \\ & + 10m_1x + 2ym_{2t})M + M_tx + (m_{3t} - 30m_1m_3)y - 60M^3 + m_4, \end{aligned} \tag{23}$$

where m_i, $i = 1, 2, 3, 4$ *are arbitrary functions of t.*

Theorem 5. *The variable coefficient potential SK decomposition solution of the potential BKP equation (5) possesses the form*

$$w_y = Mw_x + \frac{y}{30}(M + m_2)_t + \frac{1}{3}m_1x - \frac{1}{2}M^2 + m_3,\ M \equiv m_1y + m_2, \tag{24}$$

$$\begin{aligned} w_t = {} & -w_{xxxxx} + 5(M - 3w_x)w_{xxx} + 15(M - w_x)w_{xx}^2 + \left(\frac{1}{2}y^2m_{1t} + ym_{2t} + 5m_1x - \frac{5}{2}M^2 + 15m_3\right)w_x \\ & + 5m_1w + \frac{y^2}{90}(m_1y + 3m_2)_{tt} - \frac{y}{3}(My - x)m_{1t} + \frac{1}{3}(x - 2M_y)m_{2t} - \frac{10}{3}m_1(Mx + 3m_3y) + m_{3t}y + m_4 \end{aligned} \tag{25}$$

with m_i, $i = 1, 2, \ldots, 4$ *being arbitrary functions of t.*

Suppose that arbitrary functions $m_i = 0$, $i = 1, 2, \ldots, 4$, then by theorems 3 to 5, three special decompositions correspond to constant coefficient SS, KdV and SK decomposition solutions of potential BKP equation (5), respectively [32].

If a BT represents one of the different aspects of the property of integrability for a PDE, the existence of a linear combination principle further allows explicitly building some classes of solutions depending on the decomposition system (9)-(10). Consequently, a linear combination solution for the potential BKP equation (5) can be established using the outcomes of the decomposition.

Theorem 6. *If v_1 and v_2 are solutions of the variable coefficient potential KdV decompositions*

$$v_{1y}=v_{1xxx}-\frac{1}{2}a_1v_{1x}^2+\frac{1}{a_1}m_0x+n_y,\ M\equiv m_1y+m_2, \tag{26}$$

$$\begin{aligned}v_{1t}&=9v_{1xxxxx}+\frac{5}{2}a_1^2v_{1x}^3-\frac{15}{2}a_1(2v_{1x}v_{1xxx}+v_{1xx}^2)\\&+\left[15M-5m_0x-\frac{1}{2}(10m_0^2+m_{0t})y^2\right]v_{1x}-5m_0v_1+5m_0n+n_t+m_3\\&+\frac{1}{a_1}\{[(10m_0^2+m_{0t})y-15m_1]x+\frac{5}{3}y^3m_0^3+\frac{1}{6}m_0m_{0t}y^3-\frac{15}{2}m_0(m_1y+2m_2)y\},\end{aligned} \tag{27}$$

and

$$v_{2y}=v_{2xxx}-\frac{1}{2}a_2v_{2x}^2+\frac{1}{a_2}\times\left(\frac{1}{5}m_{0t}y^2-a_1n_y+m_0x+2y^2m_0^2-6M\right), \tag{28}$$

$$\begin{aligned}v_{2t}&=9v_{2xxxxx}+\frac{5}{2}a_2^2v_{2x}^3-\frac{15}{2}a_2(2v_{2x}v_{2xxx}+v_{2xx}^2)-\frac{1}{2}y^2v_{2x}m_{0t}+(15M-5m_0^2y^2\\&-5m_0x)v_{2x}-5m_0v_2+m_4+\frac{1}{a_2}\left[\frac{1}{15}m_{0tt}y^3+\frac{1}{6}(6xy+11m_0y^3)m_{0t}-3m_{1t}y^2-6m_{2t}y-a_1n_t\right.\\&\left.+5(2m_0^2y-3m_1)x+5y^3m_0^3-\frac{45}{2}m_0m_1y^2-45m_0m_2y-5m_0a_1n\right],\end{aligned} \tag{29}$$

respectively, then their linear combination

$$w=-\frac{1}{6}(a_1v_1+a_2v_2) \tag{30}$$

is a solution of the potential BKP equation (5).

In general, the functions v_1 and v_2 presented in theorem 6 do not qualify as solutions to the potential BKP equation (5), except under specific circumstances outlined in corollaries 1 and 2. However, the linear combination (30) of v_1 and v_2 does provide a solution to the potential BKP equation (5). The linear superposition solutions, characterized by fixed values of a_1 and a_2, as discussed in [32], can be considered as particular instances of theorem 6.

4. The decomposition solutions, Bäcklund transformation and Lax pair of the potential CKP equation

As a second illustration, we replicate the comprehensive procedure outlined in the preceding section to ascertain the decomposition solutions, Bäcklund transformation, and Lax pair for the potential CKP equation

$$9u_{xt}+\left(u_{x4}+15uu_{x2}+15u^3+\frac{45}{4}u_x^2-15uv-5u_{xy}\right)_{xx}-5u_{yy}=0,\ v_x=u_y, \tag{31}$$

$$9w_{xt}=5w_{yy}-(w_{x5}+15w_xw_{x3}+15w_x^3+\frac{45w_{xx}^2}{4}-15w_xw_y-5w_{xxy})_x, \tag{32}$$

in which u denotes the conservative field and w represents the potential one. In the limit where u_y approaches zero, the reduced form of the CKP equation (31) corresponds to the widely known Kaup–Kupershmidt (KK) equation [36, 37]. Likewise, a direct calculation establishes the following theorems.

Theorem 7. (***Bäcklund Transformation***) *Let v be an arbitrary solution of the potential CKP equation (32), a distinct solution w of equation (32) is subsequently defined through the Bäcklund transformation*

$$p_y=\left[p_{xx}-\frac{3p_x^2}{4p}+\frac{3}{2}p(w_x+v_x)+\frac{p^3}{4}\right]_x,\quad p=w-v, \tag{33}$$

$$\begin{aligned}p_t&=\left[p_{xxxx}-\frac{5}{2p}p_xp_{xxx}+\frac{5p}{3}(w+v)_{xxx}-\frac{5}{4p}p_{xx}^2+\left(\frac{5p^2}{4}+\frac{5p_x^2}{p^2}+\frac{5}{2}(w+v)_x\right)p_{xx}\right.\\&+\frac{5}{4}p_x(w+v)_{xx}-\frac{35p_x^4}{16p^3}-\frac{15(w+v)_x}{8p}p_x^2+\frac{15}{8}p(w_x^2+v_x^2)+\frac{p^5}{16}+\frac{5p}{6}(w+v)_y\\&\left.+\frac{5p^3}{8}(w+v)_x+\frac{5}{4}pw_xv_x\right]_x.\end{aligned} \tag{34}$$

The BT given by equations (33) and (34) appears to be a novel finding and is formulated in a conservation form. The associated Lax pair emerges from this BT, leading to the derivation of two equations

$$\psi_y = \psi_{xxx} + 3w_x\psi_x + \frac{3}{2}w_{xx}\psi, \tag{35}$$

$$\begin{aligned}\psi_t &= \psi_{xxxxx} + 5w_x\psi_{xxx} + \frac{15}{2}w_{xx}\psi_{xx} + 5\left(\frac{7}{6}w_{xxx} + w_x^2 + \frac{w_y}{3}\right)\psi_x \\ &+ 5\left(\frac{w_{xxxx}}{3} + w_xw_{xx} + \frac{w_{xy}}{6}\right)\psi\end{aligned} \tag{36}$$

followed by a relation $v = w + \ln(\partial_x^{-1}\psi^2)_x$. When $v = 0$, the BT is reduced to the modified STO decomposition solution

$$w_y = \left[w_{xx} - \frac{3w_x^2}{4w} + \frac{3}{2}ww_x + \frac{w^3}{4}\right]_x, \tag{37}$$

$$\begin{aligned}w_t = \Bigg[&w_{xxxx} - \frac{5}{2w}w_xw_{xxx} + \frac{5}{3}ww_{xxx} - \frac{5}{4w}w_{xx}^2 + 5\left(\frac{w^2}{4} + \frac{w_x^2}{w^2} + \frac{3}{4}w_x\right)w_{xx} \\ &- \frac{35w_x^4}{16w^3} - \frac{15w_x^3}{8w} + \frac{15ww_x^2}{8} + \frac{w^5}{16} + \frac{5ww_y}{6} + \frac{5w^3w_x}{8}\Bigg]_x\end{aligned} \tag{38}$$

of the potential CKP equation (32). This remarkable modified STO decomposition is intricately linked to the STO decomposition

$$f_y = \left[f_{xx} + \frac{1}{4}f^3 + \frac{3}{2}(f_xf)\right]_x, \tag{39}$$

$$f_t = \left[f_{xxxx} + 5f_xf_{xx} + 5\left(\frac{ff_{xxx}}{2} + \frac{f^2f_{xx}}{2} + \frac{f^3f_x}{4} + \frac{3ff_x^2}{4}\right) + \frac{f^5}{16}\right]_x \tag{40}$$

through the relationship $w_x = fw - w^2$, which precisely mirrors the STO decomposition observed in the potential BKP equation (5).

Theorem 8. *The function w is a solution of the potential CKP equation (32) provided that w satisfies the consistent variable coefficient SS system*

$$\begin{aligned}w_y &= w_{xxx} + \frac{3w_{xx}^2}{2W} + 3w_x^2 - \frac{9Mw_x}{2} - \frac{27}{8}M^2 + \frac{3}{2}m_{1}x \\ &+ \frac{27y}{20}(m_1y + 2m_2)_t + \frac{27m_2^2}{8} + m_3,\end{aligned} \tag{41}$$

$$\begin{aligned}w_t &= w_{xxxxx} + \frac{5w_{xx}w_{xxxx}}{W} + \frac{5w_{xxx}^2}{2W} + \left(\frac{20w_{xx}^2}{W^2} + 10w_x - \frac{15M}{2}\right)w_{xxx} \\ &+ \frac{35w_{xx}^4}{2W^3} + \frac{5(3M + 4w_x)w_{xx}^2}{4W} + 10w_x^3 - \frac{45M}{2}w_x^2 \\ &+ \left(\frac{45}{8}M^2 + \frac{45m_2^2}{8} + \frac{5m_1x}{2} + \frac{5m_3}{3}\right)w_x + \frac{(9w_xy^2 + 6xy - 27y^2M)m_{1t}}{4} \\ &+ \frac{(18w_xy + 6x - 27y(2m_1y + m_2))m_{2t}}{4} + \frac{9y^2(m_1y + 3m_2)_{tt}}{20} - \frac{15m_1Mx}{2} \\ &+ \frac{45M^3}{4} - \frac{135m_1m_2^2y}{8} + \frac{5m_1w}{2} - 5m_1m_3y + m_{3t}y + m_4,\end{aligned} \tag{42}$$

where $M \equiv m_1y + m_2$, $W \equiv 3M - 2w_x$ *and* m_i, $i = 1, 2, 3, 4$ *are arbitrary functions of t.*

Theorem 9. *The potential CKP equation (32) possesses the following variable coefficient KdV decomposition solution*

$$w_y = \frac{1}{4}w_{xxx} + \frac{3}{2}w_x^2 + \frac{3m_0x}{5} + \frac{27y^2(m_{0t} - 2m_0^2)}{50} + M,\ M \equiv m_1y + m_2, \tag{43}$$

$$\begin{aligned}w_t &= \frac{1}{16}w_{xxxxx} + \frac{5}{8}(2w_xw_{xxx} + w_{xx}^2 + 4w_x^3) + \left(m_0x - \frac{9m_0^2y^2}{5} + \frac{5}{3}M + \frac{9m_{0t}y^2}{10}\right)w_x \\ &+ m_0w + \left(\left(\frac{3m_{0t}}{5} - \frac{6m_0^2}{5}\right)y + \frac{5m_1}{9}\right)x + \left(\frac{18m_0^3}{25} - \frac{27m_0m_{0t}}{25} + \frac{9m_{0tt}}{50}\right)y^3 \\ &+ \left(\frac{m_{1t}}{2} - m_0m_1\right)y^2 + (m_{2t} - 2m_0m_2)y + m_3,\end{aligned} \tag{44}$$

where m_i, $i = 1, 2, 3, 4$ *are arbitrary functions of* t.

Theorem 10. *The variable coefficient potential KK decomposition solution of the potential CKP equation (32) possesses the form*

$$w_y = Mw_x + \frac{3y}{10}(M + m_2)_t + \frac{1}{3}m_1 x - \frac{1}{2}M^2 + \frac{m_2^2}{2} + m_3,\ M \equiv m_1 y + m_2, \tag{45}$$

$$\begin{aligned} w_t = & -\frac{w_{xxxxx}}{9} + \frac{5}{9}(M - 3w_x)w_{xxx} - \frac{5w_{xx}^2}{4} - \frac{5w_x^3}{3} + \frac{5M}{3}w_x^2 \\ & + \left(\frac{5m_1 x}{9} - \frac{5M^2}{18} + \frac{5m_2^2}{6} + \frac{5m_3}{3} + \frac{m_{1t}y^2}{2} + m_{2t}y\right)w_x + \frac{5m_1 w}{9} \\ & + \frac{y^2}{10}(m_1 y + 3m_2)_{tt} - \frac{m_{1t}y^2 M}{3} + \frac{xM_t}{3} + \frac{m_{2t}y(m_2 - 2m_1 y)}{3} \\ & + m_{3t}y - \frac{10m_1 xM}{27} - \frac{5m_1 y(m_2^2 + 2m_3)}{9} + m_4 \end{aligned} \tag{46}$$

with m_i, $i = 1, 2, \ldots, 4$ *being arbitrary functions of* t.

The proof of theorems 7-10 follows a similar structure to that presented in the appendix for the potential BKP equation (5). However, for brevity, the detailed calculations are omitted in this context. We have attempted but failed to provide the linear superposition solution of the potential CKP equation (32) from the decomposition solutions.

Overall, the decomposition technology offers advantages in:

(1) Simplifying analysis: by decomposing the higher-dimensional equation into a lower-dimensional system, the analysis becomes more manageable and can be tackled using techniques specifically designed for lower-dimensional systems. This simplification aids in understanding the behavior and properties of the system under consideration.
(2) Enhancing solvability: variable coefficient potential decompositions can lead to equations with more structured forms, making them potentially easier to solve. This can facilitate the study of the system's solutions, stability, and other important characteristics.
(3) Providing insights into system dynamics: by breaking down the original equation into two separate equations, each focusing on different aspects of the system, this separation can help in identifying and understanding the individual contributions of different factors affecting the system's behavior.
(4) Allowing for flexible parameter analysis: variable coefficient decompositions allow for a more flexible analysis of the system's behavior under varying conditions. By studying how the coefficients affect the solutions, researchers can gain a deeper understanding of the system's response to different parameters and external influences.
(5) Finding practical applications in the study of physical systems: this decomposition technique finds applications in various physical systems where the higher-dimensional equation arises, such as in the study of wave propagation, fluid dynamics, and quantum mechanics. The insights gained from the variable coefficient decompositions can be applied to understand and predict the behavior of these systems in real-world scenarios.

5. Conclusions

In this research endeavor, we have extended the findings elucidated in a preceding scholarly publication [32] by introducing a modified formal variable separation approach to derive decomposition solutions for the potential BKP and CKP equations. This heuristic method provides a unified framework for obtaining decomposition solutions of the (2+1)-dimensional PDEs. The result makes clear that we establish the connection between the potential BKP and CKP equations with several classical integrable systems. Consequently, one might therefore hope to deduce the solutions of the potential BKP and CKP equations from solutions of the classical integrable systems.

Notably, we have effectively constructed the BTs through the decomposition process. It is widely accepted that the existence of a BT serves as a sign to the integrability of a PDE. While the BTs for the (1+1)-dimensional SK and KK equations have been previously known [33], the BTs for the (2+1)-dimensional SK (4) and KK (31) equations have not been reported before. In this study, we present the explicit forms of the BTs for the potential BKP and CKP equations. Although other standard analytic techniques exist for obtaining BTs of nonlinear PDEs, the construction of BTs through decomposition proves to be an exceptionally straightforward approach.

For the potential BKP and CKP equations, the system defining the identified BTs exhibits two pivotal characteristics. Firstly, it is linearizable, culminating in the establishment of a Lax pair. Secondly, it showcases a conservation form. Furthermore, it is noteworthy that the solution to the potential BKP equation in theorem 6 possesses a distinctive trait: it adheres to the renowned linear superposition principle under specific parameter values. It is imperative to underscore that despite the apparent resemblance between the potential BKP and CKP equations, they are fundamentally disparate. No scaling transformation exists that can convert one equation into the other. Nevertheless, upon scrutiny of the system (39)-(40), the STO decomposition elucidates a connection between the potential BKP and CKP equations.

Decomposition, as a fundamental tool for formulating solutions of PDEs, proves to be as accessible to apply as we contend. We look forward to delving into additional theoretical facets associated with this approach in forthcoming research pursuits.

Acknowledgments

The work was sponsored by the National Natural Science Foundations of China (Nos. 12301315, 12235007, 11975131), and the Natural Science Foundation of Zhejiang Province (No. LQ20A010009).

Appendix

We elucidate the decomposition methodology by considering the case of the potential BKP equation (5).

Proof. By substituting (6) and (7), where $m > 3$, into the potential BKP equation (5), along with the decomposition compatibility condition (8), it is observed that the decomposition does not generally hold for $m > 3$. Consequently, we consider the specific case of $m = 3$ and subsequently set $n = 5$ in the decomposition relations (6) and (7). Then substituting (6) and (7) into (5) yields:

$$w_{x6}(1 - 5F_{x_3}^2 - 5F_{x_3} + G_{x_5}) + W = 0, \quad (A1)$$

where
$F = F(x, y, t, w, w_x, w_{x2}, w_{x3}) \equiv F(x, y, t, x_0, x_1, x_2, x_3)$,
$G = G(x, y, t, w, w_x, w_{x2}, w_{x3}, w_{x4}, w_{x5})$ and $\equiv G(x, y, t, x_0, x_1, x_2, x_3, x_4, x_5)$,
$W = W(x, y, t, x_0, x_1, \ldots, x_5)$ is a complicated expression of $x, y, t, x_0, x_1, \ldots, x_5$. With the coefficient of w_{x6} vanishing, we can deduce the following result:

$$G = (5F_{x_3} + 5F_{x_3}^2 - 1)x_5 + G_1, \quad (A2)$$

where $G_1 = G_1(x, y, t, x_0, x_1, \ldots, x_4)$ is a function of $\{x, y, t, x_0, x_1, \ldots, x_4\}$. By employing the relation (A2), (A1) undergoes a transformation, yielding the following expression:

$$\begin{aligned} & w_{x5}[G_{1x_4} - 5(F_{x_3} + 2)(x_1F_{x_0x_3} \\ & + x_2F_{x_1x_3} + x_3F_{x_2x_3} + x_4F_{x_3x_3}) \\ & - 5F_{x_2}(1 + 2F_{x_3}) - 5F_{xx_3}(2 + F_{x_3})] \\ & + W_1 = 0, \end{aligned} \quad (A3)$$

where $W_1 = W_1(x, y, t, x_0, x_1, \ldots, x_4)$ is w_{x5} independent. Eliminating the coefficient of w_{x5} yields

$$\begin{aligned} G_1 = {} & 5(F_{x_3} + 2)\left(\frac{1}{2}x_4F_{x_3x_3} + x_3F_{x_2x_3} \right. \\ & \left. + x_2F_{x_1x_3} + x_1F_{x_0x_3}\right)x_4 \\ & + 5x_4F_{x_2}(1 + 2F_{x_3}) \\ & + 5x_4F_{xx_3}(2 + F_{x_3}) + G_2, \end{aligned} \quad (A4)$$

with $G_2 \equiv G_2(x, y, t, x_0, x_1, x_2, x_3)$ being a function of $\{x, y, t, x_0, x_1, x_2, x_3\}$.

Similarly, by substituting the decomposition (6) and (7) with $m = 3$ and $n = 5$, along with the results (A2) and (A4), into the compatibility condition (8), we obtain the following expression:

$$\begin{aligned} & 5w_{x7}(1 - F_{x_3})^2(x_1F_{x_0x_3} \\ & + x_2F_{x_1x_3} + x_3F_{x_2x_3} + x_4F_{x_3x_3} + F_{xx_3}) \\ & + \Gamma = 0, \end{aligned} \quad (A5)$$

where $\Gamma = \Gamma(x, y, t, x_0, \ldots, x_6)$ is a w_{x7} independent function of the lower-order differentiations of w with respect to x. By setting the coefficient of $w_{x7}w_{x4}$ in (A5) to zero, we obtain the following result:

$$F = F_1(x, y, t, x_0, x_1, x_2)x_3 + H(x, y, t, x_0, x_1, x_2). \quad (A6)$$

Substituting (A6) into (A5) and requiring the coefficient of w_{x7} being zero result $F_1(x, y, t, x_0, x_1, x_2) = H_1(y, t)$. Thus, we have

$$F = H_1(y, t)x_3 + H(x, y, t, x_0, x_1, x_2). \quad (A7)$$

Taking account of (A7), (A3) becomes

$$\begin{aligned} & w_{x4}[G_{2x_3} - 5(H_1 + 2)x_3H_{x_2x_2} \\ & - 5(H_1 + 2)x_2H_{x_1x_2} \\ & + 5x_1(3 - H_1H_{x_0x_2} - 3H_1 - 2H_{x_0x_2}) \\ & - 10H_1H_{x_1} - 5H_{x_2}^2 - 5H_{x_1} \\ & - 5H_1H_{xx_2} - 10H_{xx_2}] + W_2 = 0 \end{aligned} \quad (A8)$$

with $W_2 = W_2(x, y, t, x_0, x_1, x_2, x_3)$. Vanishing the coefficient of w_{x4} in (A8) leads to

$$\begin{aligned} G_2 = {} & 5\left[\frac{H_1 + 2}{2}x_3H_{x_2x_2} + (H_1 + 2)x_2H_{x_1x_2} \right. \\ & - x_1(3 - H_1H_{x_0x_2} - 3H_1 - 2H_{x_0x_2}) \\ & + 2H_1H_{x_1} + H_{x_2}^2 + H_{x_1} \\ & \left. + H_1H_{xx_2} + 2H_{xx_2}\right]x_3 + J, \end{aligned} \quad (A9)$$

where $J = J(x, y, t, x_0, x_1, x_2)$. Up to now, the decomposition relation is simplified to

$$w_y = H_1w_{x3} + H, \quad (A10)$$

$$\begin{aligned} w_t = {} & (5H_1 + 5H_1^2 - 1)w_{x_5} + 5H_{x_2}(1 + 2H_1)w_{x4} \\ & + 5\left[\frac{H_1 + 2}{2}w_{x_3}H_{x_2x_2} + (H_1 + 2)w_{x_2}H_{x_1x_2} \right. \\ & - w_{x_1}(3 - H_1H_{x_0x_2} - 3H_1 - 2H_{x_0x_2}) \\ & + 2H_1H_{x_1} + H_{x_2}^2 + H_{x_1} + H_1H_{xx_2} \\ & \left. + 2H_{xx_2}\right]w_{x_3} + J \end{aligned} \quad (A11)$$

with three undetermined functions $H_1 = H_1(y, t)$, $H = H(x, y, t, x_0, x_1, x_2)$ and $J = J(x, y, t, x_0, x_1, x_2)$.

Inserting (A10) and (A11) into the potential BKP equation (5) and the compatibility condition (8), subsequently, vanishing the coefficients of w_{xk} for $k \geqslant 3$ leave the set of determining equations on $\{H, H_1, J\}$. Solving this set of equations, we can identify several cases that correspond to nontrivial solutions for the unknowns.

Case 1. When $H_1 = 0$, further calculation then leads to the expression

$$\begin{cases} H = Mw_x + \frac{y}{30}(M + m_2)_t + \frac{1}{3}m_1x - \frac{1}{2}M^2 + m_3, M \equiv m_1y + m_2, \\ J = 15(M - w_x)w_x^2 + (\frac{1}{2}y^2m_{1t} + ym_{2t} + 5m_1x - \frac{5}{2}M^2 + 15m_3)w_x + 5m_1w + \frac{y^2}{90}(m_1y + 3m_2)_{tt} \\ -\frac{y}{3}(My - x)m_{1t} + \frac{1}{3}(x - 2M_y)m_{2t} - \frac{10}{3}m_1(Mx + 3m_3y) + m_{3t}y + m_4 \end{cases} \tag{A12}$$

with m_i, $i = 1, 2, ..., 4$ being arbitrary functions of t.

Case 2. Taking $H_1 = 1$ in the result of this case, we can identify three distinct solutions.

The first one is as follows:

$$\begin{cases} H = \frac{3w_{xx}^2}{2W} + \frac{3}{2}w_x^2 - \frac{27}{4}M^2 + \frac{3}{2}m_1x + \frac{3y}{20}(m_1y + 2m_2)_t + m_3, \\ J = \frac{45}{2}w_x^3 + \frac{3}{4}[30m_1x - 135M^2 + 3y(m_1y + 2m_2)_t + 20m_3]w_x + \frac{45}{2}m_1w - \frac{3}{2}(45m_1M - m_{1t}y \\ -m_{2t})x + \frac{y^2}{20}(m_1y + 3m_2)_{tt} - \frac{27}{4}y(m_1y + 2m_2)_tM + y(m_{3t} - 45m_1m_3) + m_4 \\ +\frac{45}{2W}[(3M + w_x)w_{xx}^2] + \frac{315}{2W^3}w_{xx}^4 + \frac{405}{4}M^3, \end{cases} \tag{A13}$$

where $M \equiv m_1y + m_2$, $W \equiv 3M - 2w_x$ and m_i, $i = 1, 2, 3, 4$ are arbitrary functions of t.

The second solution is given by:

$$\begin{cases} H = 3w_x^2 + av_xw_x + \frac{a^2}{6}v_x^2 - \frac{1}{3}m_0(x + m_0y^2) - \frac{m_{0t}y^2}{30} + m_1y + m_2, \\ J = 5a(av + 3w)_xv_{xxx} + 15(av + 3w)_{xx}w_{xx} + \frac{15}{2}w_x[(av + 3w)_x^2 + 3w_x^2] + \frac{5a^2}{2}v_{xx}^2 + w_x(15m_2 \\ +15m_1y - 5m_0^2y^2 - 5m_0x - \frac{1}{2}m_{0t}y^2) - 5m_0w - \frac{xy}{3}(m_{0t} + 10m_0^2) + 5m_1x - \frac{y^3}{90}(m_{0tt} \\ +30m_0m_{0t} + 100m_0^3) + \frac{y^2}{2}(m_{1t} + 10m_0m_1) + (m_{2t} + 10m_0m_2)_y + m_4 \end{cases} \tag{A14}$$

with v satisfying

$$\begin{cases} v_y = v_{xxx} - \frac{a}{2}v_x^2 + \frac{m_0}{a}x + n_y, \\ v_t = 9v_{xxxxx} - 15av_xv_{xxx} - \frac{15a}{2}v_{xx}^2 + \frac{5a^2}{2}v_x^3 + [15m_2 + 15m_1y - (5m_0^2 + \frac{1}{2}m_{0t})y^2 - 5m_0x]v_x \\ -5m_0v + [(10m_0^2 + m_{0t})y - 15m_1]\frac{x}{a} + \frac{y^3}{6a}m_0(10m_0^2 + m_{0t}) - \frac{15y^2}{2a}m_0m_1 - \frac{15y}{a}m_0m_2 \\ +5m_0n + n_t + m_3. \end{cases} \tag{A15}$$

The third solution can be expressed as:

$$\begin{cases} H = -v_{xxx} + v_y + \frac{3}{2}(w + v)_{xx}(w - v) + \frac{3}{2}(w_x^2 - v_x^2) + \frac{1}{4}[(w - v)^3]_x, p \equiv w - v, \\ J = v_t - 9v_{xxxxx} + \frac{15}{2}pv_{xxxx} + (\frac{15}{2}w_x - \frac{105}{2}v_x)v_{xxx} - 45v_{xx}^2 + \frac{45}{2}pv_xv_{xx} + 15pv_{xy} + 45w_{xx}^2 \\ +(\frac{225}{2}pw_x - 45pv_x + \frac{45}{4}p^3)w_{xx} + \frac{45}{4}v_x^2p_x + \frac{135}{4}p^2w_xp_x + 15p_xv_y + \frac{135}{4}w_x^2p_x + \frac{45}{16}p^4p_x, \end{cases} \tag{A16}$$

in which case, the decomposition solution depends on another solution v of the potential BKP equation (5).

Case 3. $H_1 = -\frac{1}{2}$. In this particular case, the functions H and J are determined as follows:

$$\begin{cases} H = -\frac{3}{2}w_x^2 + 6Mw_x + \frac{y}{10}(m_1y + 2m_2)_t + m_1x - 6M^2 + m_3,\ M \equiv m_1y + m_2, \\ J = -\frac{45}{4}(w_{xx}^2 + 2w_x^3) + 15m_1w + \frac{3}{2}y(m_1y + 2m_2)_tw_x + 15(6M^2 + m_1x + m_3)w_x + \frac{y^2}{30}(m_1y \\ +3m_2)_{tt} - 3(y^2m_{1t} + 10m_1x + 2ym_{2t})M + M_tx + (m_{3t} - 30m_1m_3)y - 60M^3 + m_4, \end{cases} \tag{A17}$$

where $m_i, i = 1, 2, 3, 4$ are arbitrary functions of t. By substituting the obtained solutions (A12)–(A17) into the decomposition relations (A10) and (A11), the validity of theorems 1– 5 is established.

ORCID iDs

Xiazhi Hao https://orcid.org/0000-0002-4900-5107
S Y Lou https://orcid.org/0000-0002-9208-3450

References

[1] Fokas A S and Ablowitz M J 1984 On the inverse scattering transform of multidimensional nonlinear equations related to first order systems in the plane *J. Math. Phys.* **25** 2494–505
[2] Hirota R 1974 A new form of Bäcklund transformations and its relation to the inverse scattering problem *Prog. Theor. Phys.* **52** 1498–512
[3] Vakhnenko V O, Parkes E J, Morrison A J and Bäcklund A 2003 Transformation and the inverse scattering transform method for the generalised Vakhnenko equation *Chaos, Solitons & Fractals* **17** 683–92
[4] Kaur L and Wazwaz A M 2019 Lump, breather and solitary wave solutions to new reduced form of the generalized BKP equation *Int. J. Numer. Method Heat Fluid Flow* **29** 569–79
[5] Singh S, Kaur L, Sakkaravarthi K, Sakthivel R and Murugesan K 2020 Dynamics of higher-order bright and dark rogue waves in a new (2+1)-dimensional integrable Boussinesq model *Phys. Scr.* **95** 115213
[6] Wazwaz A M 2022 Derivation of lump solutions to a variety of Boussinesq equations with distinct dimensions *Int. J. Numer. Method Heat Fluid Flow* **32** 3072–82
[7] Wazwaz A M 2023 Painlevé integrability and lump solutions for two extended (3 + 1)- and (2 + 1)-dimensional Kadomtsev-Petviashvili equations *Nonlinear Dyn.* **111** 3623–32
[8] Wazwaz A M and Kaur L 2019 New integrable Boussinesq equations of distinct dimensions with diverse variety of soliton solutions *Nonlinear Dyn.* **97** 83–94
[9] Fokas A S and Anderson R L 1979 Group theoretical nature of Bäcklund transformations *Lett. Math. Phys.* **3** 117–26
[10] Conte R and Musette M 1989 Painlevé analysis and Bäcklund transformation in the Kuramoto-Sivashinsky equation *J. Phys.* A **22** 169–77
[11] Fan E G 2000 Two new applications of the homogeneous balance method *Phys. Lett.* A **265** 353–7
[12] Gao X N, Lou S Y and Tang X Y 2013 Bosonization, singularity analysis, nonlocal symmetry reductions and exact solutions of supersymmetric KdV equation *J. High Energy Phys.* **05** 29
[13] Jin Y, Jia M and Lou S Y 2013 Bäcklund transformations and interaction solutions of the Burgers equation *Chin. Phys. Lett.* **30** 020203
[14] Lou S Y 1993 Painlevé test for the integrable dispersive long wave equations in two space dimensions *Phys. Lett.* A **176** 96–100
[15] Ma W X and Abdeljabbar A 2012 A bilinear Bäcklund transformation of a (3+1)-dimensional generalized KP equation *Appl. Math. Lett.* **25** 1500–4
[16] Wazwaz A M and Kaur L 2019 Optical solitons and Peregrine solitons for nonlinear Schrödinger equation by variational iteration method *Optik* **179** 804–9
[17] Weiss J 1983 The Painlevé property for partial differential equations. II: Bäcklund transformation, Lax pair, and the Schwarzian derivative *J. Math. Phys.* **24** 1405–13
[18] Weiss J, Tabor M and Carnevale G 1983 The Painlevé property for partial differential equations *J. Math. Phys.* **24** 522–6
[19] Xu G Q, Liu Y P and Cui W Y 2022 Painlevé analysis, integrability property and multiwave interaction solutions for a new (4+1)-dimensional KdV-Calogero-Bogoyavlenkskii-Schiff equation *Appl. Math. Lett.* **132** 108184
[20] Zhang R F, Li M C and Yin H M 2021 Rogue wave solutions and the bright and dark solitons of the (3.1)-dimensional Jimbo-Miwa equation *Nonlinear Dyn.* **103** 1071–9
[21] Liu X Z 2020 A nonlocal variable coefficient KdV equation: Bäcklund transformation and nonlinear waves *Eur. Phys. J. Plus* **135** 113
[22] Xue M, Liu Q P and Mao H 2022 Bäcklund transformations for the modified short pulse equation and complex modified short pulse equation *Eur. Phys. J. Plus* **137** 500
[23] Lamb G L 1974 Bäcklund transformations for certain nonlinear evolution equations *J. Math. Phys.* **15** 2157–65
[24] Nucci M C 1988 Pseudopotentials, Lax equations and Bäcklund transformations for nonlinear evolution equations *J. Phys.* A **21** 73–9
[25] Satsuma J, Kaup D J and Bäcklund A 1977 Transformation for a higher order Korteweg-de Vries equation *J. Phys. Soc. Jpn.* **43** 692–7
[26] Wadati M, Sanuki H and Konno K 1975 Relationships among inverse method, Bäcklund transformation and an infinite number of conservation laws *Prog. Theor. Phys.* **53** 419–36
[27] Lou S Y and Chen L L 1999 Formal variable separation approach for nonintegrable models *J. Math. Phys.* **40** 6491–500
[28] Lou S Y, Tang X Y and Lin J 2001 Exact solutions of the coupled KdV system via a formally variable separation approach *Commun. Theor. Phys.* **36** 145–8
[29] Tang X Y and Lou S Y 2002 A variable separation approach to solve the integrable and nonintegrable models: coherent structures of the (2+1)-dimensional KdV equation *Commun. Theor. Phys.* **38** 1–8
[30] Li Y, Yao R X, Xia Y R and Lou S Y 2021 Plenty of novel interaction structures of soliton molecules and asymmetric solitons to (2.1)-dimensional Sawada-Kotera equation *Commun. Nonlinear Sci. Numer. Simul.* **100** 105843

[31] Yao R X, Li Y and Lou S Y 2021 A new set and new relations of multiple soliton solutions of (2.1)-dimensional Sawada-Kotera equation *Commun. Nonlinear Sci. Numer. Simul.* **99** 105820

[32] Hao X Z and Lou S Y 2022 Decompositions and linear superpositions of B-type Kadomtsev-Petviashvili equations *Math. Meth. Appl. Sci.* **45** 5774–96

[33] Musette M and Conte R 1998 Bäcklund transformation of partial differential equations from the Painlevé-Gambier classification. I. Kaup-Kupershmidt equation *J. Math. Phys.* **39** 5617–30

[34] Musette M and Verhoeven C 2000 Nonlinear superposition formula for the Kaup-Kupershmidt partial differential equation *Physica* D **144** 211–20

[35] Konopelchenko B G and Dubrovsky V G 1984 Some new integrable nonlinear evolution equations in $2+1$ dimensions *Phys. Lett.* A **102** 15–7

[36] Loris I 1999 On reduced CKP equations *Inverse Probl.* **15** 1099–109

[37] Weiss J 1984 On class of integrable systems and the Painlevé property *J. Math. Phys.* **25** 13–24