

Topical Review

Scalar dark energy models and scalar–tensor gravity: theoretical explanations for the accelerated expansion of the present Universe

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Abstract

The reason for the present accelerated expansion of the Universe stands as one of the most profound questions in the realm of science, with deep connections to both cosmology and fundamental physics. From a cosmological point of view, physical models aimed at elucidating the observed expansion can be categorized into two major classes: dark energy and modified gravity. We review various major approaches that employ a single scalar field to account for the accelerating phase of our present Universe. Dynamic system analysis was employed in several important models to find cosmological solutions that exhibit an accelerating phase as an attractor. For scalar field models of dark energy, we consistently focused on addressing challenges related to the fine-tuning and coincidence problems in cosmology, as well as exploring potential solutions to them. For scalar–tensor theories and their generalizations, we emphasize the importance of constraints on theoretical parameters to ensure overall consistency with experimental tests. Models or theories that could potentially explain the Hubble tension are also emphasized throughout this review.

Keywords: dark energy, modified gravity, quintessence, dynamic system, cosmology

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1998, two independent research groups [1, 2] studying distant type Ia supernovae (SN Ia) presented compelling evidence that the expansion of the Universe is not just expanding but actually accelerating. Subsequent observations, which included more in-depth studies of supernovae [3, 4] and independent data from the cosmic microwave background (CMB) radiation [5–11], baryon acoustic oscillation (BAO) peak length scale [12–18], clusters of galaxies [19, 20], and

large-scale structure (LSS) of the Universe [21–24], confirmed this remarkable acceleration.

According to Einstein's general theory of relativity without a cosmological constant, if the Universe was primarily filled with ordinary matter and/or radiation, which are the known constituents of the Universe, gravitational interactions would act to slow down the expansion. However, because the Universe is instead found to be accelerating, we are confronted with two intriguing possibilities, dark energy and modified gravity, both of which would have profound

implications for our understanding of the Universe and the laws of physics.

The first possibility is that approximately 70% of the energy density of the Universe exists in a new form characterized by negative pressure, referred to as dark energy. When attributing it to the matter sector, dark energy would modify the stress–energy tensor of matter in Einstein’s field equations, leading to the observed acceleration. This novel dark energy component has become central to cosmology, and understanding its nature remains one of the most significant challenges in modern cosmology and fundamental physics.

The second possibility is that general relativity, as formulated by Einstein, breaks down when it is applied to cosmological scales. In this scenario, a more comprehensive theory, modifying the Einstein–Hilbert action, would be needed to accurately describe the behavior of the Universe, which would lead to changes in the gravitational sector of the field equations.

Extra scalar fields in addition to the known Higgs field in the Standard Model of particle physics may play a critical role in fundamental physics and provide valuable insights into various aspects of the Universe [25]. Scalar fields naturally arise in unified theories, e.g. the string theory, and contribute to the overall cosmic energy density, offering a plausible explanation for the observed acceleration of the Universe [26, 27]. In the realm of gravitational theories, scalar fields are pivotal in various models of modified gravity, such as in scalar–tensor theories [28–31]. The scalar field, in addition to the metric, describes gravitational interactions, resulting in predictions that can deviate from those of Einstein’s theory of relativity. These deviations may manifest as changes in the behavior of the gravitational force at different scales and under extreme conditions, such as those found near black holes, inside neutron stars, or during the early Universe [32–36].

The mystery of cosmic acceleration is connected to several pivotal questions in cosmology and fundamental physics [37]. The cosmic acceleration may hold the key to uncovering a successor to Einstein’s theory of gravity. The relatively small energy density of the quantum vacuum could yield insights into concepts, such as supersymmetry and superstring. The mechanism governing the ongoing cosmic acceleration might relate to the primordial inflation in cosmology, whereas the quest to unravel the cause of cosmic acceleration could potentially introduce novel long-range forces or shed light on the enigmatic smallness of neutrino masses in particle physics.

This review basically focuses on the models and theories involving a single scalar field. The aim of this review is to theoretically clarify how these models drive the present Universe to undergo an accelerated expansion, with wide utilization of dynamic system analysis. We typically do not delve into whether these models conform to various observations from SN Ia, CMB, and LSS, because it usually requires perturbation calculations and further analysis. For such discussions, the readers are directed to specific articles or reviews on observational constraints, such as [38, 39]. In

section 2, we provide an overview of the historical context of the cosmological constant, tracing its birth from Einstein to its ultimate confirmation with supernova discoveries. The cosmological constant problem is also discussed in this section. Sections 3 and 4 introduce scalar field models of dark energy and modified gravity theories incorporating a single scalar field, respectively, which explain the current expansion of the Universe. We list a series of scalar dark energy models in section 3 to show that it is challenging for them to solve fine-tuning and coincidence problems simultaneously without getting into other troubles. Interacting dark energy models, which have recently received much attention, are also introduced in this section. In section 4, we provide constraints from terrestrial and Solar System tests of a general class of scalar–tensor gravity. Screening mechanisms are briefly included as well to illustrate how to maintain consistency with experimental results when there is a stronger coupling between the scalar field and matter. A brief summary is provided in the final section.

Throughout the review, we adopt natural units $c = \hbar = 1$ and have a metric signature $(-, +, +, +)$. We denote the Planck mass as $m_{\text{Pl}} = G^{-1/2} \simeq 1.22 \times 10^{19}$ GeV and the reduced Planck mass as $M_{\text{Pl}} = (8\pi G)^{-1/2} \simeq 2.44 \times 10^{18}$ GeV, where G is Newton’s gravitational constant. When the unit $\kappa \equiv 8\pi G = 1$ is employed, it is explained in advance.

2. Cosmological constant

The field equation in general relativity with the cosmological constant Λ is

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^{\text{m}}, \quad (2.1)$$

where $G_{\mu\nu}$, $g_{\mu\nu}$, and $T_{\mu\nu}^{\text{m}}$ are the Einstein tensor, metric tensor, and energy–momentum tensor for matter in the standard model of particle physics, respectively. The theory that adds a cosmological constant term can be considered as the simplest dark energy model. Introducing a cosmological constant to Einstein’s field equations is a reasonable improvement in the framework of general relativity. Because we all know that the metric itself and the Einstein tensor are the only two tensors constructed from the metric and its derivatives up to the second order, that possess vanishing covariant divergences and lead to the local conservation of the energy–momentum tensor.

If one moves the cosmological constant to the right-hand side, equation (2.1) becomes

$$G_{\mu\nu} = \kappa(T_{\mu\nu}^{\text{m}} + T_{\mu\nu}^{\Lambda}), \quad (2.2)$$

where

$$T_{\mu\nu}^{\Lambda} = -\frac{\Lambda}{\kappa} g_{\mu\nu} \quad (2.3)$$

represents the energy–momentum of a new component (dark energy) outside the standard model of particle physics. Compared with the energy–momentum tensor of a perfect

fluid

$$T_{\mu\nu}^{\text{pf}} = (\rho_{\text{pf}} + p_{\text{pf}})u_{\mu}u_{\nu} + p_{\text{pf}}g_{\mu\nu}, \quad (2.4)$$

one has

$$\rho_{\Lambda} = -p_{\Lambda} = \frac{\Lambda}{\kappa}, \quad (2.5)$$

which indicates that this new component has a constant energy density and a negative pressure. As a result, any form of matter with constant energy density will behave as a cosmological constant (and vice versa), because they enter the field equation with the same mathematical structure.

2.1. Brief history of cosmological constant

When the field equation of general relativity was proposed, most physicists, including Einstein, believed that the Universe was in a static state. However, the original field equation was not able to describe a non-evolving Universe. To address this, Einstein introduced a new constant of nature and referred to it as the cosmological constant [40]. Incorporating this constant, Einstein obtained his static Universe, which he at first failed to notice that it is unstable [41]. Einstein also believed that the existence of the cosmological constant does not result in any difference on small scales, such as within the Solar System, as long as it is sufficiently small.

In 1929, Hubble made the groundbreaking discovery that distant galaxies are indeed moving away from us, and he presented strong observational evidence for the expansion of the Universe [42]. This indicated that the Universe is dynamic, rather than static. Hence, concerning the original motivation, the cosmological constant is no longer needed.

At the end of the last century, the cosmological constant was reintroduced several times to explain problems related to the relatively young Universe age [43], as well as the peaking number counts of quasars at $z \sim 2$ [44], despite it is well known that the peak has an explanation of astrophysical origin rather than cosmological nowadays. A nonzero cosmological constant was also invoked to reconcile the flat Universe predicted by inflation and some other estimations from observations [45, 46].

The final confirmation of a nonzero value of the cosmological constant is from the accurate measurement of the luminosity distance of SN Ia as a function of redshift [1, 2]. To fit the results in the framework of general relativity, the simplest way is to assume that there exists a new cosmic component besides radiation and matter (including both ordinary matter and dark matter). The new component has a negative pressure and dominates the Universe's present energy budget.

2.2. Cosmological constant problem

Although the confrontation of equation (2.1) with the Solar System and galactic observations has already given an upper bound of the cosmological constant, a tighter constraint comes from larger-scale observations [47, 48] that Λ is of the same order of the present value of the Hubble parameter H_0 .

The corresponding energy density is calculated directly using equation (2.5):

$$\rho_{\Lambda} \approx \frac{H_0^2}{8\pi G} \sim 10^{-47} \text{ GeV}^4. \quad (2.6)$$

Theoretically, the constant energy density associated with the quantum vacuum energy is regarded as the source of the cosmological constant. Similar to the ground state of a quantum harmonic oscillator, which possesses a nonzero zero-point energy proportional to the oscillation frequency of its classical counterpart, vacuum-to-vacuum fluctuations arise from the collective contributions of an infinite number of oscillators capable of vibrating at all possible frequencies. A cutoff frequency is required to prevent the divergence of the vacuum energy density. The cutoff wave number $k_c = m_{\text{pl}}$ is generally considered for the belief of the validity of the quantum field theory and general relativity below the Planck scale. Thus, the vacuum energy density has an extremely large value:

$$\rho_{\text{vac}} \approx \frac{k_c^4}{16\pi^2} \sim 10^{74} \text{ GeV}^4, \quad (2.7)$$

which is more than 120 orders of magnitude larger than the observational bound of ρ_{Λ} . This huge discrepancy is known as the cosmological constant problem or the vacuum catastrophe.

Hints that the cosmological constant might be nonzero (see section 2.1) led Zel'dovich to consider the cosmological constant in terms of vacuum energy in 1967 [49], and since then, numerous solutions have been proposed to address this mismatch in energy density [47, 50–52]. Apart from anthropic principles [47, 53], the solutions can be broadly categorized into three groups. The first one is to establish a unified framework for quantum gravity, specifically in exploring the interplay between quantum vacuum and gravity [54–56]. The second category is to 'eliminate' the vacuum energy. In the context of supersymmetry—an extension of the standard model—the vacuum energy is exactly zero due to the equal number of fermionic and bosonic degrees of freedom [57]. In addition, an alternative approach, called Schwinger's source theory [58–60], changes the interpretation of quantum field theory formalism, and vacuum energy is no longer needed. The last is to modify gravity [37, 61], aiming to acquire an effective cosmological constant, which is actually not an exact resolution of the vacuum catastrophe. However, modified gravity seems to become more crucial because of the discovery of a tiny but nonzero cosmological constant. If a modified theory of gravity accurately describes the evolution of the Universe and explains observations well, it is plausible that at least one of the following two statements holds. (a) The vacuum energy of the Universe is either zero or extremely small. It is possible that the quantum zero-point energy of each field is offset by its higher-order corrections, or that there exists a cancellation mechanism—maybe a hidden symmetry—of vacuum energy among different fields. In such a scenario, one would need to further scrutinize the explanations of experimental evidence for vacuum energy such as the Casimir effect, the vacuum concept associated with quantum

chromodynamics, and the spontaneous symmetry breaking in the electroweak force. (b) The vacuum energy, being a purely quantum concept, does not function as a source within the classical gravity theory that influences the geometry of spacetime and other degrees of freedom in modified gravity theories, e.g. scalar fields. In this scenario, the connection between quantum field theory and gravity theory poses significant challenges, whereas the establishment of a unified framework for describing all fundamental forces remains a central pursuit in modern theoretical physics.

3. Dark energy

In the framework of general relativity, radiation and barotropic matter, as the only contents of the Universe, cannot induce an accelerated expansion. The concept of dark energy, termed by Turner [62, 63] as a potential constituent of the Universe, propels ongoing studies on accelerated expansion. In the standard lambda cold dark matter (Λ CDM) model [38], the dark energy is characterized by the presence of a cosmological constant term, which is not dynamic but exerts influence on the evolution of the Universe. Despite being the standard model in cosmology, it is confronted with various theoretical and observational challenges.

- **Fine-tuning problem:** The current energy density of the estimated cosmological constant in equation (2.6) is relatively small based on observations, whereas the theoretical value of the vacuum energy density in equation (2.7) is significantly larger. In other words, the observed value of the energy density conflicts with the possible energy scales and requires fine-tuning if the cosmological constant originates from the vacuum energy density.
- **Coincidence problem:** The Λ CDM model suggests a comparable energy density ratio between matter and dark energy [11], implying that the expansion of the Universe transitioned from a decelerated to an accelerated phase at a redshift of approximately $z \sim 0.67$ [27]. The problem of why an accelerated expansion should occur now (actually not long ago) in the very long history of the Universe is called the coincidence problem.
- **Hubble tension and more:** Recent observations have revealed that within the framework of the Λ CDM model, certain tensions exist between observations from the early Universe and late Universe [64], among which the H_0 and S_8 tensions are the most well-known. The latest observations with the James Webb Space Telescope further confirm the existence of the Hubble tension at 8σ confidence [65].

3.1. Quintessence

A dynamic scalar field as dark energy—the quintessence model—is one of the most popular models for dark energy. The audience may be familiar with a similar inflation model

that describes the very early stage of the Universe [66, 67]. However, except for the large discrepancy in the energy scale, the scalar field is the only component in inflation models, whereas it is just the fifth element of the Universe in the quintessence model. As the first and simplest model in this review, we will look at the quintessence in more detail. The evolution of the Universe in this model is displayed in section 3.1.1, and the introduction of the classification and dynamic system analysis of the quintessence model are in section 3.1.2 and section 3.1.3, respectively.

The action for the quintessence model is given by [68]

$$S[g^{\mu\nu}, \phi, \Psi_m] = S_{\text{EH}}[g^{\mu\nu}] + S_q[g^{\mu\nu}, \phi] + S_m[g^{\mu\nu}, \Psi_m], \quad (3.1)$$

where $S_{\text{EH}} = \int d^4x \sqrt{-g} R / 2\kappa$ is the Einstein–Hilbert action, $S_m = \int d^4x \sqrt{-g} \mathcal{L}_m(\Psi_m)$ is the action of all possible matter fields collectively denoted as Ψ_m , and

$$S_q = \int d^4x \sqrt{-g} \mathcal{L}_q = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right], \quad (3.2)$$

is the action for the quintessence. $V(\phi)$ is the potential of the scalar field and is always assumed to be positive. By varying action (3.7), the equations of motion for the tensor field $g_{\mu\nu}$ and scalar field ϕ are

$$G_{\mu\nu} = \kappa(T_{\mu\nu}^m + T_{\mu\nu}^q), \quad (3.3)$$

$$\square \phi = \frac{dV}{d\phi}, \quad (3.4)$$

where $\square = g^{\mu\nu} \nabla_\mu \nabla_\nu$, and $T_{\mu\nu}^m$ is the energy–momentum tensor of ordinary matter, including dust and radiation,

$$T_{\mu\nu}^m \equiv -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}, \quad (3.5)$$

and the energy–momentum tensor of quintessence is

$$T_{\mu\nu}^q = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right]. \quad (3.6)$$

Because the scalar field is minimally coupled to the metric, action (3.7) can be rewritten as

$$S[g^{\mu\nu}, \phi, \Psi_m] = S_{\text{EH}}[g^{\mu\nu}] + S'_m[g^{\mu\nu}, \Psi_m, \phi], \quad (3.7)$$

where $S'_m = S_m + S_q$. In such a case, the scalar field in action (3.7) represents the fifth element of the Universe, other than baryons, photons, neutrinos, and dark matter, which are usually considered in modern cosmology. This dynamic scalar field, as a component of the Universe, was first studied by Ratra and Peebles [69] and Wetterich [70] in 1988 and was first called quintessence in a 1998 paper by Caldwell *et al* [68].

3.1.1. Evolution of the Universe in the quintessence model.

Based on the assumptions of homogeneity and isotropy of the Universe, which is approximately true on large scales, the

Friedmann–Lemaître–Robertson–Walker (FLRW) metric,

$$ds^2 = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega_2^2 \right) \quad (3.8)$$

is used to describe the geometry of the Universe. Here, $a(t)$ is the scale factor with cosmic time t , and $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the line element of a 2-sphere. The constant K describes the geometry of the spatial part of the spacetime, with closed, flat, and open Universes corresponding to $K = +1, 0, -1$, respectively. In this review, we always assume a flat Universe, which is consistent with current observations [11].

Friedmann equations are a set of equations that govern the expansion of space in homogeneous and isotropic models of the Universe. Using the FLRW metric, one has

$$3M_{\text{Pl}}^2 H^2 = \rho_m + \rho_q, \quad (3.9)$$

$$-2M_{\text{Pl}}^2 \dot{H} = \rho_m + \rho_q + p_m + p_q, \quad (3.10)$$

where $H = \dot{a}/a$ is the Hubble parameter. The subscripts ‘q’ and ‘m’ represent quintessence and ordinary matter, respectively. Ordinary matter comprises relativistic radiation and nonrelativistic dust, denoted by the subscripts ‘r’ and ‘d’, respectively, as follows. Assuming that the scalar field depends only on the cosmic time, the energy density and pressure of the quintessence field are

$$\rho_q = \frac{1}{2} \dot{\phi}^2 + V(\phi), \quad (3.11)$$

$$p_q = \frac{1}{2} \dot{\phi}^2 - V(\phi), \quad (3.12)$$

where the overdot denotes the derivative with respect to cosmic time t . Furthermore, local conservation of the energy–momentum tensor gives

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = 0, \quad (3.13)$$

$$\dot{\rho}_q + 3H(1 + w_q)\rho_q = 0, \quad (3.14)$$

where $w_m = p_m/\rho_m$ and

$$w_q = \frac{p_q}{\rho_q} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)} \quad (3.15)$$

are parameters in equations of state (EOS). In the slow-roll limit, $\dot{\phi}^2 \ll V(\phi)$, the parameter w_q goes to -1 , and the quintessence acts just like the cosmological constant.

For a constant w_m , equation (3.13) gives

$$\rho_m = \bar{\rho}_m a^{-3(1+w_m)}, \quad (3.16)$$

while the solution to equation (3.14) is

$$\rho_q = \bar{\rho}_q \exp \left[-3 \int (1 + w_q) \frac{da}{a} \right]. \quad (3.17)$$

Here, $\bar{\rho}_m$ and $\bar{\rho}_q$ are integration constants. Figure 1 depicts the evolution of the dust and quintessence energy density parameters, as well as the effective EOS parameter of the quintessence model with an exponential potential over time. The definitions of the quantities in the plot are provided in

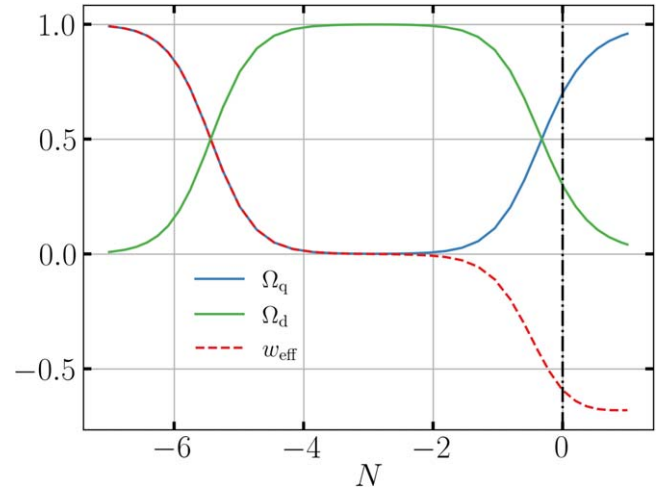


Figure 1. Evolution of the effective EOS parameter w_{eff} , dust Ω_d , and quintessence Ω_q relative energy densities with an exponential potential. The vertical dashed line denotes the present cosmological time. One has $\Omega_d \simeq 0.3$ and $\Omega_q \simeq 0.7$ today.

section 3.1.3. As observed from figure 1, to satisfy $\Omega_d \simeq 0.3$ and $\Omega_q \simeq 0.7$ at present and to experience a period of matter dominance, the initial conditions must be chosen carefully.

We are currently in a period characterized by the dominance of dark energy. In the quintessence domination, equations (3.9) and (3.10) become

$$6M_{\text{Pl}}^2 H^2 = \dot{\phi}^2 + 2V(\phi), \quad (3.18)$$

$$-2M_{\text{Pl}}^2 \dot{H} = \dot{\phi}^2. \quad (3.19)$$

The acceleration equation comes out of a combination of the Friedmann equations as

$$-3M_{\text{Pl}}^2 \frac{\ddot{a}}{a} = \frac{1}{2}(\rho_q + 3p_q) = \dot{\phi}^2 - V(\phi), \quad (3.20)$$

which shows that the Universe expands in an increasing rate when $w_q < -1/3$, or $\dot{\phi}^2 < V(\phi)$. To see how the scalar field evolves, we assume a power-law expansion, $a(t) \propto t^s$, where the accelerated expansion occurs for $s > 1$. According to equation (3.19), one obtains

$$\phi = \int dt \sqrt{-2M_{\text{Pl}}^2 \dot{H}} \propto \ln t. \quad (3.21)$$

The potential can also be expressed in terms of H and \dot{H} as

$$V = 3M_{\text{Pl}}^2 H^2 \left(1 + \frac{\dot{H}}{3H^2} \right) \propto t^{-2}. \quad (3.22)$$

Combining equations (3.21) and (3.22), the potential giving the power-law expansion is ought to be an exponential form

$$V(\phi) = \bar{V} \exp \left(-\frac{\lambda\phi}{M_{\text{Pl}}} \right), \quad (3.23)$$

where \bar{V} is a constant, and $\lambda = \sqrt{2/s}$.

3.1.2. Thawing and freezing models. In the flat FLRW background, equation (3.4) gives the evolution equation of

⁴ Here, ϕ is the azimuthal angle, not to be confused with the scalar field.

the quintessence,

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{3.24}$$

Although the presence of ordinary matter is not explicitly accounted for in equation (3.24), its influence on the Hubble parameter will impact the evolution of the quintessence field. In equation (3.24), the first term represents the acceleration of the scalar field, whereas the second term corresponds to a frictional effect, known as Hubble friction or Hubble drag, resulting from the expansion of the Universe. The third term denotes a driving force arising from the steepness of the potential.

The potential function of the scalar field is the only adjustable component in the quintessence model, the specific shape of which exerts a significant influence on cosmic evolution. To reproduce the cosmic acceleration today, we require the potential to be flat enough to satisfy the condition

$$\left| \frac{M_{\text{Pl}}^2}{V} \frac{d^2V}{d\phi^2} \right| \lesssim 1. \tag{3.25}$$

Hence, the quintessence mass squared $m_\phi^2 \equiv d^2V/d\phi^2$ needs to satisfy

$$|m_\phi^2| \lesssim \frac{V_0}{M_{\text{Pl}}^2} \sim H_0^2, \tag{3.26}$$

where V_0 and H_0 are the present values of the potential and Hubble parameter, respectively. As a result, the mass of the scalar field has to be extremely small, say $m_\phi \lesssim 10^{-33}$ eV, to be compatible with the present cosmic acceleration. Apart from this requirement, we also hope that the potential appears in some particle physics models [27].

Caldwell and Linder [71] divided quintessence models into two categories according to whether the scalar field accelerates ($\ddot{\phi} > 0$) or decelerates ($\ddot{\phi} < 0$). A coasting in the scalar field dynamics, $\ddot{\phi} = 0$, is nongeneric, as the field would need to be finely tuned to be perfectly balanced, neither accelerate due to the slope of the potential nor decelerate due to the Hubble drag.

In the accelerating region, the field that has been frozen by the Hubble friction into a cosmological constantlike state until recently and then begins to evolve gives the thawing model. The current accelerated expansion is driven by the dominance of quintessence and the scalar field in its early stages of ‘thawing’, implying that w_q , while increasing, remains smaller than $-1/3$. A representative potential of this class is the one arising from the pseudo Nambu–Goldstone boson [72],

$$V(\phi) = \mu^4 \left[1 + \cos\left(\frac{\phi}{f}\right) \right], \tag{3.27}$$

where μ and f are constants characterizing the energy scale and mass scale of spontaneous symmetry breaking, respectively. If the field mass squared $|m_\phi^2| \simeq \mu^4/f^2$ around the potential maximum is smaller than H^2 , the field is stuck there with $w_q \simeq -1$. Once H^2 drops below $|m_\phi^2|$, the scalar field starts to move, and w_q starts to increase. Likelihood analysis

using CMB shift parameters measured by WMAP7 [8] combined with SN Ia and BAO data placed a bound, $w_q|_{z=0} < -0.695$, at the 95% confidence level when the quintessence prior is assumed to be $w_q > -1$ [73].

In the decelerating region, the field that initially rolls due to the steepness of the potential but later approaches the cosmological constant state gives the freezing model. A representative potential of this class is the Ratra–Peebles potential [69, 74],

$$V(\phi) = M^{4+n}\phi^{-n}, \tag{3.28}$$

where M and n are positive constants. The potential does not possess a minimum; hence, the field rolls down the potential toward infinity. The movement of the field gradually slows down when ϕ is large enough and the system enters the phase of cosmic acceleration. Under the quintessence prior $w_q^i > -1$ for tracking solutions (see section 3.1.3), a likelihood analysis using the iterative solution in Ref. [75] resulted in bounds, $w_q^i < -0.923$ and $0.675 < \Omega_q < 0.703$, at the 95% confidence level, from the observational data of Planck 2015, SN Ia, and CMB [76]. For the Ratra–Peebles potential, this bound translates to $n < 0.17$, which means that the tracker solutions arising from the inverse power-law potential with positive integer powers are observationally excluded.

3.1.3. Dynamic system approach. The Universe comprises multiple components, each of which exerts its own influence on the dynamic evolution of the Universe. The dynamic system approach is a powerful method that helps to illustrate the global dynamics and full evolution of the Universe [77, 78]. By carefully choosing the dynamic variables, a given cosmological model can be written as an autonomous system of some differential equations, which further provides a quick answer to whether it can reproduce the observed expansion of the Universe.

An autonomous system is a system of ordinary differential equations, which does not explicitly depend on the independent variable,

$$\frac{d}{dt}\mathbf{x}(t) = f(\mathbf{x}(t)),$$

where \mathbf{x} takes values in n -dimensional Euclidean space. Many laws in physics, where t is often interpreted as time, are expressed as autonomous systems because it is always assumed that the laws of nature that hold now are identical to those for any point in the past or future. Time t is usually replaced by the scale factor a or its logarithm in cosmology because the Universe expands monotonously in most cases so that the scale can be used as a measure of time.

To describe the quintessence model, the following dimensionless variables:

$$\begin{aligned} x_K &\equiv \frac{\dot{\phi}}{\sqrt{6} M_{\text{Pl}} H}, & x_V &\equiv \frac{\sqrt{V}}{\sqrt{3} M_{\text{Pl}} H}, \\ x_r &\equiv \frac{\sqrt{\rho_r}}{\sqrt{3} M_{\text{Pl}} H}, \end{aligned} \tag{3.29}$$

were introduced in the autonomous system [79, 80]. These

variables are generally called the expansion normalized variables [81], which are widely employed in quintessence models and their generalizations, for the sake of easily closing the dynamic system. Furthermore, one defines the density parameters for the kinetic term and the potential of the scalar field,

$$\Omega_K \equiv \frac{\dot{\phi}^2/2}{3M_{\text{Pl}}^2 H^2} = x_K^2, \quad \Omega_V \equiv \frac{V}{3M_{\text{Pl}}^2 H^2} = x_V^2, \quad (3.30)$$

which give

$$\Omega_q \equiv \frac{\rho_q}{3M_{\text{Pl}}^2 H^2} = x_K^2 + x_V^2. \quad (3.31)$$

The density parameters of radiation, dust, and total ordinary fluid are given by

$$\begin{aligned} \Omega_r &\equiv \frac{\rho_r}{3M_{\text{Pl}}^2 H^2}, & \Omega_d &\equiv \frac{\rho_d}{3M_{\text{Pl}}^2 H^2}, \\ \Omega_m &\equiv \frac{\rho_m}{3M_{\text{Pl}}^2 H^2} = \Omega_r + \Omega_d. \end{aligned} \quad (3.32)$$

From equation (3.9), one has

$$\Omega_q + \Omega_m = \Omega_K + \Omega_V + \Omega_d + \Omega_r = 1, \quad (3.33)$$

which implies the flatness of the Universe. The EOS parameter w_q and the effective EOS parameter of the Universe are expressed by x_K and x_V via [80]

$$\begin{aligned} w_q &= \frac{x_K^2 - x_V^2}{x_K^2 + x_V^2}, \\ w_{\text{eff}} &\equiv \frac{p_m + p_q}{\rho_m + \rho_q} = x_K^2 - x_V^2 + \frac{1}{3}x_r^2. \end{aligned} \quad (3.34)$$

Differentiating the variables x_K , x_V , and x_r with respect to the number of e-foldings, $N = \ln a$, as well as using equations (3.9), (3.10), and (3.24), one obtains the following first-order differential equations [80]:

$$\frac{dx_K}{dN} = -3x_K + \frac{\sqrt{6}}{2}\lambda x_V^2 - x_K \frac{1}{H} \frac{dH}{dN}, \quad (3.35a)$$

$$\frac{dx_V}{dN} = -\frac{\sqrt{6}}{2}\lambda x_K x_V - x_V \frac{1}{H} \frac{dH}{dN}, \quad (3.35b)$$

$$\frac{dx_r}{dN} = -2x_r - x_r \frac{1}{H} \frac{dH}{dN}, \quad (3.35c)$$

where

$$\begin{aligned} \frac{1}{H} \frac{dH}{dN} &= \frac{\dot{H}}{H^2} \\ &= -\frac{1}{2}(3 + 3x_K^2 - 3x_V^2 + x_r^2), \end{aligned} \quad (3.36)$$

and λ is defined by

$$\lambda \equiv -\frac{M_{\text{Pl}}}{V} \frac{dV}{d\phi}. \quad (3.37)$$

If λ is constant, the differential equations are closed to become an autonomous system, and fortunately, the potential

is exactly in the form of equation (3.23). In this way, the analysis of the features of the phase space of this system, i.e. the analysis of the fixed points and the determination of the general behavior of the orbits, provides insight into the global behavior of the Universe.

Let us first clarify some mathematical concepts and their relationship to cosmic evolution. A point \mathbf{x}_c is said to be a fixed or critical point of the autonomous system if $f(\mathbf{x}_c) = 0$. The fixed points are divided into three categories based on their stability. The first class comprises the stable node and stable spirals, distinguished by whether the eigenvalues of the Jacobian matrix are purely real or complex. A stable fixed point \mathbf{x}_c , regardless of whether a stable node or a stable spiral, is an attractor. It represents the position toward which the system converges in the infinite future, i.e. $\mathbf{x} \rightarrow \mathbf{x}_c$ for $t \rightarrow \infty$. In the context of cosmic evolution, the Universe eventually moves to an attractor in phase space. The point that indicates a matter-dominated or radiation-dominated epoch should not be an attractor generally because the Universe has already entered an accelerating stage. In contrast, models with a stable fixed point representing the accelerating expansion are theoretically favored, especially those that give the current ratio of background fluid and dark energy for solving the coincidence problem. The second class refers to the unstable node, often considered as the initial point of evolution and is thus sometimes termed the ‘past attractor’. The radiation- or matter-dominated stage is encouraged to be unstable to initiate the observational acceleration. The last class is the saddle point, which neither serves as a past nor a future attractor. It may potentially represent a certain stage of cosmic evolution, but this stage is only temporary, as the system cannot actually reach the saddle point and therefore cannot remain there for an extended duration unless the system initially starts there.

By setting $dx_K/dN = dx_V/dN = dx_r/dN = 0$ in equations (3.35a), (3.35b), and (3.35c), the fixed points are acquired, and those satisfying the condition

$$x_V > 0, \quad x_r > 0$$

are listed in table 1 and plotted in figure 2 for different λ .

Point R (D) is a radiation (dust)-dominated solution that can describe the corresponding dominating epoch. Point R_{sc} (D_{sc}) is the scaling solution [79, 82] for the ratio Ω_q/Ω_r (Ω_q/Ω_d) is a nonzero constant, which can be used to hide the presence of the scalar field in the cosmic evolution, at least at the background level. The scaling solution must satisfy the condition $\lambda^2 > 3(1 + w_m)$ to ensure that $\Omega_q < 1$. Because $w_{\text{eff}} = w_m$ and usually $0 < w_m < 1$, the scaling solution usually does not describe the cosmic acceleration (unless for a double exponential potential [83] and some related others [84–86]). Notice that the scaling solution returns to the corresponding dominated solution in the limit $\lambda \rightarrow \infty$.

Points K_{\pm} represent solutions dominated by a scalar field but are unable to account for the accelerating expansion due to the wrong EOS parameter. Hence, the only fixed point that represents the present cosmic epoch is point Q , and the cosmic acceleration is realized if $\lambda^2 < 2$. As a result, the

Table 1. Properties of fixed points for the quintessence in the exponential potential (3.23) [79, 80].

Fixed point	(x_K, x_V, x_r)	Ω_r	Ω_d	Ω_q	w_q	w_{eff}
R	$(0, 0, 1)$	1	0	0	—	$\frac{1}{3}$
D	$(0, 0, 0)$	0	1	0	—	0
R_{sc}	$\left(\frac{2\sqrt{6}}{3\lambda}, \frac{2\sqrt{3}}{3\lambda}, \frac{\sqrt{\lambda^2-4}}{\lambda}\right)$	$1 - \frac{4}{\lambda^2}$	0	$\frac{4}{\lambda^2}$	$\frac{1}{3}$	$\frac{1}{3}$
D_{sc}	$\left(\frac{\sqrt{6}}{2\lambda}, \frac{\sqrt{6}}{2\lambda}, 0\right)$	0	$1 - \frac{3}{\lambda^2}$	$\frac{3}{\lambda^2}$	0	0
K_{\pm}	$(\pm 1, 0, 0)$	0	0	1	1	1
Q	$\left(\frac{\lambda}{\sqrt{6}}, \frac{\sqrt{6-\lambda^2}}{\sqrt{6}}, 0\right)$	0	0	1	$-1 + \frac{\lambda^2}{3}$	$-1 + \frac{\lambda^2}{3}$

evolution of the Universe is consistent with the trajectory $R \rightarrow D \rightarrow Q$.

If λ is not a constant, we have to consider an additional equation,

$$\frac{d\lambda}{dN} = -\sqrt{6}\lambda^2(\Gamma - 1)x_K, \tag{3.38}$$

where

$$\Gamma \equiv V \frac{d^2V}{d\phi^2} \left(\frac{dV}{d\phi} \right)^{-2}. \tag{3.39}$$

In this scenario, the fixed points obtained in the case of a constant λ can be considered as ‘instantaneous’ fixed points that change over time [87, 88], if assuming that the timescale for the variation of λ is significantly shorter than H^{-1} , the timescale that produces significant changes in the scale of the Universe.

The possibility of a transition from the fixed point D_{sc} to Q arises when λ decreases over time. If the condition $\Gamma > 1$ is satisfied, the absolute value of λ decreases toward 0. This means that the solutions finally approach the accelerated instantaneous point Q , even if $\lambda^2 > 2$, during radiation and matter domination. The condition $\Gamma > 1$ is the so-called tracking condition under which the field density eventually catches up with that of the background fluid, and a tracking solution characterized by

$$\Omega_q \simeq \frac{3(1 + w_q)}{\lambda^2} \tag{3.40}$$

is called a tracker [89]. Figure 3 shows the evolution of each component of a tracking solution. If Γ varies slowly in time, the EOS parameter of quintessence is nearly constant during the matter and radiation domination periods [89, 90],

$$w_q \simeq w_q^i = \frac{w_m - 2(\Gamma - 1)}{2\Gamma - 1}. \tag{3.41}$$

The fact that $w_q < w$ results in a slower decrease in the quintessence energy density compared to the fluid energy density ultimately leads to an attainable value of $\Omega_q = 1$. From equation (3.40), w_q is $-1 + \lambda^2/3$ in the scalar field-dominated epoch, which corresponds to the stable fixed point Q as long as $\lambda < 3(1 + w_m)$. The tracker fields correspond to attractor-like solutions, wherein the field energy density tracks the background fluid density across a wide range of initial conditions [74]. This implies that the energy density of

quintessence does not necessarily need to be significantly smaller than that of radiation or matter in the early Universe, unlike in the cosmological constant scenario. Consequently, this offers a potential solution to alleviate the coincidence problem. Another benefit of the tracker solution is that it does not require the introduction of a new mass hierarchy in fundamental parameters [74].

In contrast to the tracking solution where $\Gamma > 1$, if λ gradually increases over time, the accelerated phase of the Universe is limited at late times because the energy density of the scalar field becomes negligible compared to that of the background fluid.

3.2. Cosmological boundary

3.2.1. Phantom dark energy. The EOS parameter for the cosmological constant, $w_\Lambda = -1$, is referred to as the cosmological boundary because it not only distinguishes between the rates of accelerating expansion, whether slower or faster than exponential, but also marks the point where perturbation divergence occurs [91]. A canonical scalar field, when treated as a perfect fluid, is bound by an EOS parameter within the range $(-1, 1)$. However, current observations suggest the possibility that the equation of state parameter of dark energy is smaller than w_Λ , a scenario even favored by the data [10, 92–94].

The phantom dark energy model was proposed as a complement to the canonical quintessence model [95], offering a super-accelerating phase, the expansion faster than exponential. This can be achieved by substituting the quintessence action with an alternative:

$$\begin{aligned} S_p[g^{\mu\nu}, \phi] &= \int d^4x \sqrt{-g} \mathcal{L}_p \\ &= \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]. \end{aligned} \tag{3.42}$$

The EOS parameter of the phantom field is

$$w_p \equiv \frac{p_p}{\rho_p} = \frac{\dot{\phi}^2 + 2V(\phi)}{\dot{\phi}^2 - 2V(\phi)}, \tag{3.43}$$

then one has $w_p > 1$ for the kinetic-dominated regime ($\dot{\phi}^2/2 > V$) and $w_p < -1$ for the potential-dominated regime ($V > \dot{\phi}^2/2$).

The choice of the same expansion normalized variables (3.29) also closes the system if the exponential potential

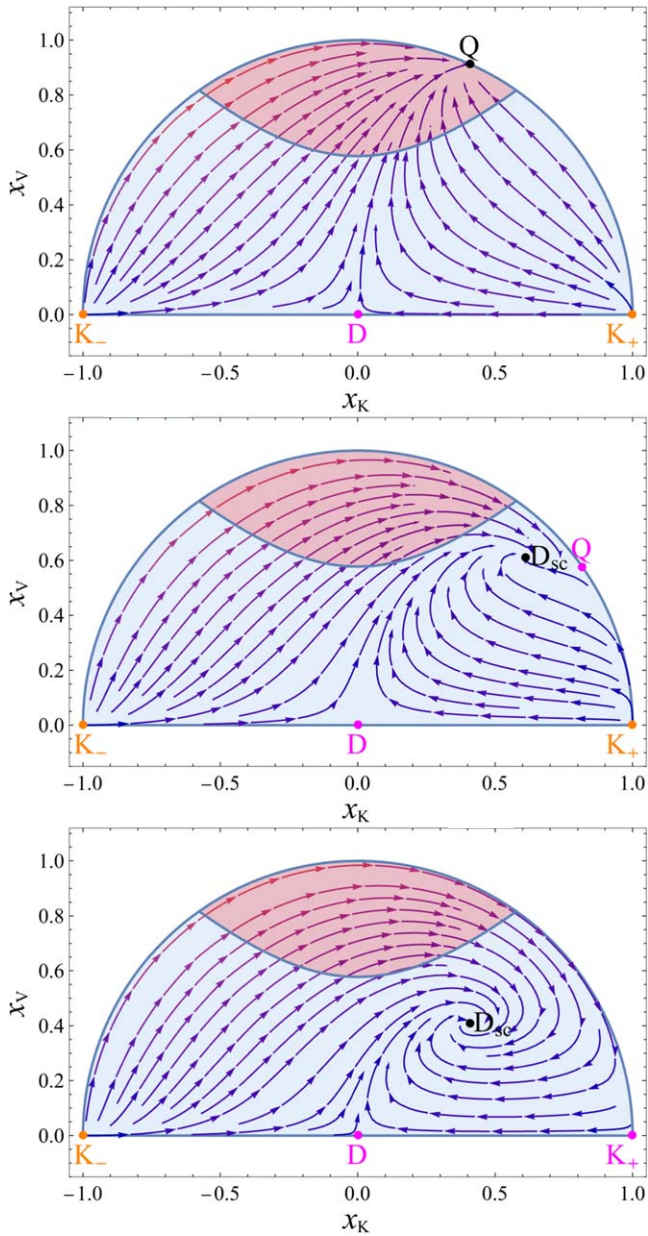


Figure 2. Phase space of x_K and x_V with $\lambda = 1$ (upper subfigure), $\lambda = 2$ (middle subfigure), and $\lambda = 3$ (bottom subfigure). Here, we assume $x_r = 0$ to reduce the phase space to two dimensions. The fixed points listed in table 1 are plotted, with yellow, magenta, and black representing stable, saddle, and unstable fixed points, respectively. Each streamline represents the evolution of the Universe under different initial conditions. The red region identifies accelerated expansion. It is clear that the scaling solution D_{sc} is not in the accelerated expansion region.

(3.23) is considered. Hence, an analogous analysis can be transplanted for phantom models. Figure 4 shows the phase space of a phantom dark energy Universe with $w_m = 0$. (A different $w_m \in [0, \frac{1}{3}]$ does not alter the following qualitative descriptions.) Only two fixed points exist in the system: the saddle point D , where the Universe contains only matter, and the stable state P , representing a phantom-dominated attractor. A heteroclinic orbit connecting point D to point P would represent the late-time translation from matter to

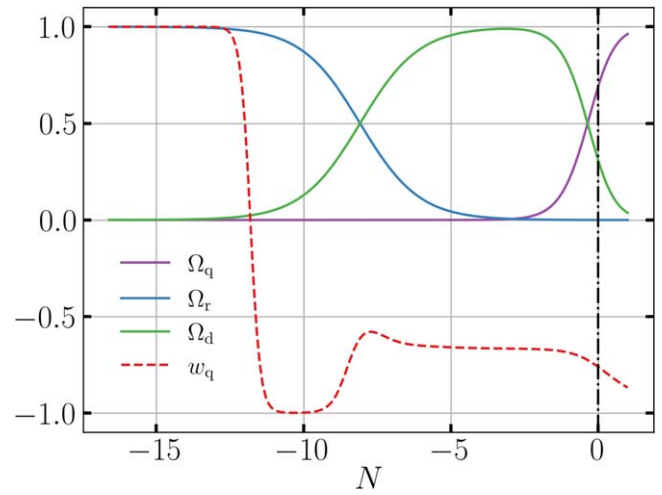


Figure 3. Evolution of Ω_r , Ω_d , Ω_q , and w_q for the potential $V(\phi) = M^2/\phi$ versus e-folding time. The vertical dashed line denotes the present cosmological time, and the present values $\Omega_d \simeq 0.3$ and $\Omega_q \simeq 0.7$ are achieved. Here, the complete curve of w_q is plotted, although it is literally no longer considered quintessence when $w_q > -1/3$. The solution enters the tracking regime in which the field energy density tracks the background fluid density. The initial conditions were chosen from [130].

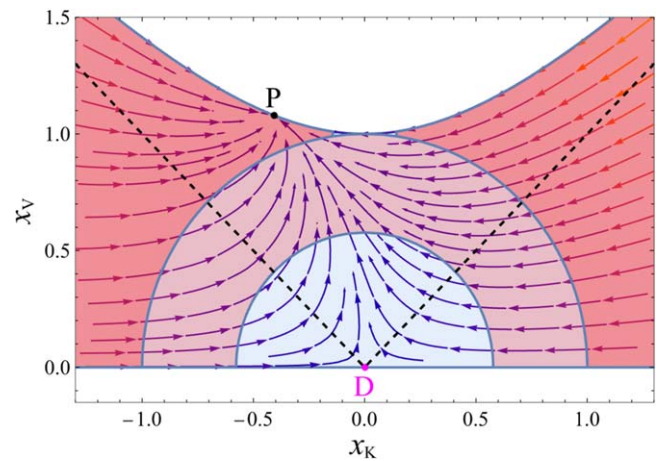


Figure 4. Phase space of x_K and x_V with values $x_r = 0$ and $\lambda = 1$. The magenta saddle fixed point D represents matter domination, whereas the black stable point P represents the phantom-dominated stage. Each streamline represents the evolution of the Universe under different initial conditions. The light red region shows where the Universe undergoes a standard accelerated expansion $-1 < w_{eff} < -1/3$, whereas the dark red region identifies expansion faster than exponential. The two dashed lines correspond to $x_V = \pm x_K$ where the phantom EOS parameter diverges.

phantom domination. On the way to the attractor, the expanding solution for the scalar field will be

$$a(t) \propto (t_\infty - t)^{-2/\lambda^2}, \tag{3.44}$$

ultimately leading to the occurrence of the ‘Big Rip’ singularity within a finite proper time t_∞ [96, 97]. A potential bound from above is necessary to suppress this singularity, because the phantom field rolls up the potential and subsequently converges toward $w_p = -1$ as the field stabilizes at the maximum of potential [98, 99].

Quantum instability always arises in phantom models due to the flipped sign of the kinetic term. Once the phantom quanta interact with other fields, even with gravity, there will be an instability of the vacuum because energy is no longer bound from below [98, 100]. Moreover, phantom models are problematic at the classical level because a strongly anisotropic CMB is expected, unless the Lorentz symmetry is broken for the phantom at an energy scale lower than 3 MeV [100]. Finally, the phantom model with an exponential potential fails to address the coincidence problem and suffers from the fine-tuning of the initial condition. Consequently, the inclusion of sophisticated potentials and matter fluids beyond the range of $[0, \frac{1}{3}]$ have to be considered [101, 102].

3.2.2. Cross the cosmological boundary. The EOS parameters for dark energy in both the quintessence and phantom models are confined to opposite sides of the cosmological boundary, preventing them from crossing over into each other's domain. A no-go theorem indeed confirms this, stating as follows [103]: for the theory of dark energy in the FLRW Universe described by a single fluid or a single scalar field ϕ with a Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi\partial^\mu\phi)$, which is minimally coupled to Einstein gravity, its EOS w_{de} cannot cross the cosmological constant boundary.

In a scenario where gravity is still governed by general relativity, additional degrees of freedom are necessary to enable dark energy to traverse the boundary effectively. The quintom model was proposed as a multifield model incorporating a quintessence-like field and a phantom-like field [104, 105],

Universe will eventually evolve to a stage where phantom dominates $w_{eff} = w_{de} \leq -1$. The w_{de} across the cosmological boundary from above to below if the matter-dominated epoch is followed by a temporary quintessence-dominated stage [105–107]. If future observations will constrain the EOS parameter of dark energy to be below -1 today but above -1 in the past, then the quintom scenario of is arguably the simplest framework where such a situation can arise.

Unfortunately, neither the coupled nor the uncoupled quintom models seem to solve the cosmic coincidence problem and the fine-tuning of the initial conditions simultaneously, as well as the fundamental problems associated with the ghost field. Furthermore, couplings between kinetic terms and scalar fields, such as $\partial_\mu\phi_q\partial^\mu\phi_p$ [108] and the so-called hessence dark energy model [109–111], have also been investigated. Finally, theories beyond general relativity, such as those incorporating higher curvature terms and nonminimally coupled scalar fields between gravity or matter, can also cross the cosmological boundary without encountering negative kinetic energy and associated quantum instability, as discussed in section 4.3.3.

3.3. Noncanonical kinetic term

In the preceding sections, some simplest dynamic dark energy models were introduced, providing as much detail as possible regarding their evolution. Here, we briefly review dark energy models from scalar fields with modified kinetic terms, which hold significant potential for addressing the problems of the Λ CDM model at once.

$$S_{qp}[g^{\mu\nu}, \phi_q, \phi_p] = \int d^4x \sqrt{-g} \mathcal{L}_{qp} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi_q \partial_\nu \phi_q + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_p \partial_\nu \phi_p - V(\phi_q, \phi_p) \right]. \tag{3.45}$$

The energy density, pressure, and corresponding EOS parameter of the quintom model are

$$\rho_{de} = \frac{1}{2} \dot{\phi}_q^2 - \frac{1}{2} \dot{\phi}_p^2 + V(\phi_q, \phi_p), \tag{3.46}$$

$$p_{de} = \frac{1}{2} \dot{\phi}_q^2 - \frac{1}{2} \dot{\phi}_p^2 - V(\phi_q, \phi_p), \tag{3.47}$$

$$w_{de} \equiv \frac{p_{qp}}{\rho_{qp}} = \frac{\dot{\phi}_q^2 - \dot{\phi}_p^2 - 2V(\phi_q, \phi_p)}{\dot{\phi}_q^2 - \dot{\phi}_p^2 + 2V(\phi_q, \phi_p)}. \tag{3.48}$$

It is clear that $w_{qp} \geq -1$ if $\dot{\phi}_q \geq \dot{\phi}_p$, and the cosmological constant scenario is recovered when $\dot{\phi}_q = \dot{\phi}_p$.

The potential $V(\phi_q, \phi_p)$ for noninteracting scalar fields can be generally written as

$$V(\phi_q, \phi_p) = V_q(\phi_q) + V_p(\phi_p), \tag{3.49}$$

where $V_q(\phi_q)$ and $V_p(\phi_p)$ are arbitrary self-interacting potentials for the quintessence and phantom, respectively. In such a case, the two scalar fields can be treated exactly as if they were single models. Because of the fact that $w_p < w_q$, the

3.3.1. Kinetically driven essence. A scalar field model of dark energy with a modified kinetic term is the so-called kinetically driven essence or k-essence, for simplicity. The initial focus of k-essence was on its application to inflation [112], and numerous theoretical models, such as low-energy effective string theory [113], ghost condensate [114], tachyon [115, 116], and Dirac–Born–Infeld (DBI) theory, [117, 118] can be classified under this dark energy model.

The action of the k-essence dark energy model is given by [112]

$$S[g^{\mu\nu}, \phi, \Psi_m] = S_{EH}[g^{\mu\nu}] + \int d^4x \sqrt{-g} P(\phi, X) + S_m[g^{\mu\nu}, \Psi_m], \tag{3.50}$$

where $P(\phi, X)$ is an arbitrary function of a scalar field ϕ and its kinetic energy

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \tag{3.51}$$

The k-essence model can achieve accelerated expansion solely through kinetic energy, even in the absence of a

potential [119, 120], distinguishing it from models with canonical kinetic energy.

The energy–momentum tensor of the k-essence is given by

$$T_{\mu\nu}^k = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}P)}{\delta g^{\mu\nu}} = P_X \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} P, \quad (3.52)$$

where the subscript X of function P represents the partial derivative with respect to X . Compared with the energy–momentum tensor of a perfect fluid (2.4), we have the pressure and energy density of the k-essence,

$$p_k = P, \quad (3.53)$$

$$\rho_k = 2XP_X - P, \quad (3.54)$$

with the four-velocity $u_\mu = \partial_\mu \phi / \sqrt{2X}$. If P depends only on ϕ , we obtain $p_k = -\rho_k$, which corresponds to general relativity with the cosmological constant. If P solely depends on X , we have $\rho_k = \rho_k(X)$. Hence, ρ_k can be rearranged to give $p_k = p_k(\rho_k)$, which is the EOS for an isentropic fluid. In the general case where $p_k = P(\phi, X)$, the hydrodynamic analogy is still useful, and the EOS parameter is acquired directly,

$$w_k \equiv \frac{p_k}{\rho_k} = \frac{P}{2XP_X - P}, \quad (3.55)$$

and w_k approaches to -1 if the condition $|XP_X| \ll |P|$ is satisfied.

In general, a noncanonical scalar field Lagrangian introduces theoretical problems at both the quantum and classical levels. Hence, the selection of the function P has limitations. The propagation sound speed c_s^k of the k-essence field is given by [121],

$$(c_s^k)^2 = \frac{\partial p_k / \partial X}{\partial \rho_k / \partial X} = \frac{P_X}{P_X + 2XP_{XX}}, \quad (3.56)$$

which should be a positive number. Furthermore, due to the positive definiteness of the Hamiltonian [122], both the denominator ξ_1 and numerator ξ_2 of $(c_s^k)^2$ in equation (3.56) should be positive. To be in accordance with the principle of causality, c_s^k is considered favorable if it does not exceed the speed of light, which gives $P_{XX} \geq 0$; however, see Ref. [123].

On account of the unknown dependence on the arbitrary function $P(\phi, X)$, it is too complicated to directly study with dynamic systems techniques. Therefore, to study their dynamics and evolution, k-essence is typically divided into several classes [78]:

$$\text{Class 1: } P(\phi, X) = XG(Y), \quad Y = \frac{X}{V(\phi)}, \quad (3.57)$$

$$\text{Class 2: } P(\phi, X) = K(\phi)\tilde{P}(X), \quad (3.58)$$

$$\text{Class 3: } P(\phi, X) = F(X) - V(\phi), \quad (3.59)$$

where G, K, \tilde{P}, F , and V are functions of their own arguments. It is worth noting that a model can be classified into one of two classes, as seen with the quintessence model, or it may not fit into either category, as exemplified by the DBI model.

The first class of k-essence exhibits a deep connection with scaling solutions that are of great relevance to the cosmic

coincidence problem. It has been proved that [122] scaling solutions appear in k-essence models only if a Lagrangian of the type (3.57) is assumed with $V(\phi) \propto e^{-\lambda\phi/M_{\text{Pl}}}$. Further research indicated that dark energy late-time solutions are invariably unstable in the presence of a scaling solution, unless they belong to the phantom type, even with a coupling to the matter sector [124, 125]. Note that the quintessence exhibits scaling solutions because it corresponds to $G(Y) = 1 - 1/Y$.

Unlike class 1, class 2 is intriguing for its tracking solutions. It has been shown that tracking solutions naturally appear for the models (3.58) during the radiation-dominated era, and a cosmological-constant-like behavior shortly after the transition to matter domination was put forward, which claims to have solved the coincidence problem without fine-tuning the parameters [126, 127]. However, a singularity associated with a diverging sound speed is present in such models [128] so that it cannot arise as a low-energy effective field theory of a causal, consistent, high-energy theory. Moreover, the Lagrangians in these models are hard to construct within the framework of particle physics.

Class 3 (3.59) has a noncanonical kinetic term that appears together with a standard self-interacting potential for the scalar field. The famous (dilatonic) ghost condensate is usually considered a subclass of this type, which is simply recommended in section 3.3.2. Interestingly, some special functions, $F(X)$ and $V(\phi)$, produce scaling solutions, even if the Lagrangian cannot be written in the form of class 1 [129].

3.3.2. (Dilatonic) ghost condensate. The quintessence and phantom models, as special cases of k-essence, have

$$P(\phi, X) = \epsilon X - V(\phi), \quad (3.60)$$

where $\epsilon = \pm 1$ represents quintessence and phantom, respectively.

The denominator and numerator of the propagation sound speed are $\xi_1 = \xi_2 = \epsilon$, indicating that the phantom model is unstable. The instabilities come from the perturbation level, where the gradient energy plays an important role. Therefore, one approach is to introduce a background action resembling that of a ghost, allowing the spatial gradient terms to address the instability issue. To implement the idea, Arkani-Hamed *et al* [114] proposed an effective field theory of a rolling ghost, which can be conceptualized as an infrared modification of gravity. They termed this modification as ghost condensate with the arbitrary function that takes the form

$$P(\phi, X) = -X + \frac{X^2}{M^4}, \quad (3.61)$$

where M is a constant having a dimension of mass. This model was later generalized to a modified version as

$$P(\phi, X) = -X + \frac{X^2}{M^4} \exp\left(\frac{\lambda\phi}{M_{\text{Pl}}}\right), \quad (3.62)$$

which is called dilatonic ghost condensate model [122].

Here, we briefly introduce the cosmic evolution of dilatonic ghost condensate dark energy model. The pressure,

energy density, and corresponding EOS parameter are

$$p_{\text{dGC}} = -X + \frac{X^2}{M^4} e^{\lambda\phi}, \quad (3.63)$$

$$\rho_{\text{dGC}} = -X + \frac{3X^2}{M^4} e^{\lambda\phi}, \quad (3.64)$$

$$w_{\text{dGC}} \equiv \frac{p_{\text{dGC}}}{\rho_{\text{dGC}}} = \frac{1 - e^{\lambda\phi} X/M^4}{1 - 3e^{\lambda\phi} X/M^4}, \quad (3.65)$$

where $M_{\text{Pl}} = 1$ is used for brevity. Quantum stability necessitates $e^{\lambda\phi} X/M^4 \geq 1/2$, resulting in the EOS parameter lying within the range $-1 \leq w_{\text{dGC}} < 1/3$. In particular, the de Sitter solution $w_{\text{dGC}} = -1$ is realized at $X = M^4 e^{-\lambda\phi}/2$. Then, it is possible to explain the present cosmic acceleration for $M \sim 10^{-3}$ eV in ghost condensate model as $\rho_{\text{GC}} = -p_{\text{GC}} = M^4/4$ at the de Sitter point. Hence, the (dilaton) ghost condensate is an example of accelerated expansion that can be generated without the assistance of a potential.

By adjusting the selection of dynamic variables to [130]

$$x_{\text{k}} \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad x_{\text{M}} \equiv \frac{\dot{\phi}^2 e^{\lambda\phi}}{2M^4}, \quad x_{\text{r}} \equiv \frac{\sqrt{\rho_{\text{r}}}}{\sqrt{3}H}, \quad (3.66)$$

we can get the autonomous system describing the dilatonic ghost condensate model. There always exists a stable point describing accelerated expansion toward which the Universe evolves, as long as $0 \leq \lambda < \sqrt{6}/3$. The corresponding sound speed is in the range $0 \leq c_{\text{s}}^{\text{dGC}} < 1/3$, which means that this model does not violate causality. The EOS parameter of this future attractor is [130]

$$w_{\text{eff}} = w_{\text{dGC}} = \frac{\lambda^2 f(\lambda) - 8}{3\lambda^2 f(\lambda) + 8}, \quad (3.67)$$

$$f(\lambda) = 1 + \sqrt{1 + \frac{16}{3\lambda^2}},$$

which suggests that the ghost condensate behaves like a cosmological constant in dynamics during the final stage of cosmic evolution.

3.3.3. Unified dark matter and dark energy. The temptation to unify dark matter and dark energy into a single entity has occurred to many cosmologists almost from the beginning to solve two big dark sector problems at once. Both a fluid and a scalar field are attempted to represent dark matter and dark energy at the same times [131–134]. Here, we introduce a k-essence model that is regarded as dark matter and dark energy simultaneously.

The Lagrangian of such a k-essence model with only a kinetic term is [132]

$$P(\phi, X) = -F_0 + F_2(X - X_0)^2, \quad (3.68)$$

where F_0 and F_2 are constants. Notice that there is an extremum X_0 of the kinetic energy, around which the pressure and energy density of k-essence are approximately

$$p_{\text{k}} \simeq -F_0, \quad (3.69)$$

$$\rho_{\text{k}} \simeq F_0 + 4F_2 X_0(X - X_0). \quad (3.70)$$

Furthermore, by substituting $X = X_0(1 + \varepsilon a^{-3})$ with

$0 < \varepsilon a^{-3} \ll 1$ (ε being a constant), the solution to the continuity equation around X_0 yields the EOS parameter

$$w_{\text{k}} = -\left(1 + \frac{4F_2 X_0^2 \varepsilon a^{-3}}{F_0}\right)^{-1}. \quad (3.71)$$

It is possible to realize $w_{\text{k}} \simeq 0$ during matter domination provided that the condition $4F_2 X_0^2/F_0 \gg 1$ is satisfied, while w_{k} approaches the de Sitter value -1 at late time due to $\varepsilon a^{-3} \rightarrow 0$. This specific model does not suffer from the problem of stability or causality if $X > X_0$, rendering it a more reliable unified model.

3.4. Interacting dark energy model

Typically, dark energy models are based on scalar fields minimally coupled to gravity and do not implement the explicit coupling of the field to background matter, as illustrated by the models introduced previously. However, there is no fundamental reason for this assumption in the absence of an underlying symmetry supposed to suppress the coupling, in particular the dark sector that we barely know anything about from a microscopic and dynamic field theory perspective. Although the interaction between dark energy and normal matter particles is heavily restricted by observations [38, 135], this is not the case for dark matter particles, and neglect of this potential interaction of dark components may result in misinterpretations of observational data. In fact, the study of the interaction between dark energy and dark matter has recently become more promising as a possible solution to the coincident problem and tensions in Λ CDM models.

3.4.1. Phenomenological description. Because there is no established approach from first principles that can specify the form of coupling between dark energy and dark matter, dark matter could manifest as either bosonic or fermionic particles, adhering to or extending beyond the standard model. Likewise, dark energy can be conceptualized as a fluid, a field, or in other forms. Constructing phenomenological models initially, grounded in intuition and experience, is consistently beneficial. These models can then be rigorously tested against diverse observational data.

The interaction between dark matter and dark energy is described by the following modified conservation equations:

$$\dot{\rho}_{\text{dm}} + 3H\rho_{\text{dm}} = Q, \quad (3.72)$$

$$\dot{\rho}_{\text{de}} + 3H(1 + w_{\text{de}})\rho_{\text{de}} = -Q, \quad (3.73)$$

where Q is the interacting kernel, and its sign determines the direction of the energy flux. Specifically, energy transfers from dark energy to dark matter when $Q > 0$, and the transmission reverses when $Q < 0$. The original model is restored if the interaction is ‘turned off’.

To illustrate how interaction between the dark components acts on cosmological dynamics, consider the time

evolution of the ratio $r \equiv \rho_{\text{dm}}/\rho_{\text{de}}$,

$$\begin{aligned} \dot{r} &= \frac{\rho_{\text{dm}}}{\rho_{\text{de}}} \left(\frac{\dot{\rho}_{\text{dm}}}{\rho_{\text{dm}}} - \frac{\dot{\rho}_{\text{de}}}{\rho_{\text{de}}} \right) = -3\Gamma H r, \\ \Gamma &= -w_{\text{de}} - \frac{1+r}{\rho_{\text{dm}}} \frac{Q}{3H}. \end{aligned} \quad (3.74)$$

It is not difficult to observe that the interaction kernel should be the product of energy density and a term with an inverse time dimension. Additionally, it is natural to assume that the kernel depends solely on relevant dynamic quantities (such as ρ_{dm} , ρ_{de} , and a) and possibly their derivatives. The simplest cases can be the following kernels:

$$Q \supset \xi_1 \rho_{\text{dm}} H, \xi_2 \rho_{\text{de}} H, \xi_3 \dot{\rho}_{\text{dm}}, \xi_4 \dot{\rho}_{\text{de}}, \dots$$

and their linear combinations. Other forms of interaction kernels, including nonlinearity and a lot of complexity, are summarized in review papers [136, 137].

Now, let us consider the case $Q = 3H(\xi_1 \rho_{\text{dm}} + \xi_2 \rho_{\text{de}})$ and examine how the interacting dark energy model can resolve the coincidence problem. For this particular kernel,

$$\Gamma = -w_{\text{de}} - \frac{(1+r)(\xi_1 r + \xi_2)}{r}, \quad (3.75)$$

and the stationary solutions are obtained imposing $r_s \Gamma(r_s) = 0$, which gives ($\xi_1 \neq 0$)

$$r_s^\pm = -\frac{w_{\text{de}} + \xi_1 + \xi_2 \pm \sqrt{(w_{\text{de}} + \xi_1 + \xi_2)^2 - 4\xi_1 \xi_2}}{2\xi_1}. \quad (3.76)$$

The coincidence problem is completely solved if one of the roots r_s^\pm is the future attractor of the system. Moreover, the evolution of the Universe should gradually approach the attractor from above, implying that the ratio of matter to dark energy decreases over time. In the special case $\xi_1 = \xi_2 = \xi$, the larger root r_s^- is a past attractor, whereas the smaller root r_s^+ is a future attractor [138, 139]. As the Universe expands, $r(t)$ will evolve from r_s^- to the stable solution r_s^+ , avoiding the coincidence problem if $r_s^+ \sim 3/7$. If $\xi_2 = 0$, the system retains a past attractor r_s^+ with no future attractor ($r_s^- = 0$ in this case). However, the variation is $|\dot{r}/r| < H_0$, slower than that in Λ CDM, which mitigates the coincidence problem [140]. For the $\xi_1 = 0$ case, it is fortunate that the only fixed point $r_s = -\xi_2/(w_{\text{de}} + \xi_2)$ is stable. Consequently, there is no more coincidence problem if we select $\xi_2 \sim 0.3$ under the assumption of $w_{\text{de}} = -1$. Figure 5 shows the phase space of this example to help us understand the conclusion.

While the initial drive behind interacting dark energy models was to solve or alleviate the coincidence problem, their focus has recently shifted toward elucidating the disparity between the observed value of the Hubble constant derived from the CMB data and the local measurements. The interacting dark energy model increases the proportion of late-time dark energy, directly boosting the Hubble expansion rate. Moreover, the reduction in dark matter entails an increase in the Hubble constant to uphold the physical ratio of dark matter energy density in accordance with the CMB constraints. In addition, dark matter decaying into dark energy also reduces the structural growth of matter at a late time, thereby alleviating the S_8 problem. Consequently, the

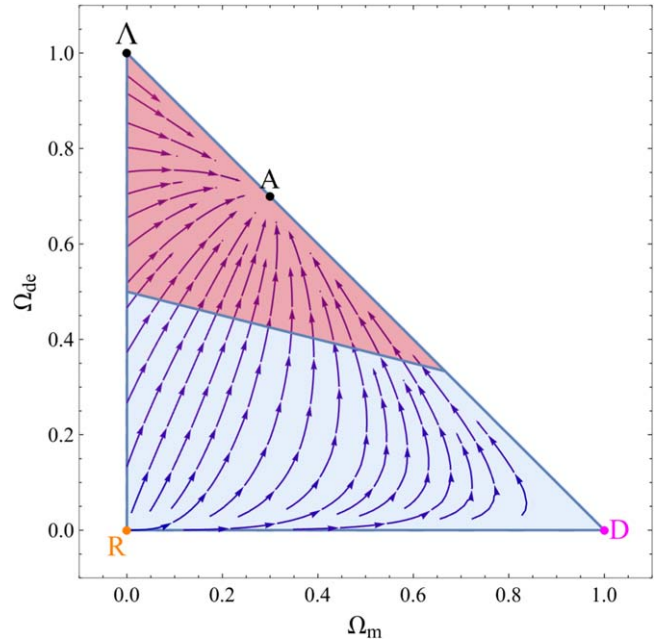


Figure 5. Diagram of Ω_m - Ω_{de} with values $w_{\text{de}} = -1$ and $\xi_2 = 0.3$ in the case where $Q = 3\xi_2 H \rho_{\text{de}}$. The radiation part is not included in Ω_m in this plot. The black point A is the only stable fixed point in the system representing the future attractor with $\Omega_m = 0.3$ and $\Omega_{\text{de}} = 0.7$. The black point Λ is the future attractor of the Λ CDM model, with $\Omega_m = 0$ and $\Omega_{\text{de}} = 1$. The orange point R and magenta point D represent the unstable radiation domination and saddle matter domination, respectively. Each streamline represents the evolution of the Universe under different initial conditions. The red region identifies accelerated expansion.

interacting dark energy model emerges as a multifaceted potential solution to the Hubble tension.

3.4.2. Coupled scalar field. If focusing our attention on dark energy in the form of quintessence, one has the coupled quintessence model with the modified Klein–Gordon equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = -\frac{Q}{\phi}. \quad (3.77)$$

The basic kernel $Q \propto \rho_{\text{dm}} \dot{\phi}$ naturally emerges in certain modified gravity theories (such as $f(R)$ gravity [141, 142]) in the Einstein frame, through which dynamics are introduced in section 4.2.1.

In general, all the aforementioned dark energy models can be generalized by introducing a coupling in the dark sector, following the same approach as with quintessence. The coupled phantom models with exponential potentials and frequently analyzed couplings demonstrate that scaling solutions cannot remain stable [27, 143], as is the case with quintessence. Therefore, they are unable to resolve the cosmic coincidence problem.

The situation is different for the more general k-essence models. Scaling solutions typically exist in Lagrangians of the form $P(\phi, X) = Q^2(\phi)XG(XQ^2(\phi)e^{\lambda\phi/M_{\text{Pl}}})$, where $Q(\phi)$ is the coupling function and G an arbitrary function [125]. However, it has been demonstrated that an evolutionary trajectory capable of resolving the coincidence problem is prohibited due to the

singularity associated with both the dynamic variables and sound speed [125]. As additional resources, the study of coupled tachyonic models can be found in [144–146], whereas coupled DBI scalar field models are discussed in [147, 148].

4. Scalar–tensor gravity

Physicists introduced dark energy to explain the accelerating phase of the Universe, but the nature of this bizarre substance is unknown, and the validity has also been questioned. Because the evidence for dark energy comes entirely from its gravitational effects, which are inferred assuming the validity of general relativity, the consideration of modifications to general relativity, such that the necessity of dark energy is obviated, is another rational approach [37]. In this section, we focus on scalar–tensor gravity, which is one of the most studied modified theories of gravity.

4.1. Introduction to scalar–tensor gravity

The central idea behind the scalar–tensor gravity theory is the incorporation of a scalar field alongside the metric tensor field to participate in the gravitational interaction. This addition of the scalar field allows for variations in the strength of gravity over spacetime, opening new avenues for understanding the fundamental forces of the Universe. The scalar–tensor theory has since evolved and been refined in various forms and models, becoming a focal point for research in cosmology, astrophysics, and fundamental physics [30].

The origins of the scalar–tensor gravity can be traced back to the pioneering work in the mid-20th century. One of the earliest proponents of this theory was the German physicist Jordan [149], who in 1955 introduced the concept of a scalar field coupled to gravity as a means to unify gravity with electromagnetism. Brans and Dicke further developed and formalized the theory in 1961 [150], now known as the Brans–Dicke (BD) theory. The action for the theory is

$$S_{\text{BD}}[g^{\mu\nu}, \Phi, \Psi_m] = \int d^4x \sqrt{-g} \left(\Phi \frac{R}{2\kappa} - \frac{\omega_{\text{BD}}}{2\Phi} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right) + S_m[g^{\mu\nu}, \Psi_m], \tag{4.1}$$

where w_{BD} is the only parameter in the theory. Here, we add a coefficient of $\frac{1}{2}$ to the Lagrangian of the gravitational part of the traditional BD theory. Shortly afterward, Bergmann [151] and Wagoner [152] separately generalized the action to

$$S_J[g^{\mu\nu}, \Phi, \Psi_m] = \int d^4x \sqrt{-g} \left[\Phi \frac{R}{2\kappa} - \frac{W(\Phi)}{2\Phi} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - U(\Phi) \right] + S_m[g^{\mu\nu}, \Psi_m], \tag{4.2}$$

by letting the coupling parameter w_{BD} be a function of the scalar field $W(\Phi)$ and adding a potential term $U(\Phi)$. The subscript ‘J’ represents the Jordan frame.

A suitable conformal transformation

$$g_{\mu\nu} \rightarrow g_{\mu\nu}^* = A^{-2}(\phi) g_{\mu\nu} \tag{4.3}$$

is able to take us from the Jordan frame to the Einstein frame

[153]. We denote the scalar field in the Einstein frame as ϕ instead of the field Φ in the Jordan frame. The action in the Einstein frame is

$$S_E[g_*^{\mu\nu}, \phi, \Psi_m] = \int d^4x \sqrt{-g_*} \left[\frac{R_*}{2\kappa_*} - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] + S_m[A^{-2}(\phi) g_*^{\mu\nu}, \Psi_m], \tag{4.4}$$

where $\kappa_* = 8\pi G_*$, and G_* is a bare gravitational constant. The physical quantity marked with an asterisk indicates that it is in the Einstein frame. The potential $V(\phi)$ in the Einstein frame is related to the potential $U(\Phi)$ in the Jordan frame by the rescaling function $A(\phi)$ as

$$V(\phi) = A^4(\phi) U(\phi). \tag{4.5}$$

The gravity theory is usually not invariant under conformal transformation, including Einstein’s general relativity. A natural question arises: which conformal frame is physical? Unfortunately, as of now, the answer remains unknown. In sections 4.2 and 4.3, the Einstein frame and Jordan frame are both considered, respectively. Cosmologies within these two frames are discussed in the corresponding sections.

4.2. Einstein frame: coupled quintessence

The canonical form of action (4.4) is acquired if $A(\phi)$ satisfies

$$A^2(\phi) = \frac{\kappa}{\kappa_* \Phi}, \tag{4.6}$$

$$\alpha^2(\phi) = \frac{1}{2[2\kappa W(\Phi) + 3]}, \tag{4.7}$$

where $\alpha(\phi) \equiv \text{dln } A/\text{d}\phi$. The relationship between scalar fields Φ and ϕ in Jordan and Einstein frames is easily acquired from equations (4.6) and (4.7) as

$$\frac{\text{d}\phi}{\text{d}\Phi} = \pm \frac{\sqrt{\kappa W(\Phi) + 3/2}}{\Phi}. \tag{4.8}$$

From now on, for simplicity, we will use the unit $\kappa = 1$.

The equations of motion derived from action (4.4) read

$$G_{\mu\nu}^* = \kappa_* (T_{\mu\nu}^{*\text{m}} + T_{\mu\nu}^{*\phi}), \tag{4.9}$$

$$\square_* \phi = \frac{\text{d}V}{\text{d}\phi} - \alpha T_m^*, \tag{4.10}$$

where $T_m^* \equiv g_*^{\mu\nu} T_{\mu\nu}^{*\text{m}}$ is the trace of $T_{\mu\nu}^{*\text{m}}$, and the energy–momentum tensors of the background fluid and scalar field are

$$T_{\mu\nu}^{*\text{m}} = -\frac{2}{\sqrt{-g_*}} \frac{\delta(\sqrt{-g_*} \mathcal{L}_m)}{\delta g_*^{\mu\nu}}, \tag{4.11}$$

$$T_{\mu\nu}^{*\phi} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}^* \left[\frac{1}{2} g_*^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right]. \tag{4.12}$$

When $\alpha = 0$, the field equations of the scalar–tensor gravity in the Einstein frame, equations (4.9) and (4.10), along with the energy–momentum tensors in equations (4.11) and (4.12), are in the same form of the field equations (3.3) and (3.4) and the energy–momentum tensors in equations (3.5) and (3.6) for the quintessence model, respectively, so that the gravity is entirely

described by the metric tensor. The modification of gravity comes from the direct coupling term $\alpha(\phi)T_m^*$ in equation (4.10) between the fluid and scalar field ϕ . Hence, the local conservation of the energy–momentum tensor no longer holds in the Einstein frame, and it changes to

$$\nabla_\mu^* T_{*\text{m}}^{\mu\nu} = \alpha(\phi)T_m^* \nabla_{*\text{m}}^\nu \phi. \tag{4.13}$$

As a result, the test particle no longer moves along the geodesics of $g_{\mu\nu}^*$, as if it suffers from a kind of ‘fifth force’. According to equation (4.13), the original quintessence model is recovered as $\alpha(\phi)$ vanishes, which corresponds to the case where the function $W(\Phi)$ in the Jordan frame diverges. In this subsection, we discuss the results in the Einstein frame, and the asterisk notation of the physical quantity is omitted in the following part of this subsection for simplicity. However, we should keep in mind that we are in the Einstein frame.

4.2.1. Dynamics of coupled quintessence. The fact that the energy density of dark energy is of the same order as that of dark matter in the present Universe suggests that there may be some relationship between them. Among the various coupled dark energy models [80], the coupled quintessence includes an interaction between a scalar field ϕ and dust matter—nonrelativistic matter whose pressure is zero—in the form [154, 155]

$$\begin{aligned} \nabla^\mu T_{\mu\nu}^\phi &= -\alpha(\phi)T_d \nabla^\nu \phi, \\ \nabla^\mu T_{\mu\nu}^d &= \alpha(\phi)T_d \nabla^\nu \phi, \end{aligned} \tag{4.14}$$

where $T_{\mu\nu}^\phi$ and $T_{\mu\nu}^d$ are the energy–momentum tensors of the scalar field and dust. The coupling of the scalar field and radiation vanishes due to the vanishing trace of $T_{\mu\nu}^r$, the energy–momentum tensor of the radiation. Generally speaking, the coupling strength $\alpha(\phi)$ between the scalar field and various components of the Universe is not necessarily uniform [156–159], and baryons are usually treated as uncoupled to the scalar field to avoid an extra long-range force in addition to gravity.

We consider only the case where the coupling α is a constant and the potential is exponential for simplicity;⁵ the

⁵ In the general scalar–tensor theory with action (4.29), the constant coupling condition is approximately satisfied, according to equation (4.56), in section 4.3 if

$$\left(\frac{dF}{d\varphi}\right)^2 \gg \frac{2}{3}\zeta F. \tag{4.15}$$

Then, one gets the function $F(\phi) = \exp(\pm\sqrt{2/3}\phi)$ and the constant coupling $\alpha \approx \pm 1/\sqrt{6}$. Furthermore, a simple way to construct the exponential potential $V(\phi)$ is assuming that the coupling function $F(\varphi)$ and the potential function $U(\varphi)$ satisfy the relation

$$U(\varphi) \propto F^k(\varphi). \tag{4.16}$$

which holds, for instance, when both U and F are power law or exponential, but is also valid for much more complicated functions, such as products of power law and exponential forms. The relation (4.16) was first used by Amendola *et al* [160] to reveal the relationship between the scalar–tensor theory and the coupled quintessence, as well as to study the nonminimal coupling gravity in the strong coupling limit. However, the solutions of cosmic evolution are ruled out by the present constraints on the variability of the gravitational coupling, and they only allow for an energy density $\Omega_\phi \simeq 0.04$ in the form of quintessence [161].

Friedmann equations are equations (3.9) and (3.10), which are the same as for the quintessence. However, the evolution of the dust and scalar field obeys different equations:

$$\dot{\rho}_d + 3H\rho_d = \alpha\rho_m\dot{\phi}, \tag{4.17}$$

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = -\alpha\rho_m\dot{\phi}, \tag{4.18}$$

where ρ_ϕ and p_ϕ have already been shown in equations (3.11) and (3.12).

Introducing the same expansion normalized variables x_K , x_V , and x_r defined in equation (3.29), we have an altered differential equation

$$\begin{aligned} \frac{dx_K}{dN} &= -3x_K + \frac{\sqrt{6}}{2}\lambda x_V^2 - x_K \frac{1}{H} \frac{dH}{dN} \\ &\quad - \frac{\sqrt{6}}{2}\alpha(1 - x_K^2 - x_V^2 - x_r^2) \end{aligned} \tag{4.19}$$

for x_K , whereas the equations for x_V and x_r do not change, so as the EOS parameters and density parameters defined in section 3.1.3. Except for the fixed points in table 1, additional points of the coupled quintessence are listed in table 2, and figure 6 shows fixed points in the phase space with $\lambda = 1$ and different coupling constants α .

In the coupled quintessence model, the fixed point D' (D'_{sc}) replaces D (D_{sc}) in the uncoupled case, and D' (D'_{sc}) returns to D (D_{sc}) when the coupling α vanishes. Notice that both D' and D'_{sc} are scaling solutions, but D' returns to a domination solution, whereas D'_{sc} returns to a scaling solution as $\alpha \rightarrow 0$. The fixed point R_{ex} represents an extra phase of the radiation domination.

Two fixed points, Q and D'_{sc} , are possible to represent the present accelerating phase of the Universe, giving two feasible evolutions. The first trajectory is the sequence $R \rightarrow D' \rightarrow D'_{sc}$, which can give rise to a global attractor with $\Omega_d \sim 0.3$ and $\Omega_q \sim 0.7$, so that it can be used to solve the coincidence problem. However, the coupled quintessence with an exponential potential does not allow for such a cosmological evolution because the condition $\alpha^2 \ll 1$ is required to have point D' compatible with observations, whereas large values of $|\alpha|$ are needed to get the late-time cosmic acceleration [155]. A solution to this problem is to consider a step-like function of the coupling α [162].

The other evolution route is $R \rightarrow D' \rightarrow Q$, where the phase that is intermediate between the radiation-dominated epoch and accelerated epoch is D' , a saddle point in the phase space that replaces point D in the uncoupled model. The presence of the ϕ -matter domination era changes the background expansion history of the Universe as $a \propto t^{2/(3+2\alpha^2)}$. Therefore, one gets a smaller sound horizon at the decoupling epoch, as well as a larger growth rate of matter perturbations relative to the uncoupled quintessence. According to them, the CMB data and the Lyman- α power spectra would put an upper bound $\alpha \sim 0.1$ on the coupling strength [163, 164].

4.2.2. Chameleon mechanism. Although the general couplings between the background fluid and scalar field are not necessarily universal, the preferential coupling of dark

Table 2. Properties of additional fixed points for the coupled quintessence with a constant α and potential (3.23) [130, 155].

Fixed point	(x_K, x_V, x_r)	Ω_r	Ω_d	Ω_ϕ	w_ϕ	w_{eff}
R_{ex}	$\left(-\frac{1}{\sqrt{6}\alpha}, 0, \frac{\sqrt{2\alpha^2-1}}{\sqrt{2}\alpha}\right)$	$1 - \frac{1}{2\alpha^2}$	$\frac{1}{3\alpha^2}$	$\frac{1}{6\alpha^2}$	1	$\frac{1}{3}$
D'	$\left(-\frac{\sqrt{6}\alpha}{3}, 0, 0\right)$	0	$1 - \frac{2\alpha^2}{3}$	$\frac{2\alpha^2}{3}$	1	$\frac{2\alpha^2}{3}$
D'_{sc}	$\left(\frac{\sqrt{6}}{2(\alpha+\lambda)}, \frac{\sqrt{2\alpha^2+2\alpha\lambda+3}}{2(\alpha+\lambda)}, 0\right)$	0	$\frac{\lambda^2+\alpha\lambda-3}{(\alpha+\lambda)^2}$	$\frac{\alpha^2+\alpha\lambda+3}{(q+\lambda)^2}$	$-\frac{\alpha(\alpha+\lambda)}{\alpha(\alpha+\lambda)+3}$	$-\frac{\alpha}{\alpha+\lambda}$

energy to dark matter looks like ‘conceptual fine-tuning’. If the scalar field couples to baryons, unless the coupling is weak enough, a long-range fifth force should be observed. Nevertheless, the experimental bounds on the coupling would not be effectively applied on a large cosmological scale if some type of screening mechanism works.

The chameleon mechanism [165, 166] effectively shields the fifth force by mediating the dynamics of the scalar field with the matter density of the environment. Consequently, the behavior of the scalar field varies in different environments, analogous to a chameleon adjusting its color in response to various surroundings.

Considering the chameleon mechanism, the motion equation of the scalar field (4.10) is rewritten as

$$\square\phi = \frac{dV}{d\phi} - \alpha e^{\alpha\phi} \hat{\rho}_d, \tag{4.20}$$

where $\hat{\rho}_d \equiv e^{-\alpha\phi} \rho_d$, satisfying

$$\dot{\hat{\rho}}_d + 3H\hat{\rho}_d = 0, \tag{4.21}$$

thus $\hat{\rho}_d$ is conserved in the Einstein frame. A nontrivial assumption of the chameleon mechanism is that the matter density considered in the Klein–Gordon equation of motion is the conserved $\hat{\rho}_d$ in the Einstein frame, which is independent of ϕ . According to equation (4.20), the dynamics of the chameleon is not governed by $V(\phi)$ alone, but by the effective potential,

$$V_{\text{eff}}(\phi, \hat{\rho}_d) = V(\phi) + \hat{\rho}_d e^{\alpha\phi}, \tag{4.22}$$

which is a function that also depends on the dust density of the environment.

The value of ϕ at the minimum of the potential, ϕ_{min} , and the mass squared of small fluctuations around the minimum, $m_{\text{eff}}^2 = \partial_\phi^2 V_{\text{eff}}(\phi_{\text{min}})$, depend on the environment as well, provided such a minimum exists. The denser the environment, the more massive is the chameleon. In turn, the effective mass determines the reach of the Yukawa-type potential for the interaction, in a form $\propto e^{-m_{\text{eff}} r}/r$. The larger the effective mass, the weaker is the fifth force associated with the chameleon field, because it results in a faster decay of the interaction with the distance. For a massive object, the fifth force from deep within the exterior profile is Yukawa-suppressed. Consequently, only the contribution from within a thin shell beneath the surface significantly affects the exterior profile, a phenomenon known as the thin-shell effect. Because the chameleon effectively couples only to the shell, whereas the gravity couples to the entire bulk of the object, the chameleon force on an exterior test mass is suppressed compared with the gravitational force. To satisfy the Solar System tests, the Milky Way galaxy must be screened,

which gives the condition [166]

$$\ln \frac{A(\phi_0)}{A(\phi_{\text{MW}})} \lesssim 10^{-6}. \tag{4.23}$$

Here, ϕ_0 represents the cosmic background scalar field today, ϕ_{MW} denotes the ambient field value at the center of the Milky Way, and $A(\phi)$ is the function in the conformal transformation (4.3).

The prototypical chameleon potential considers the Ratra–Peebles potential (3.28); thus, the effective potential is given by, up to an irrelevant constant,

$$V_{\text{eff}}(\phi) = \frac{M^{4+n}}{\phi^4} + \alpha \hat{\rho}_d \phi, \tag{4.24}$$

where it uses the fact that $\phi \ll 1$ over the relevant field range. For a positive coupling α , it displays a minimum at $\phi_{\text{min}} \propto \hat{\rho}_d^{-1/(n+1)}$ and an effective mass $m_{\text{eff}} \propto \hat{\rho}_d^{(n+2)/(n+1)}$ that increases when the ambient matter gets denser.

The most stringent constraint on this model arises from laboratory tests of the inverse-square law, which impose an upper limit of approximately 50 μm in the range of the fifth force, assuming gravitational strength coupling [167]. The constraints on deviations from general relativity, including effective principles as well as post-Newtonian tests in the Solar System and observations of binary pulsars, translate to an upper limit on M of approximately 10^{-3} eV, which coincides with the dark energy scale [165, 168].

An intriguing observation is that the vacuum expectation value of the chameleon field increases in high-density environments. In other words, the effective cosmological constant also increases, consequently leading to a higher local Hubble expansion rate. This suggests that the chameleon dark energy model could potentially serve as a model to elucidate the Hubble tension (and even the S_8 tension) [169].

4.3. Jordan frame: extended quintessence

Quintessence modeled by a nonminimally coupled scalar field is called an extended quintessence. The action for the extended quintessence was given by Perrotta *et al* [170],

$$S[g^{\mu\nu}, \varphi, \Psi_m] = \int d^4x \sqrt{-g} \left[\frac{1}{2} f(\varphi, R) - \frac{1}{2} \zeta(\varphi) g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \right] + S_m[g^{\mu\nu}, \Psi_m], \tag{4.25}$$

where f is a general function of the scalar field φ and Ricci scalar R , and ζ is a function of φ only. Action (4.25) includes a wide variety of theories, such as the $f(R)$ gravity [141, 171],

where

$$f(\varphi, R) = f(R), \quad \zeta(\varphi) = 0, \quad (4.26)$$

the BD gravity, where

$$f(\varphi, R) = \varphi R, \quad \zeta(\varphi) = \frac{\omega_{\text{BD}}}{\varphi}, \quad (4.27)$$

and the dilaton gravity [172], where

$$f(\varphi, R) = 2e^{-\varphi}R - 2U(\varphi), \quad \zeta(\varphi) = -2e^{-\varphi}. \quad (4.28)$$

Here, we focus on particular cases when $f(\varphi, R) = F(\varphi)R - 2U(\varphi)$, and the action becomes

$$S[g^{\mu\nu}, \varphi, \Psi_m] = \int d^4x \sqrt{-g} \left[\frac{1}{2}F(\varphi)R - \frac{1}{2}\zeta(\varphi)g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - U(\varphi) \right] + S_m[g^{\mu\nu}, \Psi_m]. \quad (4.29)$$

The related equations of motion are

$$FG_{\mu\nu} = T_{\mu\nu}^m + \zeta\partial_\mu\varphi\partial_\nu\varphi - \frac{\zeta}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi - g_{\mu\nu}U + (\nabla_\mu\nabla_\nu - g_{\mu\nu}\square)F, \quad (4.30)$$

$$\zeta\square\varphi + \partial_\varphi\zeta \cdot g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + \frac{\partial_\varphi F}{2}R = \partial_\varphi U. \quad (4.31)$$

4.3.1. Nonminimal coupling theory. To illustrate the dynamics of the extended quintessence scenario, we consider the nonminimal coupling theory,

$$F(\varphi) = 1 - \xi\varphi^2, \quad \zeta(\varphi) = 1, \quad (4.32)$$

where ξ is the nonminimal coupling constant. Special values of the constant ξ have received particular attention in the literature. For example, $\xi = 1/6$ corresponds to conformal coupling because the Klein–Gordon equation (4.31) and the physics of φ are conformally invariant if $U = 0$ or $U \propto \varphi^4$ [173]. In addition, $\xi = 0$ and $|\xi| \gg 1$ are the minimal coupling and strong coupling, respectively.

One defines the energy–momentum tensor of the scalar field as⁶ [176]

$$T_{\mu\nu}^\varphi = \partial_\mu\varphi\partial_\nu\varphi - g_{\mu\nu}\left[\frac{1}{2}g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi + U(\varphi)\right] + \xi(g_{\mu\nu}\square - \nabla_\mu\nabla_\nu)(\varphi^2) + \xi\varphi^2G_{\mu\nu}. \quad (4.33)$$

The field equation of the tensor field becomes

$$G_{\mu\nu} = T_{\mu\nu}^m + T_{\mu\nu}^\varphi. \quad (4.34)$$

The cosmological dynamics is then governed by

$$3H^2 = \rho_m + \rho_\varphi, \quad (4.35)$$

$$-2\dot{H} = \rho_m + \rho_\varphi + p_m + p_\varphi, \quad (4.36)$$

where

⁶ There are different possible definitions of the effective energy–momentum tensor for the scalar field in scalar–tensor theories, and the effective energy density, pressure, and EOS are unavoidably linked to one of these definitions. See Ref. [174] for the nonminimal coupling theory and Ref. [175] for the BD theory.

$$\rho_\varphi = \frac{1}{2}\dot{\varphi}^2 + V + 3H\xi\varphi(2\dot{\varphi} + H\varphi), \quad (4.37)$$

$$p_\varphi = \frac{1}{2}\dot{\varphi}^2 - V - \xi[(2\dot{H} + 3H^2)\varphi^2 + 4H\varphi\dot{\varphi} + 2\varphi\ddot{\varphi} + 2\dot{\varphi}^2]. \quad (4.38)$$

The equation of motion for the scalar field in a flat FLRW background is given by

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dU_{\text{eff}}}{d\varphi} = 0, \quad (4.39)$$

where

$$U_{\text{eff}} = U + \frac{\xi}{2}R\varphi^2, \quad (4.40)$$

and $R = 6(2H^2 + \dot{H})$ is the Ricci curvature scalar. At a sufficiently early time when the curvature is high enough, the nonminimal coupling term dominates over the self-interaction potential U . The field φ then settles down to a slow-roll regime, where the friction term, $3H\dot{\varphi}$, balances the term $\xi R\varphi$. After that, φ starts to roll fast, then the coupling can be ignored, and the field behaves as a minimally coupled field.

To have an accelerated expansion with $0 < \xi < 1/6$, one needs a potential $U(\varphi)$ that does not grow with φ faster than the function

$$C(\varphi) \equiv \bar{U}\varphi^k \exp\left(\frac{\varphi^2}{6}\right), \quad (4.41)$$

where \bar{U} is a constant, and $k \approx 0.26/\xi$ by assuming $\Omega_m \simeq 0.3$ and $\Omega_\varphi \simeq 0.7$ today [29]. If instead $\xi < 0$, then $U(\varphi)$ must grow faster than $C(\varphi)$. In this case, k is negative, and upon a rescaling of φ , function (4.41) reduces to a supergravity potential [177].

When the Ratra–Peebles potential is used with $\xi \neq 0$, the tracking solution (see equation (3.41))

$$w_\varphi = \frac{nw_m - 2}{n + 2} \quad (4.42)$$

is still valid, where n is the parameter in the Ratra–Peebles potential (3.28). This shows that the scaling solution does not depend on the coupling ξ , and the solution is always stable because the value of ξ only determines the nature of the stable point [176]. Relation (4.42) generalizes the one found for minimally coupled scalar fields [69, 74, 178].

However, the value of ξ cannot be arbitrary because the scalar field is ultralight. It mediates a long-range force constrained by Solar System experiments. From the effects induced on photon trajectories [25] and equation (4.48), one has

$$\omega_{\text{BD}} = \frac{F_0}{F_0'} > 4 \times 10^4, \quad (4.43)$$

which gives

$$|\xi| < \frac{4.3 \times 10^{-3}}{\sqrt{n(n+2)}}. \quad (4.44)$$

Here, we write the abstract function $F(\varphi)$ instead of a specific shape because restrictions (4.43), as well as equation (4.45),

are generally valid for any form of F . In addition, a constraint from the time variation of the gravitational constant [179]

$$\left. \frac{\dot{G}_{\text{eff}}}{G_{\text{eff}}} \right|_0 = \frac{\dot{F}_0}{F_0} \leq 10^{-11} \text{ yr}^{-1} \quad (4.45)$$

yields a constraint on ξ [180]

$$-10^{-2} \lesssim \xi \lesssim 10^{-1} \quad (4.46)$$

for the Ratra–Peebles potential.

The scalar field, as an additional source of fluctuations, can cause new and observable effects in CMB and the formation of LSSs [181]. In addition, the time variation of the potential U_{eff} between the last scattering surface and the present time would enhance the integrated Sachs–Wolfe effect [182]. The non-minimal coupling in extended quintessence models also modifies the positions of the acoustic peak multipoles, which can be $\sim 10\%$ to $\sim 30\%$ with respect to the standard quintessence [170].

4.3.2. Brans–Dicke theory. Before going through the general scalar–tensor theory, it is helpful to analyze a simple case—the Brans–Dicke theory—to acquire an advanced impression of how nonminimally coupling diverges from the minimal approach in cosmic evolution. Usually, to meet constraint (4.43), a non-self-interacting BD scalar cannot be in the form of quintessence [29], and one needs a BD theory with a potential,

$$S[g^{\mu\nu}, \Phi, \Psi_m] = \int d^4x \sqrt{-g} \left[\frac{\Phi}{2} R - \frac{\omega_{\text{BD}}}{2\Phi} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - U(\Phi) \right] + \int d^4x \sqrt{-g} \mathcal{L}_m. \quad (4.47)$$

Notice that action (4.47) is equivalent to the action of a nonminimal coupling theory by applying a redefinition of the scalar field $\Phi \rightarrow F(\varphi)$ that satisfies

$$F \left(\frac{dF}{d\varphi} \right)^{-2} = \omega_{\text{BD}}. \quad (4.48)$$

To avoid the obvious fraction term in the action, we need the so-called string frame representation [183] of a dilatonic BD theory by rescaling the scalar field $\Phi = e^{\psi}$ into

$$S[g^{\mu\nu}, \psi, \Psi_m] = \int d^4x \sqrt{-g} e^{\psi} \left[\frac{R}{2} - \frac{\omega_{\text{BD}}}{2} g^{\mu\nu} \partial_\mu \psi \partial_\nu \psi - V(\psi) + e^{-\psi} \mathcal{L}_m \right], \quad (4.49)$$

where $V(\psi) = U(\Phi)/\Phi$. Choosing

$$y_{\text{K}} \equiv \frac{\dot{\psi}}{\sqrt{6}H}, \quad x_{\text{V}} \equiv \frac{\sqrt{V}}{\sqrt{3}H}, \quad (4.50)$$

and λ (defined in equation (3.37)) as variables of the phase space in the autonomous system, the density parameters are defined as follows:

$$\Omega_{\text{K}}^{\text{eff}} \equiv \omega_{\text{BD}} y_{\text{K}}^2 - \sqrt{6} y_{\text{K}}, \quad (4.51)$$

$$\Omega_{\text{V}} \equiv x_{\text{V}}^2, \quad (4.52)$$

$$\Omega_{\text{m}}^{\text{eff}} \equiv \frac{e^{-\psi} \rho_{\text{m}}}{3H^2}. \quad (4.53)$$

From the mathematical expression, the effective kinetic energy density, $\Omega_{\text{K}}^{\text{eff}}$ —unlike Ω_{K} in the quintessence model—has not to be positive, and it is rational in physics because the scalar field is a part of gravity in a modified theory rather than a part of cosmic contents.

To investigate the asymptotic dynamics of the BD cosmological model, it is useful to consider the vacuum cosmology with $\Omega_{\text{m}}^{\text{eff}} = 0$. The autonomous ordinary differential equations of the BD theory are derived from the equations of motion [184],

$$\begin{aligned} \frac{dy_{\text{K}}}{dN} &= (\omega_{\text{BD}} y_{\text{K}}^2 - \sqrt{6} y_{\text{K}} - 1) \\ &\times \left[3y_{\text{K}} - \frac{(1 + \lambda)(3y_{\text{K}} + \sqrt{6})}{2\omega_{\text{BD}} + 3} \right], \end{aligned} \quad (4.54a)$$

$$\frac{d\lambda}{dN} = -\sqrt{6} \lambda^2 (\Gamma - 1) y_{\text{K}}, \quad (4.54b)$$

where $N = \ln a$ is the e-folding number, and Γ —assumed to be written as a function of λ —is defined in equation (3.39). We only consider a nonnegative potential $V(\psi)$, i.e. $\Omega_{\text{V}} \geq 0$, which restricts the range of y_{K} in the phase space,

$$\begin{aligned} y_- &\leq y_{\text{K}} \leq y_+, \\ y_{\pm} &= \frac{\sqrt{3} \pm \sqrt{2\omega_{\text{BD}} + 3}}{\sqrt{2}\omega_{\text{BD}}}. \end{aligned} \quad (4.55)$$

The bounds of λ , if any, are set by the concrete form of the potential.

The system has four fixed points listed in table 3. The parameter q in the table is the deceleration parameter. For the solutions in the table, the General Relativity-de Sitter (GR-dS) solution is a potential-dominated solution, which corresponds to the cosmological constant domination in general relativity. In contrast, the Brans Dicke-de Sitter (BD-dS) phase, which does not arise in the minimally coupled model, is a scaling solution, and it arises even if matter exists. The last two stiff-dilaton solutions are dominated by the effective kinetic term.

Further investigation shows that the GR-dS solution is stable. However, Garcia-Salcedo *et al* [184] showed that the BD cosmology does not have the Λ CDM phase as a universal attractor unless the given potential approaches to the exponential form $V \propto e^{\psi}$ as an asymptote⁷. According to

⁷ It has also been shown that very specific conditions on the coupling function $W(\Phi)$ are to be imposed for the given scalar–tensor gravity to have a GR-dS limit [185, 186].

Table 3. Properties of fixed points for the vacuum BD Universe [31].

Fixed point	y_K	λ	Ω_V	q	Γ
GR-dS	0	-1	1	-1	1
BD-dS	$\frac{1}{\sqrt{6(1+\omega_{BD})}}$	0	$\frac{12+17\omega_{BD}+6\omega_{BD}^2}{6(1+\omega_{BD})^2}$	-1	—
Stiff dilaton	y_{\pm}	0	0	$2 + \sqrt{6}y_{\pm}$	—

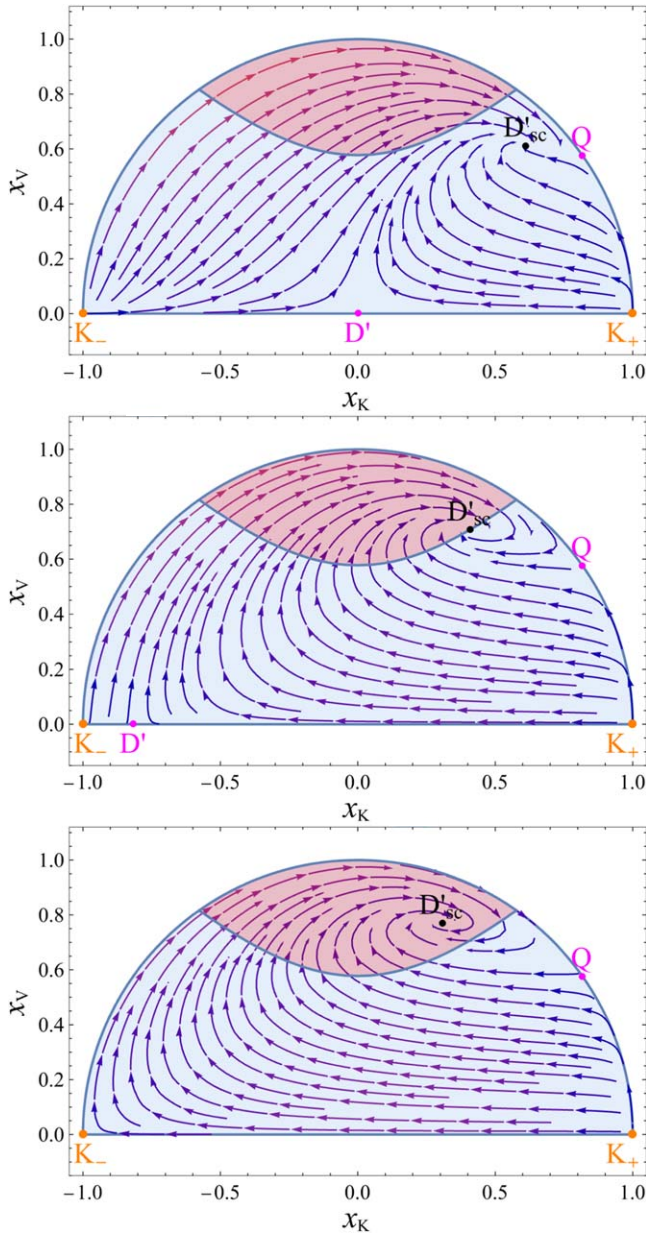


Figure 6. Phase space of x_K and x_V with $\alpha = 0$ (upper subfigure), $\alpha = 1$ (middle subfigure), and $\alpha = 2$ (bottom subfigure). Here, we assume that $x_r = 0$ and $\lambda = 1$. The fixed points of the coupled quintessence are plotted, with yellow, magenta, and black representing the stable, saddle, and unstable fixed points, respectively. Each streamline represents the evolution of the Universe under different initial conditions. The red region indicates accelerated expansion. Each point on the curve $x_K^2 + x_V^2 = 1$ corresponds to $\Omega_q = 1$. Because the future attractor D_{sc}' in the bottom subfigure lies in the red region but not on the semicircle, it offers a solution to the coincidence problem.

table 3, the GR-dS solution corresponds to the exponential potential $V(\psi) \propto e^{\psi}$, which amounts to the quadratic potential $U(\Phi) \propto \Phi^2$ in terms of the standard BD field Φ in action (4.47).

4.3.3. Dynamics of the extended quintessence. Now, let us concentrate on the general scalar-tensor theory. To analyze the dynamics, it is convenient to use the scalar ϕ in the Einstein frame instead of the original φ . The suitable conformal transformation described by the parameter

$$\alpha \equiv -\frac{\partial_\phi F}{2F} = -\frac{\partial_\varphi F}{2F} \left[\frac{3}{2} \left(\frac{\partial_\varphi F}{F} \right)^2 + \frac{\zeta}{F} \right]^{-1/2}, \quad (4.56)$$

takes action (4.29) into

$$S[g^{\mu\nu}, \phi, \Psi_m] = \int d^4x \sqrt{-g} \left[\frac{1}{2} F(\phi) R - \frac{1}{2} (1 - 6\alpha^2) F(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) \right] + S_m[g^{\mu\nu}, \Psi_m]. \quad (4.57)$$

The rescaled scalar field of the extended quintessence in the Einstein frame is

$$\phi = \int d\varphi \sqrt{\frac{3}{2} \left(\frac{\partial_\varphi F}{F} \right)^2 + \frac{\zeta}{F}}. \quad (4.58)$$

In the uncoupled limit $\alpha \equiv d \ln A / d\phi \rightarrow 0$, action (4.57) reduces to the action of the quintessence model.

For simplicity, we treat α as a constant from now on,⁸ leading to

$$F(\phi) = e^{-2\alpha\phi}. \quad (4.61)$$

In a flat FLRW background, the variation of action (4.57)

⁸ Usually, α is not a constant. For instance, the coupling α is field dependent in the nonminimal coupling theory,

$$\alpha(\varphi) = \frac{\xi\varphi}{\sqrt{1 - \xi(1 - 6\xi)\varphi^2}}. \quad (4.59)$$

Here, $\alpha = \xi\varphi$ for $|\xi| \ll 1$, and $\alpha = \pm 1/\sqrt{6}$ in the strong coupling limit $|\xi| \gg 1$. Nevertheless, action (4.57) with a constant α is able to represent some meaningful theories. The BD theory with a potential in equation (4.47) is equivalent to the case in which the parameter ω_{BD} is related to α via the relation

$$2\omega_{BD} + 3 = \frac{1}{2\alpha^2}. \quad (4.60)$$

In the general relativistic limit $\alpha \rightarrow 0$, we have $\omega_{BD} \rightarrow \infty$ as expected. In addition, $f(R)$ theory is acquired in the metric (Palatini) formalism when $\alpha = -1/\sqrt{6}$ ($\alpha^2 \rightarrow \infty$), and the dilaton gravity (4.28) is recovered when $\alpha = 1/\sqrt{2}$.

Table 4. Properties of fixed points for the extended quintessence with a constant α and potential (3.23) in the absence of radiation [187].

Fixed point	(x_K, y_V)	Ω_d	w_{eff}
D''	$\left(\frac{\sqrt{6}\alpha}{3(2\alpha^2-1)}, 0\right)$	$\frac{3-2\alpha^2}{3(1-2\alpha^2)^2}$	$\frac{4\alpha^2}{3(1-2\alpha^2)}$
D''_{sc}	$\left(\frac{\sqrt{6}}{2\lambda}, \sqrt{\frac{3+2\alpha\lambda-6\alpha^2}{2\lambda^2}}\right)$	$1 - \frac{3-12\alpha^2+7\alpha\lambda}{\lambda^2}$	$-\frac{2\alpha}{\lambda}$
K'_{\pm}	$\left(\frac{1}{\sqrt{6}(\alpha+1)}, 0\right)$	0	$\frac{3 \mp \sqrt{6}\alpha}{3(1 \pm \sqrt{6}\alpha)}$
Q'	$\left(\frac{\sqrt{6}(4\alpha-\lambda)}{6(4\alpha^2-\alpha\lambda-1)}, \sqrt{\frac{6-\lambda^2+8\alpha\lambda-16\alpha^2}{6(4\alpha^2-\alpha\lambda-1)^2}}\right)$	0	$-\frac{20\alpha^2-9\alpha\lambda-3+\lambda^2}{3(4\alpha^2-\alpha\lambda-1)}$
Q_{ds}	(0, 1)	0	-1

with respect to the metric $g_{\mu\nu}$ and the scalar field ϕ leads to

$$3FH^2 = \frac{1}{2}(1-6\alpha^2)F\dot{\phi}^2 + U - 3H\dot{F} + \rho_d + \rho_r, \quad (4.62)$$

$$2F\dot{H} = -(1-6\alpha^2)F\dot{\phi}^2 - \dot{F} + H\dot{F} - \rho_d - \frac{4}{3}\rho_r, \quad (4.63)$$

$$(1-6\alpha^2)F\left(\ddot{\phi} + 3H\dot{\phi} + \frac{\dot{F}}{2F}\dot{\phi}\right) + \frac{dU}{d\phi} + \alpha FR = 0, \quad (4.64)$$

where the overdot denotes the derivative of the cosmic time t , and ρ_d (ρ_r) is the energy density of dust (radiation). We introduce the following variables of the autonomous system:

$$\begin{aligned} x_K &\equiv \frac{\dot{\phi}}{\sqrt{6}H}, & y_V &\equiv \frac{1}{H}\sqrt{\frac{U}{3F}}, \\ y_r &\equiv \frac{1}{H}\sqrt{\frac{\rho_r}{3F}} \end{aligned} \quad (4.65)$$

and density parameters for the scalar field, nonrelativistic matter, and radiation

$$\begin{aligned} \Omega_\phi &\equiv (1-6\alpha^2)x_K^2 + y_V^2 + 2\sqrt{6}\alpha x_K, \\ \Omega_d &\equiv \frac{\rho_d}{3FH^2}, & \Omega_r &\equiv y_r^2. \end{aligned} \quad (4.66)$$

A similar relation

$$\Omega_\phi + \Omega_d + \Omega_r = 1 \quad (4.67)$$

comes from equation (4.62). The deceleration parameter q and the effective EOS parameter of the Universe are

$$\begin{aligned} q &= -1 - \frac{\dot{H}}{H^2}, \\ w_{\text{eff}} &= -1 - \frac{2}{3}\frac{\dot{H}}{H^2}, \end{aligned} \quad (4.68)$$

where

$$\begin{aligned} \frac{\dot{H}}{H^2} &= -\frac{1-6\alpha^2}{2}(3+3x_K^2-3y_V^2+y_r^2-6\alpha^2x_K^2+2\sqrt{6}\alpha x_K) \\ &\quad + 3\alpha(\lambda y_V^2-4\alpha). \end{aligned} \quad (4.69)$$

Using equations (4.62)–(4.64), one obtains the differential equations for x_K , y_V , and y_r [187],

$$\begin{aligned} \frac{dx_K}{dN} &= \frac{\sqrt{6}}{2}(\lambda y_V^2 - \sqrt{6}x_K) \\ &\quad + \frac{\sqrt{6}\alpha}{2}[(5-6\alpha^2)x_K^2 + 2\sqrt{6}\alpha x_K - 3y_V^2 + y_r^2 - 1] \\ &\quad - x_K \frac{\dot{H}}{H^2}, \end{aligned} \quad (4.70a)$$

$$\frac{dy_V}{dN} = \frac{\sqrt{6}}{2}(2\alpha - \lambda)x_K y_V - y_V \frac{\dot{H}}{H^2}, \quad (4.70b)$$

$$\frac{dy_r}{dN} = \sqrt{6}\alpha x_K y_r - 2y_r - y_r \frac{\dot{H}}{H^2}, \quad (4.70c)$$

where $N = \ln a$, and $\lambda = -\partial_\phi U/U$ as stated previously.

The fixed points of the system for a constant λ in the absence of radiation ($y_r = 0$) are listed in table 4. The dust-dominated epoch can be realized by either point D'' or point D''_{sc} . If point D'' is responsible for matter domination, the condition $\alpha^2 \ll 1$ is required, leading to $\Omega_m \simeq 1 + 10\alpha^2/3 > 0$ and $w_{\text{eff}} \simeq 4\alpha^2/3$. When $\alpha^2 \ll 1$, the scalar field-dominated point Q' yields an accelerated expansion of the Universe provided that $-\sqrt{2} + 4\alpha < \lambda < \sqrt{2} + 4\alpha$ [187]. The scaling solution D''_{sc} can give rise to EOS, $w_{\text{eff}} \simeq 0$ for $|\alpha| \ll |\lambda|$. In this case, however, the condition $w_{\text{eff}} < -1/3$ for the point Q' gives $\lambda^2 < 2$. The energy fraction of the matter for point D''_{sc} does not satisfy the condition $\Omega_m \simeq 1$. As a result, the only way to describe the evolution from matter domination to scalar field domination in the phase diagram is to trace from point D'' to point Q' . Note that the fixed points in table 4 tend to become the corresponding fixed points in table 1 as $\alpha \rightarrow 0$ in the limit of the general relativity.

Unsurprisingly, the GR-dS phase arises in the extended quintessence model. If $\lambda = 4\alpha$, the fixed point Q' turns into the GR-dS point Q_{ds} , with which corresponds $\Omega_d = 0$ and $w_{\text{eff}} = -1$. Similar to the quintessence model mentioned in section 3.1.3, we have to consider the varying λ case to make it possible for the Universe to evolve from the scaling solution D''_{sc} to the acceleration phase Q' or Q_{ds} . A feasible potential was found by Tsujikawa *et al* [187].

The dynamics of the extended quintessence system are much richer than those of the minimally coupled case [175, 188–191]. One can have spontaneous bouncing, spontaneous entry into and exit from inflation, and superacceleration ($\dot{H} > 0$) in extended models [190]. The phantom EOS and

boundary crossing of cosmological constants naturally arise in scalar–tensor theories with large couplings [187]. Even in the limit $|\alpha| \ll 1$, the phantom EOS can be realized without introducing a ghost field [192–195].

4.3.4. Cosmological attractor to general relativity. In section 4.3.2, we demonstrate that the Λ CDM model serves as an attractor in the BD theory with certain potentials. Additionally, we highlight in this section that, even within more intricate scalar–tensor theories, general relativity itself can emerge as an attractor.

From a cosmological perspective, conformal frames are not equivalent to each other. However, once the physical conformal frame is determined, the different conformal frames are mathematically equivalent. Usually, people regard the Jordan frame as the physical one, as baryons follow the geodesic, and we can directly compare the results with observations. In this case, the Einstein frame is a mathematical treatment that can sometimes be used to avoid complicated calculations in the original Jordan frame, e.g. Damour’s famous work [196, 197].

According to the transformation (4.3), it is easy to notice that

$$\begin{aligned} T_{\mu\nu}^{*m} &= A^2 T_{\mu\nu}^m, & T_{*m\nu}^\mu &= A^4 T_{m\nu}^\mu, \\ T_{*m}^{\mu\nu} &= A^6 T_m^{\mu\nu}, \end{aligned} \quad (4.71)$$

and the transformation of the trace is

$$T_m^* = A^4 T_m. \quad (4.72)$$

One also defines t_* and a_* as

$$t_* \equiv \int A^{-1}(t) dt, \quad (4.73)$$

$$a_* \equiv A^{-1} a, \quad (4.74)$$

so that the line element is still in the form of the FLRW metric

$$ds_*^2 = -dt_*^2 + a_*^2(t_*) \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega_2^2 \right). \quad (4.75)$$

Substituting metric (4.75) into equations of motion (4.9) and (4.10), we obtain

$$3H_*^2 = \kappa_* (\rho_m^* + \rho_\phi), \quad (4.76)$$

$$-2\dot{H}_* = \kappa_* (\rho_m^* + p_m^* + \rho_\phi + p_\phi), \quad (4.77)$$

$$\ddot{\phi} + 3H_* \dot{\phi} + \frac{dV}{d\phi} = -\alpha (\rho_m^* - 3p_m^*), \quad (4.78)$$

where the overdot represents the derivative with respect to the cosmic time t_* , and $H_* \equiv \dot{a}_*/a_*$ is the Hubble parameter in the Einstein frame. The Bianchi identity

$$\begin{aligned} d(\rho_m^* a_*^3) + p_m^* da_*^3 \\ = (\rho_m^* - 3p_m^*) a_*^3 d \ln A(\phi) \end{aligned} \quad (4.79)$$

tells us that the Einstein frame scaling laws are $\rho_r \propto a_*^{-4}$ for the radiation and $\rho_d \propto A(\phi) a_*^{-3}$ for the dust.

Defining the e-folding number $N_* \equiv \ln a_*$ in the Einstein frame and introducing the parameters [198]

$$w(N_*, \phi) \equiv \frac{p_m^*}{\rho_m^*}, \quad v(N_*, \phi) \equiv \frac{V}{\rho_m^*}, \quad (4.80)$$

the equation of motion of the scalar field becomes

$$\begin{aligned} \frac{4(1+v)}{6 - \kappa_* \phi'^2} \phi'' + (1 - w + 2v) \phi' \\ + \frac{2}{\kappa_*} \left[(1 - 3w)\alpha + v \frac{d \ln V}{d\phi} \right] = 0, \end{aligned} \quad (4.81)$$

where the prime denotes the derivative with respect to N_* .

Each scalar–tensor theory is specified by a particular choice for $\alpha(\phi)$ and $V(\phi)$. For example, a constant $\alpha(\phi) = \alpha_{\text{BD}}$ and $V(\phi) = 0$ select the traditional BD theory (4.1). The mechanism of attraction toward general relativity can be illustrated by the simplest case, where $\alpha(\phi) = \beta_{\text{DEF}} \phi$, known as the Damour–Esposito–Farèse (DEF) theory [199], where in cosmology one usually considers the case $\beta_{\text{DEF}} > 0$. Choosing $\kappa_* = 2$ and $V = 0$, equation (4.81) takes the form of the equation of motion of a particle with a velocity-dependent mass $m(\phi') = 2/(3 - \phi'^2)$ in a parabolic potential, $V_*(\phi) = (1 - 3w)\beta_{\text{DEF}}\phi^2/2$, and subjecting to a damping force proportional to $1 - w$ [196]. Notice that the potential V_* vanishes in the radiation-dominated epoch regardless of the value of α . The exact solution to the initial condition $(\phi, \phi')|_{N_* = N_i} = (\phi_i, \phi'_i)$ for the radiation epoch is [196]

$$\begin{aligned} \phi(N_*) &= \phi_i - \sqrt{3} \ln \left(\frac{K_i e^{-N_*} + \sqrt{1 + K_i^2 e^{-2N_*}}}{K_i + \sqrt{1 + K_i^2}} \right), \\ K_i &= \frac{\phi'_i}{\sqrt{3 - \phi_i'^2}}. \end{aligned} \quad (4.82)$$

The duration of the radiation-dominated epoch $\Delta N_*^{\text{rad}} \sim 20$ is long enough to slow down the particle, even if it is ultrarelativistic as long as $\phi'_i \lesssim \sqrt{3}$ [200, 201]. The particle is stationary at the beginning of the matter-dominated epoch. Hence, it is easy to realize that at late times, the field ϕ will settle down at the minimum of the potential V_* with $\phi = 0$, where $\alpha = \beta_{\text{DEF}} \phi = 0$, and the theory flows toward general relativity as the Universe evolves. Figure 7 illustrates the evolution of α as the Universe evolves, revealing that the coupling between matter and scalar fields is currently negligibly small.

Another interesting feature of the DEF theory is that when the gravitational field gets stronger, the compact star may scalarize spontaneously and become a scalarized star if the parameter $\beta_{\text{DEF}} \lesssim -4$ [34, 199]. Spontaneous scalarization and general relativity attractor share different parameter spaces in the DEF theory and some generalized theories [198, 200, 202]. Adding another extra coupling between the scalar field and the Gauss–Bonnet term is a feasible solution to exhibit spontaneous scalarization in the strong-field regime,

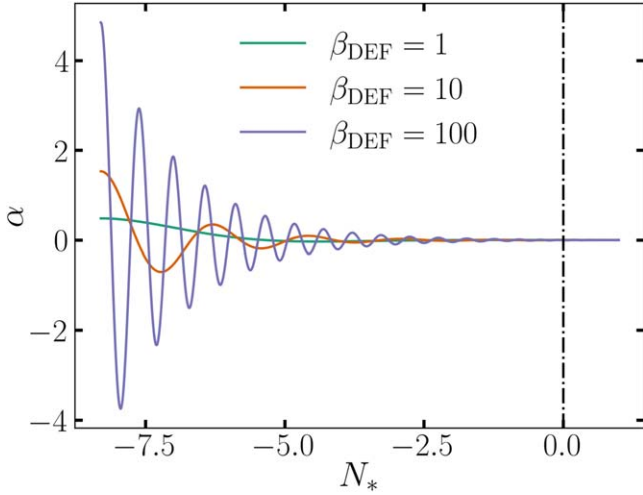


Figure 7. Evolution of the coupling α between matter and scalar field with different β_{DEF} in the DEF theory. The initial conditions are $\phi' = 0$ and ϕ saturating the constraint of the speed-up factor during nucleosynthesis [252]. The current coupling $\alpha_0 = \alpha|_{N_*=0}$ is negligibly small.

and the spacetime behaves like in general relativity at infinity as a cosmological attractor [203, 204].

To explain the accelerated expansion, the potential $V(\phi)$ cannot be zero. If one is interested in a scalar field with an energy density of the same order as that of matter today, then the term $d\ln V/d\phi$ in equation (4.81) turns out to be subdominant with respect to $(1 - 3w)\alpha$ during the radiation domination epoch and most of the matter domination epoch. At late times, the scalar potential term starts to dominate, and gravity is approximately described by general relativity as $\alpha \rightarrow 0$. By choosing the coupling function as

$$\alpha(\phi) = -Be^{-\beta\phi} \quad (4.83)$$

and the Ratra–Peebles potential (3.28) for the scalar field, one obtains an approximate solution during the radiation and matter domination [205]

$$\phi(a_*) \simeq \frac{1}{\beta} \ln \left[\beta B \ln \left(\frac{2}{3} + \frac{a_*}{a_*^{\text{eq}}} \right) + C \right], \quad (4.84)$$

where a_*^{eq} is the value of the scale factor at the equivalence epoch (the cosmic time when $\rho_r = \rho_d$), and C is a constant. This solution is an attractor in phase space and behaves as a subdominant contribution to the energy density that starts increasing during the matter-dominated era and eventually comes to dominate in the present era. The Doppler peaks are shifted toward higher multipoles, and their height is changed with respect to minimally coupled models, providing a signature to look for in future CMB experiments [205].

4.4. Recapitulation and possible extension

A general action describing various couplings to the scalar field for different components of the Universe in the Einstein

frame is given by

$$\begin{aligned} S[g_{\mu\nu}^*, \phi, \Psi_m^{(i)}] &= \int d^4x \sqrt{-g_*} \left[\frac{R_*}{2\kappa_*} - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \\ &+ S_m[g_{\mu\nu}^{(i)}, \Psi_m^{(i)}], \end{aligned} \quad (4.85)$$

where $\Psi_m^{(i)}$ is the matter field coupled to the Jordan frame metric $g_{\mu\nu}^{(i)}$ related to the Einstein frame metric $g_{\mu\nu}^*$ via

$$g_{\mu\nu}^{(i)} = A_i^2(\phi) g_{\mu\nu}^*. \quad (4.86)$$

Action (4.85) reduces to the general relativity if $A_i = 1$, which means that each component of the Universe does not couple to the metric in the Einstein frame. In this case, metrics in the Jordan frame and Einstein frame are the same. In the framework of general relativity, the quintessence model relies on a self-interacting potential of the scalar field and a violation of the strong energy condition to provide the necessary cosmic acceleration. During a slow-roll phase of the massive scalar field, when its kinetic energy is much less than its self-interacting one, negative pressures are achieved, producing the desired cosmic acceleration. The potential, as the only degree of freedom in the quintessence model, can be quite sophisticated and has to be well shaped to reproduce the cosmic evolution or exhibit interesting tracking properties. The coincidence problem is somehow moved to the specific choice of both an appropriate shape and energy scale for the self-interaction potential. Most of the potentials used in quintessence models come out of an effective theory of high-energy physics, leaving a glimpse into new physics. However, deriving such quintessence models (and also the k-essence extension models) from high-energy physics in a self-consistent framework is a challenging task, as combining all the cosmological, gravitational, and particle physics aspects raises many technical difficulties.

The quintessence scalar field couples minimally to gravitation and does not couple directly to ordinary matter. Hence, quintessence only modifies the background cosmic expansion, and all the observations of the gravity theory without the quintessence component remain the same. If matter fields couple to both scalar field and gravitation directly, the new ‘fifth’ interaction yields a violation of the strong equivalence principle [206]. The nonminimal coupling changes the physical coupling constants or inertial masses and, as a consequence, modifies the way energies have been weighted in the cosmic history. In the case where the scalar field directly affects the couplings to the gravitational field and therefore makes the gravitational constant vary, one gets the scalar–tensor theories of gravity,

$$\begin{aligned} S[g_{\mu\nu}^*, \phi, \Psi_m] &= \int d^4x \sqrt{-g_*} \left[\frac{R_*}{2\kappa_*} - \frac{1}{2} g_*^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right] \\ &+ S_m[A^2(\phi) g_{\mu\nu}^*, \Psi_m]. \end{aligned} \quad (4.87)$$

If we consider that approximately 70% of the missing energy is the whole contribution of such a nonminimally coupled scalar field, it might be difficult to match the present tests of gravity without assuming that the nonminimal couplings to be extremely weak. In fact, the coupled quintessence model corresponds to the case $A_b = 1$ for baryons, whereas A_{dm} for dark matter varies with ϕ , which reveals a nonuniversal coupling. This nonuniversal coupling solves the extremely small coupling problem, but the weak equivalent principle is severely violated. However, in chameleon cosmology, both baryons and dark matter are nonminimally coupled to the quintessence field as $A_b = A_{dm} = e^{\alpha\phi}$, with the mass of the scalar field being environmentally dependent. This offers the possibility of accounting for the present bounds of local tests of the strong and weak equivalent principles on Earth and in the Solar System by the larger mass of the scalar field in a denser neighborhood than in the cosmological low-density limit. To do so, a specific self-interaction potential and a consequent violation of the strong energy condition have to be reintroduced. Another alternative model proposes that both A_b and A_{dm} depend on the scalar field but are not identical [207]. The self-interacting potential $V(\phi)$ is unnecessary in this model if both couplings are assumed to be of order unity, and the cosmic acceleration is a generic prediction of the competition between different nonminimal couplings. Moreover, the attractor to which gravitation is driven depends on the ratio between the energy densities of the ordinary matter and dark matter.

The most general four-dimensional scalar–tensor theories having second-order equations of motion are the Horndeski theory [208, 209]

$$S_H[g_{\mu\nu}, \phi, \Psi_m] = \sum_{i=2}^5 \int d^4x \sqrt{-g} \mathcal{L}_i + S_m[g_{\mu\nu}, \Psi_m], \quad (4.88)$$

where the Lagrangian densities are

$$\mathcal{L}_2 = G_2(\phi, X), \quad (4.89a)$$

$$\mathcal{L}_3 = G_3(\phi, X) \square \phi, \quad (4.89b)$$

$$\mathcal{L}_4 = G_4(\phi, X)R + G_{4X}[(\square\phi)^2 - \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi], \quad (4.89c)$$

$$\begin{aligned} \mathcal{L}_5 = & G_5(\phi, X)G_{\mu\nu} \nabla^\mu \nabla^\nu \phi - \frac{G_{5X}}{6} \\ & \times [(\square\phi)^3 - 3 \square \phi \nabla_\mu \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi + 2 \nabla^\mu \nabla_\alpha \phi \nabla^\alpha \nabla_\beta \phi \nabla^\beta \nabla_\mu \phi]. \end{aligned} \quad (4.89d)$$

This theory is termed the Horndeski theory because Kobayashi *et al* [210] found that action (4.88) is equivalent to the one discovered by Horndeski in 1974 [211]. Here, G_i ($i = 2, 3, 4, 5$) are arbitrary functions of the scalar field ϕ and its kinetic energy X , whereas $G_{i\phi}$ and G_{iX} represent the derivatives of functions G_i with respect to ϕ and X , respectively. Because of the second-order property, there is no Ostrogradski instability [212] associated with the Hamiltonian unbounded from below.

In a flat FLRW background, the EOS parameter of the scalar field ϕ is

$$w_\phi = -1 + \frac{2(q_t - M_{pl}^2)\dot{H} - D_6\ddot{\phi} + D_7\dot{\phi}}{\rho_\phi}, \quad (4.90)$$

where

$$q_t = 2G_4 - 2\dot{\phi}^2 G_{4X} + \dot{\phi}^2 G_{5\phi} - H\dot{\phi}^3 G_{5X}, \quad (4.91)$$

and the definition of $D_{6,7}$ and energy density ρ_ϕ can be found in Ref. [213]. Because the evolution of w_ϕ is different depending on dark energy models, it is possible to distinguish them from observations, such as SN Ia, CMB, BAO, LSSs, and weak lensing.

The propagation speed squared of the tensor perturbations in the Horndeski theory is [210]

$$c_t^2 = \frac{1}{q_t}(2G_4 - \dot{\phi}^2 G_{5\phi} - \dot{\phi}^2 \ddot{\phi} G_{5X}). \quad (4.92)$$

The detection of gravitational waves by GW170817 [214] from a binary neutron star merger together with the short gamma-ray burst GRB 170817A [215] constrained the propagation speed c_t of the gravitational waves to be [216]

$$-3 \times 10^{-15} < c_t - 1 < 7 \times 10^{-16}. \quad (4.93)$$

If any fine-tuning among functions is forbidden, one has $G_{4X} = G_{5\phi} = G_{5X} = 0$, then the Horndeski theory is restricted to be of the form [217–220]

$$\begin{aligned} S[g_{\mu\nu}, \phi, \Psi_m] = & \int d^4x \sqrt{-g} [G_2(\phi, X) + G_3(\phi, X) \square \phi + G_4(\phi)R] \\ & + S_m[g_{\mu\nu}, \Psi_m]. \end{aligned} \quad (4.94)$$

Each dark energy model or scalar–tensor gravity mentioned previously corresponds to this reduced action (4.94) with specific functions G_i ($i = 1, 2, 3$). The k-essence model (3.50) is given by the choice

$$G_2 = P(\phi, X), \quad G_3 = 0, \quad G_4 = \frac{M_{pl}^2}{2}, \quad (4.95)$$

and the extended quintessence model (4.29) is acquired if

$$G_2 = \zeta(\phi)X - U(\phi), \quad G_3 = 0, \quad G_4 = \frac{M_{pl}^2}{2}F(\phi). \quad (4.96)$$

There are other modified gravity theories dubbed Galileon theories [208, 209, 221] containing scalar derivative self-interactions $G_3 \neq 0$. In the original Galileon theory [208], the field equations of motion are invariant under the Galilean shift $\partial_\mu \phi \rightarrow \partial_\mu \phi + b_\mu$ in the Minkowski spacetime. In a curved spacetime, the Lagrangian of covariant Galileon theories [209] is constructed to keep the equations of motion up to the second order while recovering the Galilean shift symmetry in the Minkowski limit. For covariant Galileons, there exist self-accelerating de Sitter attractors responsible for late-time cosmic acceleration [222–224]. However, the covariant Galileons with quartic and quintic Lagrangians do not pass the test of tensor perturbation without fine-tuning parameters.

The generally considered function of G_3 is the form of cubic Galileon $X\Box\phi$, which arises in the Dvali–Gabadadze–Porrati braneworld model due to the mixture between longitudinal and transverse gravitons [225] and also in the DBI decoupling theory with bulk Lovelock invariants [226]. This derivative self-interaction can suppress the propagation of the fifth forces in local regions of the Universe with the Vainshtein mechanism [227] while modifying the gravitational interaction at cosmological distances [228–231].

Two kinds of solutions displace the present cosmic acceleration in minimally coupled cubic Galileon theories ($G_4 = M_{\text{pl}}^2/2$). One is the solution with a constant X in the absence of potential [223, 224], but it is in tension with the observational data of redshift space distortions, weak lensing, and integrated Sachs–Wolfe–galaxy cross-correlations [232]. The other is the solution in cubic Galileon with a linear potential $V \propto \phi$ or other specific form [233], and it is the potential that drives the late-time accelerated expansion [221]. In recent years, the Galileon ghost condensate with $G_2 = X + c_2 X^2$ has received more attention for alleviating the problem of tracker solutions found in other cubic Galileon theories and improving compatibility with Planck CMB data [234]. Furthermore, the cosmology of nonminimally coupled cubic Galileons has also been studied [224, 235], including constraints on the coupling from Solar System and binary pulsar tests [236–239].

It is conceivable that the evolution of the Universe within the Horndeski theory undergoes a diverse and dynamic process. Various phenomena, such as late-time tracking, scaling, quintessence, and phantom behaviors, are within the realm of possibility, which offer promising avenues for addressing fine-tuning and coincidence problems. However, the intricate equations of motion arising from these models prevent simple applications of dynamic systems theories, and extremely complicated analyses are required, often dealing with non-compact high-dimensional autonomous systems.

5. Brief outlook

We have presented a concise overview of two distinct physical mechanisms—dark energy and modified gravity—to elucidate the accelerated expansion of the Universe. The technique of dynamic system is applied to typical models—quintessence, coupled quintessence, and extended quintessence—to qualitatively describe the evolution of the Universe and the emergence of an accelerated expansion stage. The primary focus of the discussion revolves around constant couplings and exponential potentials, while more specific and sophisticated models are left to be explored in the original literature.

By assuming that the vacuum energy is either negligible or does not contribute to the gravitational field equation, modern study of cosmological constant problem on the classical level inquiries into identifying mechanisms responsible for the current acceleration of the Universe, so as to avoid the original fine-tuning problem in the Λ CDM model. Nevertheless, for theories, such as quintessence, to accurately

portray the present state of the Universe, characterized by roughly equal proportions of matter and dark energy, the initial conditions of the Universe often need to be finely tuned to an extremely narrow range, thus presenting another instance of fine-tuning. A solution to the so-called coincidence problem would be to establish that the state where $\Omega_{\text{m}} \simeq 0.3$ and $\Omega_{\text{de}} \simeq 0.7$ is indeed the ultimate fate of the Universe, such as the coupled quintessence model. Two additional strategies that potentially alleviate the coincidence problem include expanding the range of initial conditions to accommodate existing observations of the evolution of the Universe and permitting the Universe to evolve at a gradual pace toward a state vastly different from its current configuration, with tracking solutions and some interacting dark energy models as corresponding examples. Unlike the other two issues, the Hubble tension and more tensions observed in cosmological observations represent genuine problems that demand resolution but continue to harbor uncertainty. The interacting dark energy and chameleon dark energy discussed in this review have opportunity to solve these tensions. While these tensions are receiving increased attention, the scope of this review does not delve deeply into the early Universe, and more information can be found in comprehensive review papers [240–242].

At present, dark energy models are not particularly favored over the Λ CDM model from the current observational data, although they are better equipped to deal with various challenges. Furthermore, there is no concrete observational signature for nonminimally coupled theories, despite the fact that they usually have much richer properties in dynamics. In the coming decades, LSS and CMB surveys will possess the capability to probe dynamic dark energy and deviations from general relativity with greater precision [243–246]. Additionally, new methods for observing cosmology, such as gamma-ray burst detection [247, 248] and gravitational wave observation [249, 250], will contribute to the discourse. These advancements hopefully will either rule out some dark energy and modified gravity models or yield breakthroughs in cosmology and physics by confirming new fundamental ingredients.

Numerous models exist to elucidate the current accelerated expansion of the Universe through the employment of scalar fields; however, we are unable to discuss all of them due to space limitations. Certainly, there are models that explain the expansion in terms of fluids or other fields beyond scalar fields, whereas some specific theoretical models also propose alternatives to the solutions discussed in this review [251]. We encourage the readers to explore additional reviews and specialized literature for supplementary insights, both within and beyond the scope of this review. The references we quote do not represent all the research in the field, and we regret any oversight of significant literature.

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