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Hybrid bidirectional quantum communication with different levels of control with simulation

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Abstract
In this paper, we develop a quantum communication protocol for the simultaneous preparation of a two-qubit and a three-qubit state at the positions of two different parties situated spatially apart. For one party, Alice, it is a remote state preparation of a known two-qubit state while for the other party, Bob, it is a joint remote state preparation with the help of a third party, Eve. The protocol is executed in a hybrid form bi-directionally in the presence of two controllers, Charlie and David. There is a hierarchy in the process through different levels of control under which the actions by Alice and Bob are performed. There is a need for a ten-qubit entangled channel connecting the five parties. The generation of this channel through a circuit is discussed. The protocol is executed on the IBM Quantum platform. We also study the effect of noise on our protocol. Here, amplitude-damping, bit-flip and phase-flip noisy environments are considered and the corresponding variations of fidelity are theoretically and numerically analyzed.

Keywords: remote state preparation, joint remote state preparation, hierarchy, entangled state, quantum circuit, controlled protocol

(Some figures may appear in colour only in the online journal)

1. Introduction

Quantum entanglement is a pivotal concept in quantum information technology on which most of the applications of quantum information theory are dependent. With the rapid development of quantum technology in recent times, quantum communication theory is becoming more attractive to physicists, mathematicians and engineers throughout the globe. The use of quantum entanglement has been widely explored in the different fields of quantum communication such as quantum teleportation (QT) [1, 2], remote state preparation (RSP) [3, 4], joint remote state preparation (JRSP) [5, 6], quantum secure dense coding [7, 8], quantum key distribution [9, 10] and quantum secret sharing [11, 12].

QT protocol was introduced by Bennett *et al* [1] in 1993 through which an arbitrary single-qubit quantum state was transferred to a distant party using a pre-shared maximally entangled state as a quantum channel with the help of some classical communication. Teleportation protocols can be classified into various types based on their nature. These include bidirectional quantum teleportation [13, 14], controlled bidirectional quantum teleportation [15], probabilistic teleportation [16], cyclic quantum teleportation [17], multi-hop quantum teleportation [18], conference teleportation (CT) [19], etc. They are designed to serve different purposes in the science of quantum communication. The requirements of

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entangled resources also vary with these variants of teleportation.

RSP protocols constitute another important class of protocols similar to QT, first introduced by Pati [20] in 2000. Its central feature lies in the knowledge of the quantum state to be transmitted. In RSP protocols, the sender knows the quantum state that the sender wants to prepare on the receiver's end whereas in QT the quantum state is arbitrary at least within a specified class. Following the work of Pati [20], several RSP protocols have been published [21–24].

Furthermore, for security and confidentiality reasons, it may be undesirable for a single party to access all the information about the quantum state to be transmitted. In addition, there are situations where the complete information of the state is not available to a single party due to some technical constraints. Instead it is distributed between two parties. In these cases, a new protocol called JRSP was introduced by Xia *et al* [25] in 2007 in which two parties in possession of partial information cooperate to prepare the state at the receiver's end. Several variants of JRSP intended for different purposes have appeared in works such as [26–30].

In a protocol for performing multiple tasks of quantum communication, it may be the case that the importance is not the same for all the tasks. There may be different levels of importance attached to them and therefore it may be desirable to treat them differently within the framework of the same integrated protocol. The above consideration has produced the concept of hierarchical quantum communication protocol.

Hierarchy was introduced in quantum information splitting (QIS) by Gottesman [31] who introduced a scheme where the participants were endowed with different levels of access to the split information. QIS protocols with different hierarchical concepts between the participants were discussed by many authors such as Wang *et al* [32], Shukla *et al* [33], Ma *et al* [34], etc. Hierarchical teleportation was discussed by Bich and An [35] in which teleportation under different levels of control was considered. Hierarchy in the context of RSP protocol was discussed by Ma *et al* [36]. JRSP protocol for the preparation of a single-qubit state with hierarchy of power amongst three receivers was discussed by Chen *et al* in [37]. Hierarchical JRSP in a noisy environment using a five-qubit cluster state has been discussed by Shukla *et al* in [38]. Furthermore, a bidirectional teleportation scheme with different levels of control creating hierarchy has been proposed by Cao *et al* [39] while a hierarchically controlled asymmetric bidirectional teleportation scheme for a single- and a two-qubit state was discussed by Yang [40]. Several more recent works discussed both teleportation and RSP protocols with parties possessing different levels of powers for the purpose of producing the states intended for transfer at their respective ends for which some references are [41–45].

In the present study, we consider one party, Alice, who intends to create a two-qubit state at the site of Bob while the parties Bob and Eve together intend to produce a three-qubit state at the site of Alice. The two-qubit state is fully known to Alice while the information of the three-qubit state is divided between Bob and Eve. The protocol is a bidirectional protocol being a hybrid combination of an RSP and a JRSP scheme. There are two controllers, Charlie and David. The RSP intended by Alice is controlled by Charlie only while the accomplishment of JRSP requires the action of both the controllers, Charlie and David. The protocol is represented through a quantum circuit and is run on an IBM quantum platform using IBM's Qiskit package. Furthermore, we consider the effect of noise on our protocol. Since entanglement is fragile in nature, it is very much affected whenever there is a scope of interaction with the environment. In particular, this occurs when the qubits constituting parts of the quantum channel we used are distributed amongst different parties. The quantum noise is represented through Kraus operators. As particular cases, we consider amplitude-damping, bit-flip and phase-flip noise and study their effect on our protocol by analysis of fidelity, which is identically of unit value when there is no noise.

2. Main protocol

Let us consider a scenario where Alice aims to remotely prepare a known two-qubit state $|\phi\rangle$ in Bob's laboratory and at the same time, Bob and Eve concurrently seek to jointly prepare a known entangled state $|\psi\rangle$ in Alice's laboratory. Here, the states $|\phi\rangle$ and $|\psi\rangle$ are defined as follows:

$$|\phi\rangle = a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle), \tag{1}$$

$$|\psi\rangle = x_0|000\rangle + y_0 \mathrm{e}^{\mathrm{i}\theta}|111\rangle, \tag{2}$$

where, the coefficients a, b, x_0 and y_0 are all assumed to be real and satisfy the normalization conditions $a^2 + b^2 = 1$ and $x_0^2 + y_0^2 = 1$. In addition, it is assumed that θ adheres to the relation $0 \leqslant \theta < 2\pi$. Specifically, Alice possesses knowledge of coefficients a and b, whereas Bob holds information regarding coefficients x_0 and y_0, and only Eve possesses knowledge of θ.

In our proposed protocol, the state that Bob and Eve intend to jointly prepare at Alice's site is more important, which is overseen by controllers Charlie and David. In contrast, the state that Alice aims to remotely prepare at Bob's site is less important and is only controlled by Charlie.

To accomplish the task defined above, all the involved parties, namely, Alice, Bob, Charlie, David and Eve share a ten-qubit entangled state, which is given by:

$$\begin{aligned}|\Xi\rangle = \frac{1}{2\sqrt{2}}[&|0000111000\rangle + |0110111000\rangle \\ &- |0001000111\rangle - |0111000111\rangle - |1011000011\rangle \\ &- |1101000011\rangle + |1010111100\rangle \\ &+ |1100111100\rangle]_{A_1B_1B_3B_2A_2A_3A_4CDE}.\end{aligned} \tag{3}$$

Qubits A_1, A_2, A_3 and A_4 belong to Alice, qubits B_1, B_2 and B_3 are in the hands of Bob, Charlie possesses qubit C, David owns qubit D and Eve has qubit E.

In the following, the proposed protocol is described step by step.

Step 1: Alice performs a single-qubit projective measurement on her qubit A_1 in the basis $\{|\nu^1\rangle, |\nu^2\rangle\}$, which is given as:

$$|\nu^1\rangle_{A_1} = \sqrt{2}\,(a|0\rangle + b|1\rangle),$$
$$|\nu^2\rangle_{A_1} = \sqrt{2}\,(b|0\rangle - a|1\rangle). \quad (4)$$

Since Alice possesses knowledge of a and b, she can make a choice of basis as above. After the measurement, Alice publicly announces her measurement results via a classical channel.

Step 2: At the same time, Bob executes a projective measurement on his qubit B_2 in the basis $\{|\xi^1\rangle, |\xi^2\rangle\}$, which is defined as:

$$|\xi^1\rangle = x_0|0\rangle + y_0|1\rangle,$$
$$|\xi^2\rangle = y_0|0\rangle - x_0|1\rangle. \quad (5)$$

Bob's choice is possible since he has knowledge of x_0 and y_0. After the measurement, Bob publicly announces his measurement results via a classical channel.

Step 3: Eve performs a projective measurement on her particle E in the basis $\{|\tau_1^j\rangle, |\tau_2^j\rangle\}$, which is dependent on Bob's measurement results as follows. If Bob's measurement yields $|\xi^1\rangle$, then Eve selects $\{|\tau_1^1\rangle, |\tau_2^1\rangle\}$ as her basis. Otherwise, she opts for $\{|\tau_1^2\rangle, |\tau_2^2\rangle\}$. The measurement basis $\{|\tau_1^j\rangle, |\tau_2^j\rangle\}$ is defined as:

$$\begin{pmatrix} |\tau_1^j\rangle \\ |\tau_2^j\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} G_j \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix},\ j = 1, 2, \quad (6)$$

where

$$G_1 = \begin{pmatrix} 1 & \mathrm{e}^{-\mathrm{i}\theta} \\ 1 & -\mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} \text{ and } G_2 = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta} & 1 \\ \mathrm{e}^{-\mathrm{i}\theta} & -1 \end{pmatrix}. \quad (7)$$

This choice is possible since Eve has knowledge of the parameter θ.

Step 4: There are two controllers, Charlie and David. If they are satisfied with all the measurement results from Alice, Bob and Eve, they measure their respective single-qubit and share their outcome with the respective parties. The creation of state $|\phi\rangle$ given in (1) in Bob's hands, needs assistance from the controller, Charlie. However, the joint creation of the state $|\psi\rangle$ given in (2), at Alice's place, needs assistance from both the controllers, Charlie and David.

Both the controllers, Charlie and David, measure their respective qubits C and D in the basis given by:

$$|\eta^1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),\ |\eta^2\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \quad (8)$$

and Charlie shares his measurement results with Alice and Bob through a classical channel, while David shares his measurement results with Alice via a classical channel.

Step 5: Finally, Alice and Bob apply a unitary operation to reconstruct the desired respective states. According to the classical information about the measurement results from Bob, Eve, Charlie and David, Alice needs to apply a unitary operator on her qubits A_2,A_3 and A_4 to reconstruct the state $|\psi\rangle$ given in (2). At the same time, to correspond to the classical messages from Alice and Charlie, Bob needs to apply a unitary operation on his qubits B_1 and B_2 to reconstruct the state $|\phi\rangle$ given in (1).

This is the end of the protocol.

Now, we illustrate the protocol in detail.

Equation (3) can be rewritten as:

$$|\Xi\rangle = \frac{1}{2\sqrt{2}} \sum_{i,j,k=1}^{2} |\nu^i\rangle_{A_1} |\xi^j\rangle_{B_2} |\tau_k^j\rangle_E |M_{ijk}\rangle_{B_1B_3A_2A_3A_4CD}, \quad (9)$$

where $|\nu^i\rangle's$ are given in equation (4), $|\xi^j\rangle's$ are given in equation (5) and $|\tau_k^j\rangle's$ are given in equations (6) and (7). In addition, $|M_{ijk}\rangle's$ are given by:

$$\begin{aligned}
|M_{111}\rangle = (&ax_0|0011100\rangle + ax_0|1111100\rangle \\
&- ay_0\mathrm{e}^{\mathrm{i}\theta}|0000011\rangle - ay_0\mathrm{e}^{\mathrm{i}\theta}|1100011\rangle \\
&- by_0\mathrm{e}^{\mathrm{i}\theta}|0100001\rangle - by_0\mathrm{e}^{\mathrm{i}\theta}|1000001\rangle \\
&+ bx_0|0111110\rangle + bx_0|1011110\rangle), \\
|M_{112}\rangle = (&ax_0|0011100\rangle + ax_0|1111100\rangle \\
&+ ay_0\mathrm{e}^{\mathrm{i}\theta}|0000011\rangle \\
&+ ay_0\mathrm{e}^{\mathrm{i}\theta}|1100011\rangle \\
&+ by_0\mathrm{e}^{\mathrm{i}\theta}|0100001\rangle + by_0\mathrm{e}^{\mathrm{i}\theta}|1000001\rangle \\
&+ bx_0|0111110\rangle + bx_0|1011110\rangle), \\
|M_{121}\rangle = (&ay_0\mathrm{e}^{\mathrm{i}\theta}|0011100\rangle + ay_0\mathrm{e}^{\mathrm{i}\theta}|1111100\rangle \\
&+ ax_0|0000011\rangle + ax_0|1100011\rangle \\
&+ bx_0|0100001\rangle + bx_0|1000001\rangle \\
&+ by_0\mathrm{e}^{\mathrm{i}\theta}|0111110\rangle \\
&+ by_0\mathrm{e}^{\mathrm{i}\theta}|1011110\rangle), \\
|M_{122}\rangle = (&ay_0\mathrm{e}^{\mathrm{i}\theta}|0011100\rangle + ay_0\mathrm{e}^{\mathrm{i}\theta}|1111100\rangle \\
&- ax_0|0000011\rangle - ax_0|1100011\rangle \\
&- bx_0|0100001\rangle - bx_0|1000001\rangle \\
&+ by_0\mathrm{e}^{\mathrm{i}\theta}|0111110\rangle \\
&+ by_0\mathrm{e}^{\mathrm{i}\theta}|1011110\rangle), \\
|M_{211}\rangle = (&bx_0|0011100\rangle + bx_0|1111100\rangle \\
&- by_0\mathrm{e}^{\mathrm{i}\theta}|0000011\rangle \\
&- by_0\mathrm{e}^{\mathrm{i}\theta}|1100011\rangle \\
&+ ay_0\mathrm{e}^{\mathrm{i}\theta}|0100001\rangle + ay_0\mathrm{e}^{\mathrm{i}\theta}|1000001\rangle \\
&- ax_0|0111110\rangle - ax_0|1011110\rangle), \\
|M_{212}\rangle = (&bx_0|0011100\rangle + bx_0|1111100\rangle + by_0\mathrm{e}^{\mathrm{i}\theta}|0000011\rangle \\
&+ by_0\mathrm{e}^{\mathrm{i}\theta}|1100011\rangle \\
&- ay_0\mathrm{e}^{\mathrm{i}\theta}|0100001\rangle - ay_0\mathrm{e}^{\mathrm{i}\theta}|1000001\rangle \\
&- ax_0|0111110\rangle - ax_0|1011110\rangle), \\
|M_{221}\rangle = (&by_0\mathrm{e}^{\mathrm{i}\theta}|0011100\rangle + by_0\mathrm{e}^{\mathrm{i}\theta}|1111100\rangle \\
&+ bx_0|0000011\rangle + bx_0|1100011\rangle \\
&- ax_0|0100001\rangle - ax_0|1000001\rangle \\
&- ay_0\mathrm{e}^{\mathrm{i}\theta}|0111110\rangle \\
&- ay_0\mathrm{e}^{\mathrm{i}\theta}|1011110\rangle), \\
|M_{222}\rangle = (&by_0\mathrm{e}^{\mathrm{i}\theta}|0011100\rangle + by_0\mathrm{e}^{\mathrm{i}\theta}|1111100\rangle \\
&- bx_0|0000011\rangle - bx_0|1100011\rangle \\
&+ ax_0|0100001\rangle + ax_0|1000001\rangle \\
&- ay_0\mathrm{e}^{\mathrm{i}\theta}|0111110\rangle \\
&- ay_0\mathrm{e}^{\mathrm{i}\theta}|1011110\rangle).
\end{aligned}$$

From the above equation (9), it can be concluded that after Steps 1, 2 and 3 are performed, if Alice's, Bob's and Eve's measurement outcomes are $|\nu^i\rangle_{A_1}$, $|\xi^j\rangle_{B_2}$ and $|\tau_k^j\rangle_E$, respectively, then the state of the remaining qubits collapses into the state $|M_{ijk}\rangle_{B_1B_3A_2A_3A_4CD}$ with probability $\frac{1}{8}$. Suppose Alice's, Bob's and Eve's measurement results are $|\nu^1\rangle_{A_1}$, $|\xi^1\rangle_{B_2}$ and $|\tau_1^1\rangle_E$, respectively, then the reduced state of the remaining qubits B_1, B_3 A_2, A_3, A_4, C and D is:

$$\begin{aligned}|M_{111}\rangle = (&ax_0|0011100\rangle + ax_0|1111100\rangle - ay_0\mathrm{e}^{\mathrm{i}\theta}|0000011\rangle \\ &- ay_0\mathrm{e}^{\mathrm{i}\theta}|1100011\rangle \\ &- by_0\mathrm{e}^{\mathrm{i}\theta}|0100001\rangle - by_0\mathrm{e}^{\mathrm{i}\theta}|1000001\rangle \\ &+ bx_0|0111110\rangle + bx_0|1011110\rangle)_{B_1B_3A_2A_3A_4CD}.\end{aligned} \tag{10}$$

The above state $|M_{111}\rangle$ can be written as:

$$\begin{aligned}|M_{111}\rangle = &\frac{1}{\sqrt{2}}|\eta^1\rangle_C \otimes [a(|00\rangle + |11\rangle) \\ &+ b(|01\rangle + |10\rangle)]_{B_1B_3} \\ &\otimes (x_0|1110\rangle - y_0\mathrm{e}^{\mathrm{i}\theta}|0001\rangle)_{A_2A_3A_4D} \\ &+ \frac{1}{\sqrt{2}}|\eta^2\rangle_C \otimes [a(|00\rangle - |11\rangle) \\ &+ b(|01\rangle + |10\rangle)]_{B_1B_3} \\ &\otimes (x_0|1110\rangle + y_0\mathrm{e}^{\mathrm{i}\theta}|0001\rangle)_{A_2A_3A_4D}.\end{aligned} \tag{11}$$

Controller Charlie measures his qubit C in the measurement basis $\{|\eta^1\rangle, |\eta^2\rangle\}$ described in Step 4. There are two possible outcomes in Charlie's measurement. From equation (11), it can be seen that irrespective of Charlie's measurement results, the state of qubits B_1 and B_3 is decoupled from the rest of the qubits. Thus, Bob can reconstruct the desired state $|\phi\rangle$ with the help of the controller, Charlie; there is no need for assistance from controller David. However, at this point, Alice cannot reconstruct the state $|\psi\rangle$ because David's qubit D remains entangled with qubits A_2, A_3 and A_4.

If Charlie's measurement result is $|\eta^1\rangle_C$ then the state of qubits B_1 and B_3 is $a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle)$, which is Bob's desired state. At the same time, the joint state of qubits A_2, A_3, A_4 and D is given by:

$$|q\rangle = (x_0|1110\rangle - y_0\mathrm{e}^{\mathrm{i}\theta}|0001\rangle)_{A_2A_3A_4D}. \tag{12}$$

The above state $|q\rangle$ can be written as:

$$\begin{aligned}|q\rangle = &\frac{1}{\sqrt{2}}|\eta^1\rangle_D \otimes (x_0|111\rangle - y_0\mathrm{e}^{\mathrm{i}\theta}|000\rangle)_{A_2A_3A_4} \\ &+ \frac{1}{\sqrt{2}}|\eta^2\rangle_D \otimes (x_0|111\rangle + y_0\mathrm{e}^{\mathrm{i}\theta}|000\rangle)_{A_2A_3A_4}.\end{aligned} \tag{13}$$

Now, the controller, David, makes a measurement on his qubit D in the basis described in Step 4. David will obtain two possible outcomes, $|\eta^1\rangle_D$ and $|\eta^2\rangle_D$, with equal probability. David shares his measurement outcomes with Alice and then, based on that, Alice applies a unitary operator on his qubits A_2, A_3 and A_4 to reconstruct the desired state $|\psi\rangle$.

If David's measurement result is $|\eta^1\rangle_D$, then the reduced state of qubits A_2, A_3 and A_4 is $(x_0|111\rangle - y_0\mathrm{e}^{\mathrm{i}\theta}|000\rangle)$, which is not the desired state. Thus, Alice needs to apply $(\sigma_z\sigma_x)_{A_2} \otimes (\sigma_x)_{A_3} \otimes (\sigma_x)_{A_4}$ as a unitary operator, then the state will become the desired state $|\psi\rangle$.

If David's measurement outcome is $|\eta^2\rangle_D$, then the reduced state of qubits A_2, A_3 and A_4 is $(x_0|111\rangle + y_0\mathrm{e}^{\mathrm{i}\theta}|000\rangle)$, which is also not the desired state. Thus, Alice needs to apply $(\sigma_x)_{A_2} \otimes (\sigma_x)_{A_3} \otimes (\sigma_x)_{A_4}$ as a unitary operator, then the state will become the desired state $|\psi\rangle$.

All possible reduced states and unitary operators corresponding to the measurement outcomes of the parties are given in tables 1 and 2.

3. Generation of the entangled channel

The generation of the quantum channel is important because, without the quantum channel, it is impossible to perform quantum information transfer. The generation of the ten-qubit quantum channel used in our protocol can be done by taking the initial state of all the ten qubits as $|0\rangle$ and applying different quantum gates on the qubits. The steps are the following:

Step 1: Consider the initial state of the ten qubits as:

$$\begin{aligned}|\Xi_0\rangle = &|0\rangle_{q_0} \otimes |0\rangle_{q_1} \otimes |0\rangle_{q_2} \otimes |0\rangle_{q_3} \otimes |0\rangle_{q_4} \\ &\otimes |0\rangle_{q_5} \otimes |0\rangle_{q_6} \otimes |0\rangle_{q_7} \otimes |0\rangle_{q_8} \otimes |0\rangle_{q_9}.\end{aligned} \tag{14}$$

Step 2: A Hadamard gate (H) is applied to each of the three qubits q_0, q_1 and q_4. An X-gate is applied to qubit q_3, then the state $|\Xi_0\rangle$ is transformed into the following state:

$$\begin{aligned}|\Xi_1\rangle = &\frac{1}{2\sqrt{2}}(|0\rangle + |1\rangle)_{q_0} \otimes (|0\rangle + |1\rangle)_{q_1} \\ &\otimes |0\rangle_{q_2} \otimes |0\rangle_{q_3} \otimes (|0\rangle + |1\rangle)_{q_4} \\ &\otimes |0\rangle_{q_5} \otimes |0\rangle_{q_6} \otimes |0\rangle_{q_7} \otimes |0\rangle_{q_8} \otimes |0\rangle_{q_9}.\end{aligned} \tag{15}$$

Step 3: Four controlled-not (CNOT) gate operations are performed on the pairs of qubits (q_1, q_2), (q_4, q_3), (q_4, q_5) and (q_4, q_6), where q_1 and q_4 serve as control qubits and q_2, q_3, q_5 and q_6 are the target qubits. Then, the above state $|\Xi_1\rangle$ becomes:

$$\begin{aligned}|\Xi_2\rangle = &\frac{1}{2\sqrt{2}}(|0001000000\rangle + |0000111000\rangle \\ &+ |0111000000\rangle + |0110111000\rangle \\ &+ |1001000000\rangle + |1000111000\rangle \\ &+ |1111000000\rangle + |1110111000\rangle).\end{aligned} \tag{16}$$

Step 4: A Z-gate is applied to qubit q_3. Two CNOT gates are applied on the qubit pairs (q_0, q_2) and (q_0, q_7), where qubit q_0 acts as a controlled qubit and qubits q_2 and q_7 are target

Table 1. Reduced state of Bob's qubits B_1 and B_3 and unitary operators for Bob's execution corresponding to the measurement results of Alice and Charlie.

Alice's measurement result ($|\nu^i\rangle$)	Charlie's measurement result ($|\eta^l\rangle$)	State of qubits B_1 and B_3	Bob's operation $\sigma^{il}_{B_1B_3}$
$|\nu^1\rangle$	$|\eta^1\rangle_C$	$\{a(|00\rangle+|11\rangle)+b(|01\rangle+|10\rangle)\}_{B_1B_3}$	$I\otimes I$
$|\nu^1\rangle$	$|\eta^2\rangle_C$	$\{a(|00\rangle+|11\rangle)-b(|01\rangle+|10\rangle)\}_{B_1B_3}$	$\sigma_z\otimes\sigma_z$
$|\nu^2\rangle$	$|\eta^1\rangle_C$	$\{b(|00\rangle+|11\rangle)-a(|01\rangle+|10\rangle)\}_{B_1B_3}$	$\sigma_z\otimes\sigma_x\sigma_z$
$|\nu^2\rangle$	$|\eta^2\rangle_C$	$\{b(|00\rangle+|11\rangle)+a(|01\rangle+|10\rangle)\}_{B_1B_3}$	$I\otimes\sigma_x$

Table 2. Reduced state of Alice's qubits A_2, A_3 and A_4 and unitary operators for execution by Alice corresponding to the measurement results of Bob, Eve, Charlie and David.

Bob's and Eve's measurement result ($|\xi^j\rangle$, $|\tau^j_k\rangle$)	Charlie's measurement result ($|\eta^l\rangle$)	David's measurement result ($|\eta^m\rangle$)	State of qubits A_2, A_3 and A_4	Alice's operation $\sigma^{jklm}_{A_2A_3A_4}$
($|\xi^1\rangle$, $|\tau^1_1\rangle$)	$|\eta^1\rangle_C$	$|\eta^1\rangle_D$	$x_0|111\rangle-y_0e^{i\theta}|000\rangle$	$\sigma_z\sigma_x\otimes\sigma_x\otimes\sigma_x$
		$|\eta^2\rangle_D$	$x_0|111\rangle+y_0e^{i\theta}|000\rangle$	$\sigma_x\otimes\sigma_x\otimes\sigma_x$
($|\xi^1\rangle$, $|\tau^1_1\rangle$)	$|\eta^2\rangle_C$	$|\eta^1\rangle_D$	$x_0|111\rangle+y_0e^{i\theta}|000\rangle$	$\sigma_x\otimes\sigma_x\otimes\sigma_x$
		$|\eta^2\rangle_D$	$x_0|111\rangle-y_0e^{i\theta}|000\rangle$	$\sigma_z\sigma_x\otimes\sigma_x\otimes\sigma_x$
($|\xi^1\rangle$, $|\tau^2_1\rangle$)	$|\eta^1\rangle_C$	$|\eta^1\rangle_D$	$x_0|111\rangle+y_0e^{i\theta}|000\rangle$	$\sigma_x\otimes\sigma_x\otimes\sigma_x$
		$|\eta^2\rangle_D$	$x_0|111\rangle-y_0e^{i\theta}|000\rangle$	$\sigma_z\sigma_x\otimes\sigma_x\otimes\sigma_x$
($|\xi^1\rangle$, $|\tau^2_1\rangle$)	$|\eta^2\rangle_C$	$|\eta^1\rangle_D$	$x_0|111\rangle-y_0e^{i\theta}|000\rangle$	$\sigma_z\sigma_x\otimes\sigma_x\otimes\sigma_x$
		$|\eta^2\rangle_D$	$x_0|111\rangle+y_0e^{i\theta}|000\rangle$	$\sigma_x\otimes\sigma_x\otimes\sigma_x$
($|\xi^2\rangle$, $|\tau^1_2\rangle$)	$|\eta^1\rangle_C$	$|\eta^1\rangle_D$	$x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$I\otimes I\otimes I$
		$|\eta^2\rangle_D$	$-x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$\sigma_x\sigma_z\sigma_x\otimes I\otimes I$
($|\xi^2\rangle$, $|\tau^1_2\rangle$)	$|\eta^2\rangle_C$	$|\eta^1\rangle_D$	$-x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$\sigma_x\sigma_z\sigma_x\otimes I\otimes I$
		$|\eta^2\rangle_D$	$x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$I\otimes I\otimes I$
($|\xi^2\rangle$, $|\tau^2_2\rangle$)	$|\eta^1\rangle_C$	$|\eta^1\rangle_D$	$-x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$\sigma_x\sigma_z\sigma_x\otimes I\otimes I$
		$|\eta^2\rangle_D$	$x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$I\otimes I\otimes I$
($|\xi^2\rangle$, $|\tau^2_2\rangle$)	$|\eta^2\rangle_C$	$|\eta^1\rangle_D$	$x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$I\otimes I\otimes I$
		$|\eta^2\rangle_D$	$-x_0|000\rangle+y_0e^{i\theta}|111\rangle$	$\sigma_x\sigma_z\sigma_x\otimes I\otimes I$

qubits. Then, the state $|\Xi_2\rangle$ is transformed into:

$$|\Xi_3\rangle=\frac{1}{2\sqrt{2}}(-|0001000000\rangle+|0000111000\rangle-|0111000000\rangle+|0110111000\rangle-|1011000100\rangle+|1010111100\rangle-|1101000100\rangle+|1100111100\rangle). \quad (17)$$

Step 5: Three CNOT gates are executed on the pairs of qubits (q_3, q_7), (q_3, q_8) and (q_3, q_9) with qubit q_3 acting as the control qubit and qubits q_7, q_8 and q_9 as the target qubits. Then, the quantum state $|\Xi_3\rangle$ becomes:

$$|\Xi_4\rangle=\frac{1}{2\sqrt{2}}(-|0001000111\rangle+|0000111000\rangle-|0111000111\rangle+|0110111000\rangle-|1011000011\rangle+|1010111100\rangle-|1101000011\rangle+|1100111100\rangle), \quad (18)$$

which is used as a quantum channel $|\Xi\rangle$ in this paper. The complete circuit diagram for generating the entangled channel (3) is shown in figure 1.

4. Protocol under noisy environments

Quantum noise is an unavoidable phenomenon in the practical scenario. When the qubits are distributed after the generation of the quantum entangled state to the involved parties, quantum noise comes into play. Suppose the quantum channel used in our protocol is prepared by Alice in her laboratory and she keeps particles A_1, A_2, A_3 and A_4 for herself and distributes particles B_1, B_2 and B_3 to Bob, particle C to Charlie, D to David and E to Eve through quantum channels. Only particles B_1, B_2, B_3, C, D and E are affected by noise. We have used 'Wolfram Mathematica' software for the calculation of noise and its effects on the protocol.

The quantum channel (3) can be rewritten as:

$$|\Xi\rangle = \frac{1}{2\sqrt{2}}[|0111000000\rangle + |0111110000\rangle - |0000001111\rangle - |0000111111\rangle - |1000011011\rangle - |1000101011\rangle + |1111010100\rangle + |1111100100\rangle]_{A_1A_2A_3A_4B_1B_3B_2CDE}. \tag{19}$$

The evolution of the quantum channel under the influence of quantum noise can be depicted through the transformation:

$$T(\rho) = \sum_{i,j,k,l,m,n} K_{ijklmn}|\Xi\rangle_{A_1A_2A_3A_4B_1B_3B_2CDE}\langle\Xi|K^{\dagger}_{ijklmn}, \tag{20}$$

where $\sum_{i,j,k,l,m,n}(K^{\dagger}_{ijklmn}K_{ijklmn}) = I$, $K_{ijklmn} = I_{A_1}\otimes I_{A_2}\otimes I_{A_3}\otimes I_{A_4}\otimes X_i\otimes X_j\otimes X_k\otimes X_l\otimes X_m\otimes X_n$ and $X_i's$ are the Kraus operators corresponding to different noises. All the parties do their respective jobs described in the previous section. Suppose that Alice's, Bob's, Eve's, Charlie's and David's measurement outcomes are $|\nu^i\rangle_{A_1}$, $|\xi^j\rangle_{B_2}$, $|\tau^j_k\rangle_E$, $|\eta^l\rangle_C$ and $|\eta^m\rangle_D$, respectively. Then, the output state of the protocol can be expressed as:

$$\rho^{\text{out}}_{ijklm} = \text{Tr}_{A_1B_2ECD}\left(\frac{U_{ijklm}[T(\rho)]U^{\dagger}_{ijklm}}{\text{tr}(U^{\dagger}_{ijklm}U_{ijklm}[T(\rho)])}\right), \tag{21}$$

where the partial trace $\text{Tr}_{A_1B_2ECD}$ is performed over qubits (A_1, B_2, E, C, D) and U_{ijklm} is defined as follows:

$$\begin{aligned}U_{ijklm} = &\{I_{A_1}\otimes\sigma^{il}_{B_1B_3}\otimes I_{B_2}\otimes\sigma^{jklm}_{A_2A_3A_4}\\ &\otimes I_C\otimes I_D\otimes I_E\}\\ &\{I_{A_1}\otimes I_{B_1}\otimes I_{B_3}\otimes I_{B_2}\otimes I_{A_2}\otimes I_{A_3}\\ &\otimes I_{A_4}\otimes I_C\otimes|\eta^m\rangle_D\langle\eta^m|\otimes I_E\}\\ &\{I_{A_1}\otimes I_{B_1}\otimes I_{B_3}\otimes I_{B_2}\otimes I_{A_2}\otimes I_{A_3}\\ &\otimes I_{A_4}\otimes|\eta^l\rangle_C\langle\eta^l|\otimes I_D\otimes I_E\}\\ &\{I_{A_1}\otimes I_{B_1}\otimes I_{B_3}\otimes I_{B_2}\otimes I_{A_2}\otimes I_{A_3}\otimes I_{A_4}\\ &\otimes I_C\otimes I_D\otimes|\tau^j_k\rangle_E\langle\tau^j_k|\}\\ &\{I_{A_1}\otimes I_{B_1}\otimes I_{B_3}\otimes|\xi^j\rangle_{B_2}\langle\xi^j|\otimes I_{A_2}\\ &\otimes I_{A_3}\otimes I_{A_4}\otimes I_C\otimes I_D\otimes I_E\}\\ &\{|\nu^i\rangle_{A_1}\langle\nu^i|\otimes I_{B_1}\otimes I_{B_3}\otimes I_{B_2}\otimes I_{A_2}\\ &\otimes I_{A_3}\otimes I_{A_4}\otimes I_C\otimes I_D\otimes I_E\},\end{aligned} \tag{22}$$

where $i, j, k, l, m \in \{1, 2\}$. Unitary operators $\sigma^{il}_{B_1B_3}$ and $\sigma^{jklm}_{A_2A_3A_4}$ are given in table 1 and table 2, respectively.

The influence of noisy channels on the quantum protocol can be measured by fidelity. When Alice's, Bob's, Eve's, Charlie's and David's measurement outcomes are $|\nu^i\rangle_{A_1}$, $|\xi^j\rangle_{B_2}$, $|\tau^j_k\rangle_E$, $|\eta^l\rangle_C$ and $|\eta^m\rangle_D$, respectively, fidelity of the protocol is given by:

$$F_{ijkl} = \langle\Psi|\rho^{\text{out}}_{ijklm}|\Psi\rangle, \tag{23}$$

where $|\Psi\rangle$ represents the ideal output state. In the present

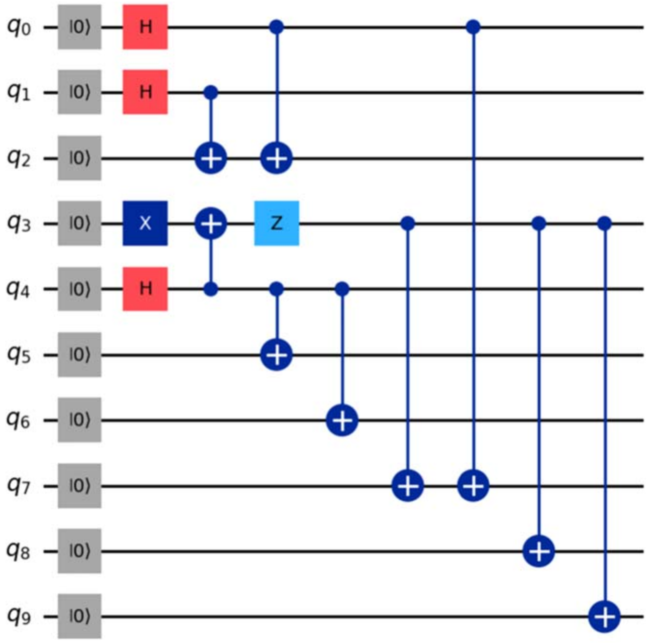


Figure 1. Circuit diagram for the generation of the entangled channel.

protocol, the ideal output state is:

$$\begin{aligned}|\Psi\rangle &= (x_0|000\rangle + y_0 e^{i\theta}|111\rangle)_{A_2A_3A_4}\\ &\quad\otimes(a(|00\rangle + |11\rangle) + b(|01\rangle + |10\rangle))_{B_1B_3}\\ &= [ax_0|00000\rangle + ay_0 e^{i\theta}|11100\rangle\\ &\quad + ax_0|00011\rangle + ay_0 e^{i\theta}|11111\rangle\\ &\quad + bx_0|00001\rangle + by_0 e^{i\theta}|11101\rangle\\ &\quad + bx_0|00010\rangle + by_0 e^{i\theta}|11110\rangle]_{A_2A_3A_4B_1B_3}.\end{aligned} \tag{24}$$

Since final output state $\rho^{\text{out}}_{ijklm}$ is obtained by applying unitary operation by the receivers, after the measurement is performed by all the parties on respective qubits, the final output states $\rho^{\text{out}}_{ijklm}$ for $i, j, k, l, m \in \{1, 2\}$ are the same, and the total fidelity of the protocol is defined as:

$$F = \frac{1}{32}\sum_{i,j,k,l,m} F_{ijkl} = F_{11111}. \tag{25}$$

In the following, we analyze cases of specific types of noises applying the general case discussed above.

4.1. Amplitude-damping noisy environment

Amplitude-damping (AD) noise typically occurs when a quantum system interacts with its surrounding environment, causing the system to lose energy and transition to a lower-energy state. In the case of a two-level quantum system (a qubit), the Kraus operators for amplitude-damping noise are typically represented by a set of operators denoted as:

$$X_0 = \begin{pmatrix}1 & 0\\ 0 & \sqrt{1-q}\end{pmatrix} \text{ and } X_1 = \begin{pmatrix}0 & \sqrt{q}\\ 0 & 0\end{pmatrix}, \tag{26}$$

where q is the noise intensity rate in an AD noisy environment.

The quantum channel is transformed to $T_{\rm AD}(\rho)$ according to equation (20). Then, all the parties do their measurement, as in the illustration. Suppose Alice's, Bob's, Eve's, Charlie's and David's measurement outcomes are $|\nu^1\rangle_{A_1}$, $|\xi^1\rangle_{B_2}$, $|\tau_1^1\rangle_E$, $|\eta^1\rangle_C$ and $|\eta^1\rangle_D$, respectively. Then, the output state according to formula (21) of the protocol can be represented as:

$$\rho_{\rm AD}^{\rm out}=\rho_{11111}^{\rm out}=\sum_{i=1}^{64}|R_i\rangle_{A_2A_3A_4B_1B_3}\langle R_i|, \tag{27}$$

where the expressions of $|R_i\rangle$ (for $i=1$ to 64) are given in appendix A, table A1.

According to formulas (23) and (25), the fidelity is calculated as:

$$\begin{aligned}F^{\rm AD}=&\frac{1}{2}[2+q\{-2+b^2(4-6q)+8b^4(-1+q)+q\}\\&+qy_0^2\{(-3+2q)(2+(-2+q)q)\\&-2b^2(-2+q)(4-9q+6q^2)\\&+8b^4(-1+q)(4+q(-7+2q))\\&-2(-1+q)(2-2q+(-1+2b^2)\\&\times(4b^2(-2+q)(-1+q)-q^2))y_0^2\}].\end{aligned} \tag{28}$$

The fidelity $F^{\rm AD}$ is dependent on the AD noise parameter q, on Alice's known state amplitude b and on Bob's known parameter y_0. The variations of the fidelity are given in figure 2.

4.2. Bit-flip noisy environment

In a noisy environment susceptible to bit-flip (BF) errors, these errors occur with a probability s (where $0\leqslant s\leqslant 1$), and the state remains unchanged with a probability of $(1-s)$. Here, s is the noise parameter. The Kraus operators associated with bit-flip noise are:

$$X_0=\begin{pmatrix}\sqrt{1-s}&0\\0&\sqrt{1-s}\end{pmatrix}\text{and }X_1=\begin{pmatrix}0&\sqrt{s}\\\sqrt{s}&0\end{pmatrix}. \tag{29}$$

The quantum channel is transformed according to formula (20), which is a mixed state. Then, all the parties measure their qubits according to the protocol. After the measurement is complete, suppose Alice's, Bob's, Eve's, Charlie's and David's measurement outcomes are $|\nu^1\rangle_{A_1}$, $|\xi^1\rangle_{B_2}$, $|\tau_1^1\rangle_E$, $|\eta^1\rangle_C$ and $|\eta^1\rangle_D$, respectively. The output state according to formula (21) of the protocol can be represented as:

$$\rho_{\rm BF}^{\rm out}=\rho_{11111}^{\rm out}=\sum_{i=1}^{64}|H_i\rangle_{A_2A_3A_4B_1B_3}\langle H_i|, \tag{30}$$

where the expressions of $|H_i\rangle$ (for $i=1$ to 64) are given in appendix B, table B1.

According to formulas (23) and (25), the fidelity is calculated as:

$$\begin{aligned}F^{\rm BF}=&-(2(1-4b^2)^2(s-1)s+1)\\&\times(4s(y_0^2-1)y_0^2\cos^2(\theta)+s-1).\end{aligned} \tag{31}$$

We observe that the fidelity $F^{\rm BF}$ of the protocol is influenced by the noise parameter s as well as the coefficients of the input states. Figure 3 illustrates how the fidelity varies with different parameters.

4.3. Phase-flip noisy environment

In this environment filled with noise, the relative phase undergoes a flip with a probability of p (where $0\leqslant p\leqslant 1$) and remains unchanged with a probability of $(1-p)$. Here, p is the noise parameter. The Kraus operators associated with phase-flip (PF) noise are:

$$X_0=\begin{pmatrix}\sqrt{1-p}&0\\0&\sqrt{1-p}\end{pmatrix}\text{and }X_1=\begin{pmatrix}\sqrt{p}&0\\0&-\sqrt{p}\end{pmatrix}. \tag{32}$$

According to formula (20), the quantum channel is transformed. If Alice's, Bob's, Eve's, Charlie's and David's measurement outcomes are $|\nu^1\rangle_{A_1}$, $|\xi^1\rangle_{B_2}$, $|\tau_1^1\rangle_E$, $|\eta^1\rangle_C$ and $|\eta^1\rangle_D$, respectively, then the output state according to formula (21) of the protocol can be represented as:

$$\rho_{\rm PF}^{\rm out}=\rho_{11111}^{\rm out}=\sum_{i=1}^{64}|G_i\rangle_{A_2A_3A_4B_1B_3}\langle G_i|, \tag{33}$$

where the expressions of $|G_i\rangle$ (for $i=1$ to 64) are given in appendix C, table C1.

According to formulas (23) and (25), the fidelity is calculated as:

$$\begin{aligned}F^{\rm PF}=&-16(p-1)p(y_0^2-1)y_0^2\\&\times(4b^4(4(p-1)p(2(p-1)p+1)+1)\\&+b^2(8p(p(-2(p-2)p-3)+1)-2)\\&+(2(p-1)p+1)^2)-2\\&\times(8b^4-4b^2-1)(p-1)p+1.\end{aligned} \tag{34}$$

It can be seen that fidelity $F^{\rm PF}$ of the protocol depends on the noise parameter p and on the coefficients of the input states. Variations of the fidelity with different parameters are presented in figure 4.

5. Execution of the protocol on the IBM quantum platform

In order to test the feasibility of our proposed scheme, we initiated a simulation experiment using IBM's Qiskit Aer (1.0) on the IBM quantum platform. This involved constructing a quantum circuit comprising ten qubits and ten classical bits, which we subsequently executed utilizing the 'Aer Simulator'. Following the execution, we captured and analyzed the resultant output states of both Alice and Bob. The entire process, along with its associated circuit diagram, is depicted in figure 5.

The comprehensive description of the entire process of the quantum circuit diagram is described part by part in the following. In figure 5, qubits q_0, q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8 and q_9 represent qubits A_1, B_1, B_3, B_2, A_2, A_3, A_4, C, D and E, respectively.

Part I involves the preparation of an entangled channel.

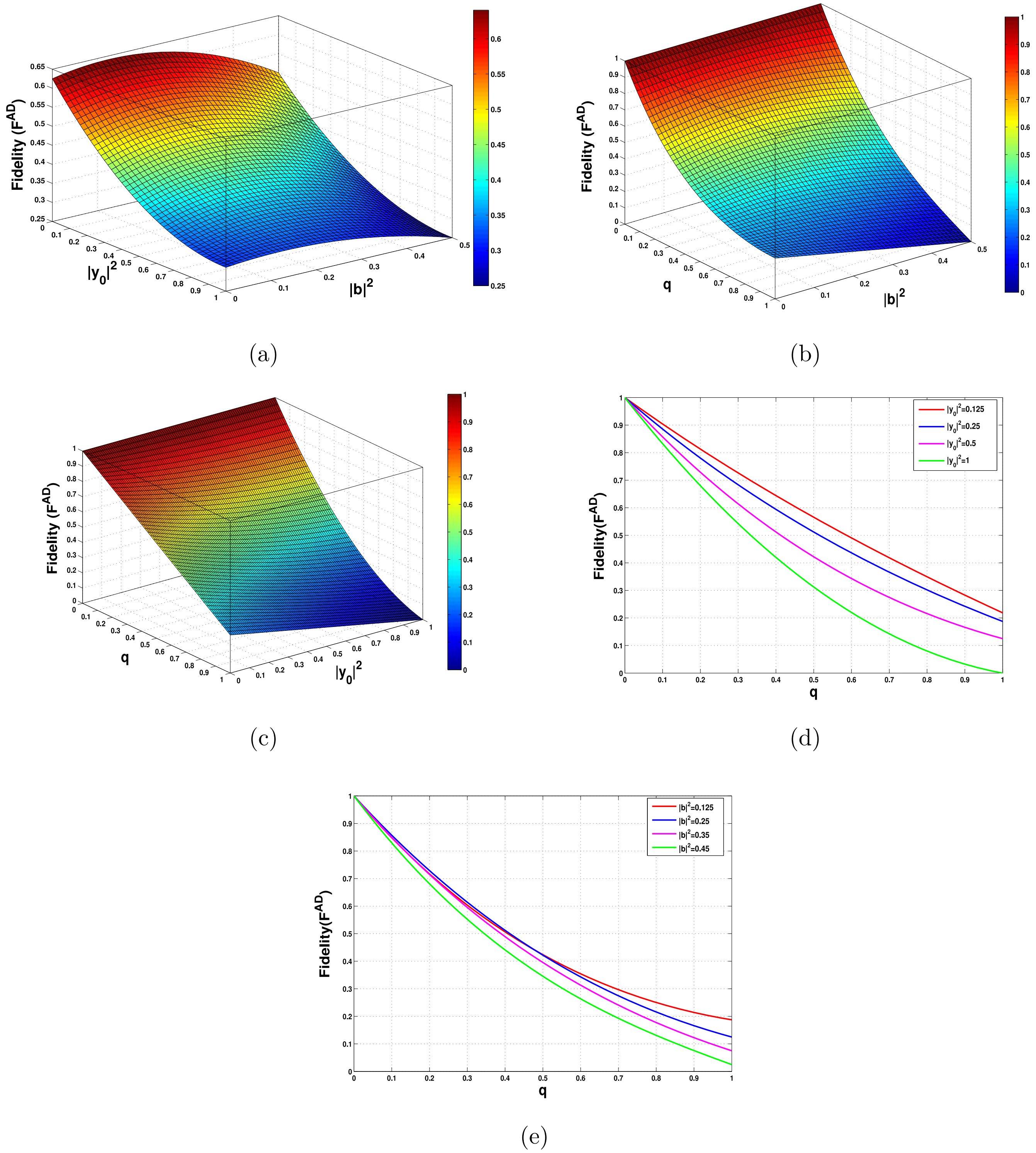


Figure 2. (a) Variation of fidelity F^{AD} with $|y_0|^2$ and $|b|^2$ when $q = 0.5$. (b) Variation of fidelity F^{AD} with $|b|^2$ and q when $|y_0|^2 = 0.5$. (c) Variation of fidelity F^{AD} with $|y_0|^2$ and q when $|b|^2 = 0.25$. (d) Variation of fidelity F^{AD} with q when $|b|^2 = 0.25$ and $|y_0|^2 = 0.125$, 0.25, 0.5, 1. (e) Variation of fidelity F^{AD} with q when $|y_0|^2 = 0.5$ and $|b|^2 = 0.125$, 0.25, 0.35, 0.45.

Part II is for the projective measurement made by Alice. Here, quantum gate U_1 is employed on qubit q_0 for the basis defined in equation (4), where U_1 is defined as:

$$U_1 = \begin{pmatrix} \sqrt{2}\,a & \sqrt{2}\,b \\ \sqrt{2}\,b & -\sqrt{2}\,a \end{pmatrix}.$$

Part III of the circuit involves Bob and Eve conducting a projection measurement on qubits B_2 and E. Quantum gate U_2 utilized by Bob is represented by the matrix:

$$U_2 = \begin{pmatrix} x_0 & y_0 \\ y_0 & -x_0 \end{pmatrix}.$$

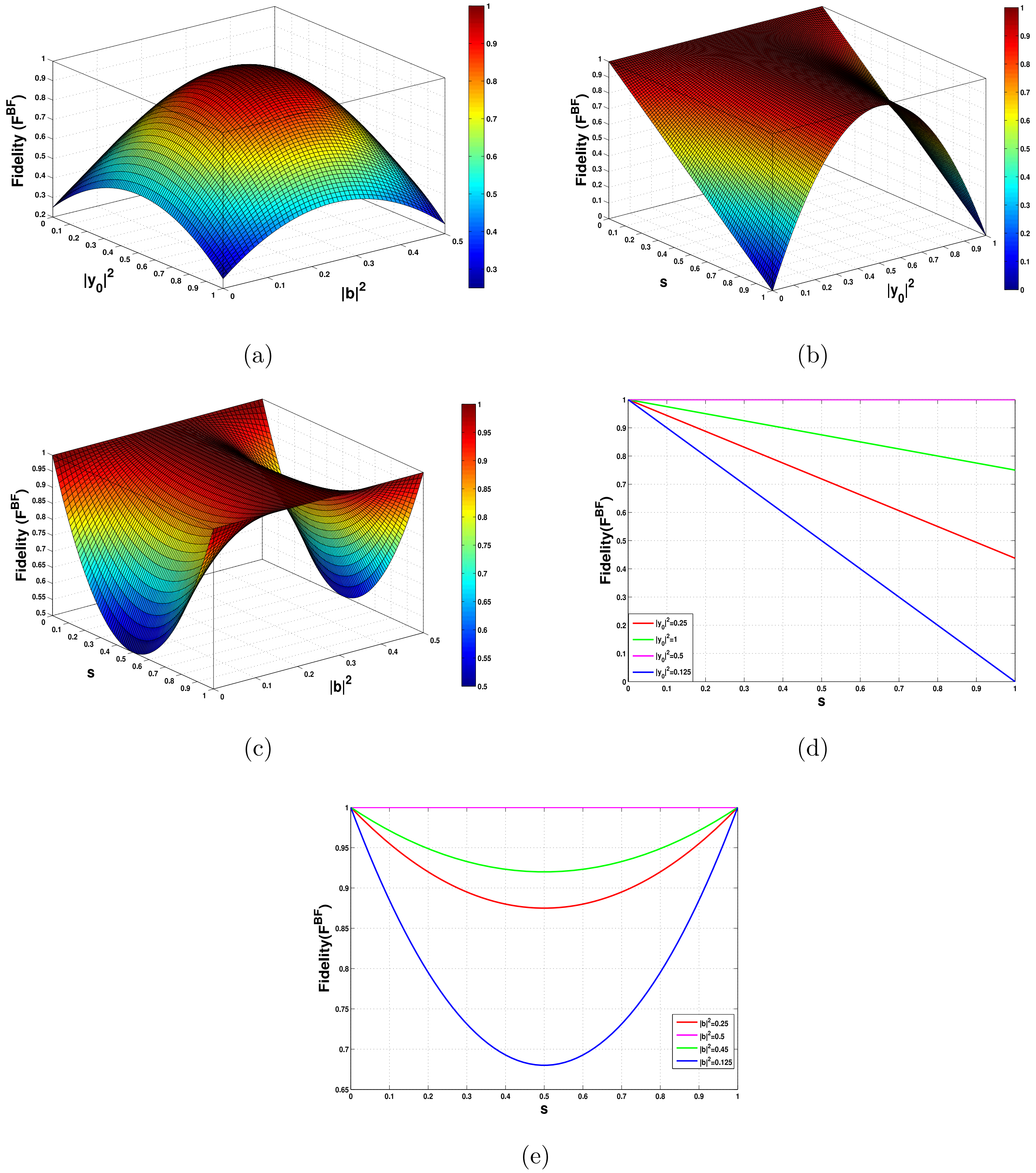


Figure 3. (a) Variation of fidelity F^{BF} with $|y_0|^2$ and $|b|^2$ when $s = 0.5$. (b) Variation of fidelity F^{BF} with $|b|^2$ and s when $|y_0|^2 = 0.5$. (c) Variation of fidelity F^{BF} with $|y_0|^2$ and s when $|b|^2 = 0.25$. (d) Variation of fidelity F^{BF} with s when $|b|^2 = 0.25$ and $|y_0|^2 = 0.25$, 1, 0.5, 0.125. (e) Variation of fidelity F^{BF} with s when $|y_0|^2 = 0.5$ and $|b|^2 = 0.25$, 0.5, 0.45, 0.125.

Based on the Bob's measurement outcome, Eve performs either gate U_3 or U_4. Quantum gates U_3 and U_4 are given as:

$$U_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{e}^{\mathrm{i}\theta} \\ 1 & -\mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}, \quad U_4 = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 1 \\ \mathrm{e}^{\mathrm{i}\theta} & -1 \end{pmatrix}.$$

Part IV of the circuit illustrates the circuit employed by Charlie and David to perform the measurement using the basis $\{|\eta^1\rangle, |\eta^2\rangle\}$.

Part V of the circuit involves the unitary operation executed by Bob based on the measurement outcomes of Alice and Charlie.

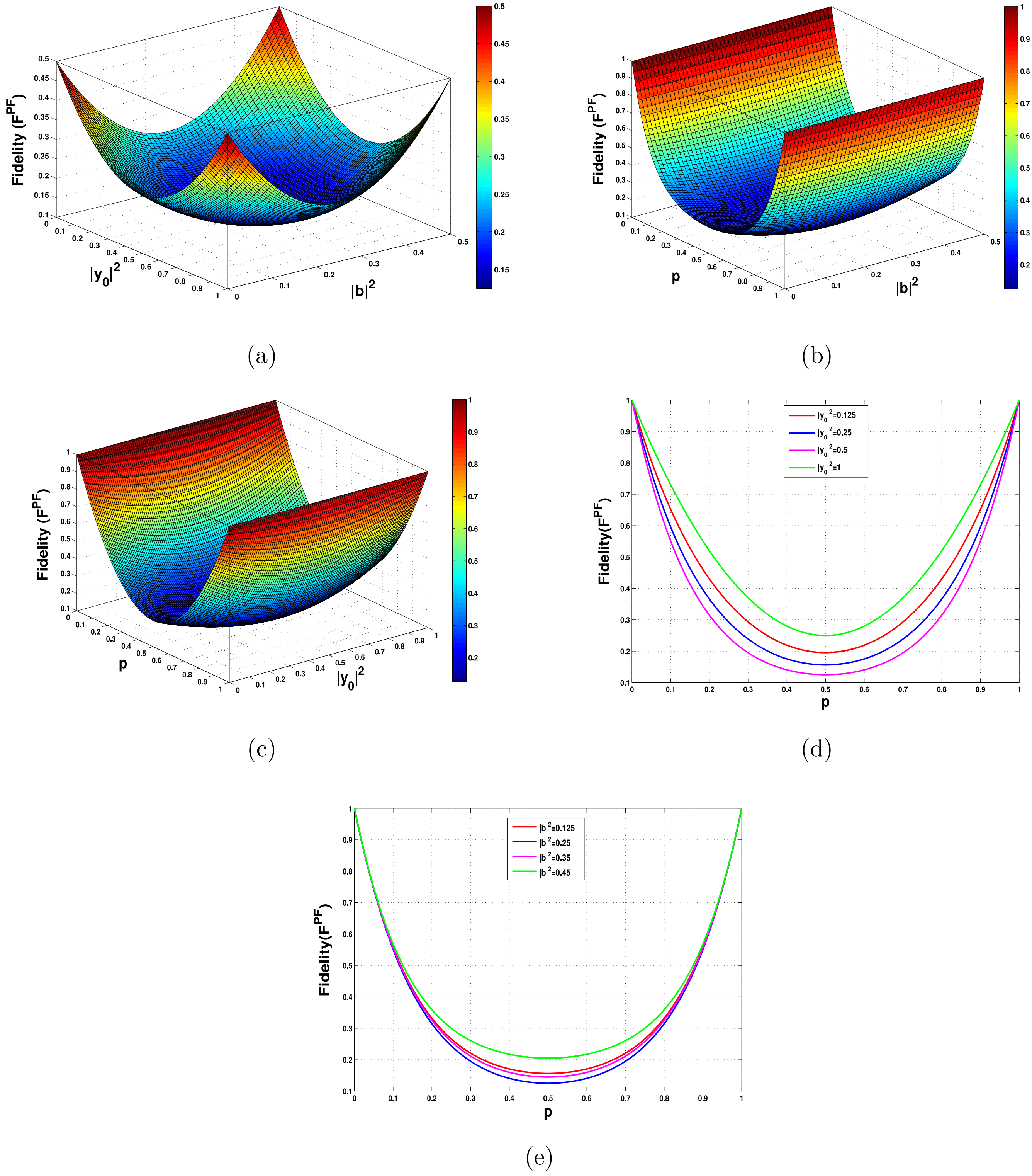


Figure 4. (a) Variation of fidelity F^{PF} with $|y_0|^2$ and $|b|^2$ when $p = 0.5$. (b) Variation of fidelity F^{PF} with $|b|^2$ and p when $|y_0|^2 = 0.5$. (c) Variation of fidelity F^{PF} with $|y_0|^2$ and p when $|b|^2 = 0.25$. (d) Variation of fidelity F^{PF} with p when $|b|^2 = 0.25$ and $|y_0|^2 = 0.125$, 0.25, 0.5, 1. (e) Variation of fidelity F^{PF} with p when $|y_0|^2 = 0.5$ and $|b|^2 = 0.125$, 0.25, 0.35, 0.45.

Part VI of the circuit comprises the unitary operation carried out by Alice based on the measurement outcomes of Bob, Eve, Charlie and David.

Parts VII and VIII of the circuit represent the measurement verification conducted after the protocol is completed.

Experiment 1: Suppose Alice knows the information about the state $|\phi\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle) + \frac{1}{\sqrt{6}}(|01\rangle + |10\rangle)$ and Bob and Eve jointly know the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, that is the values $x_0 = \frac{1}{\sqrt{2}}$, $y_0 = \frac{1}{\sqrt{2}}$ are known to Bob only while

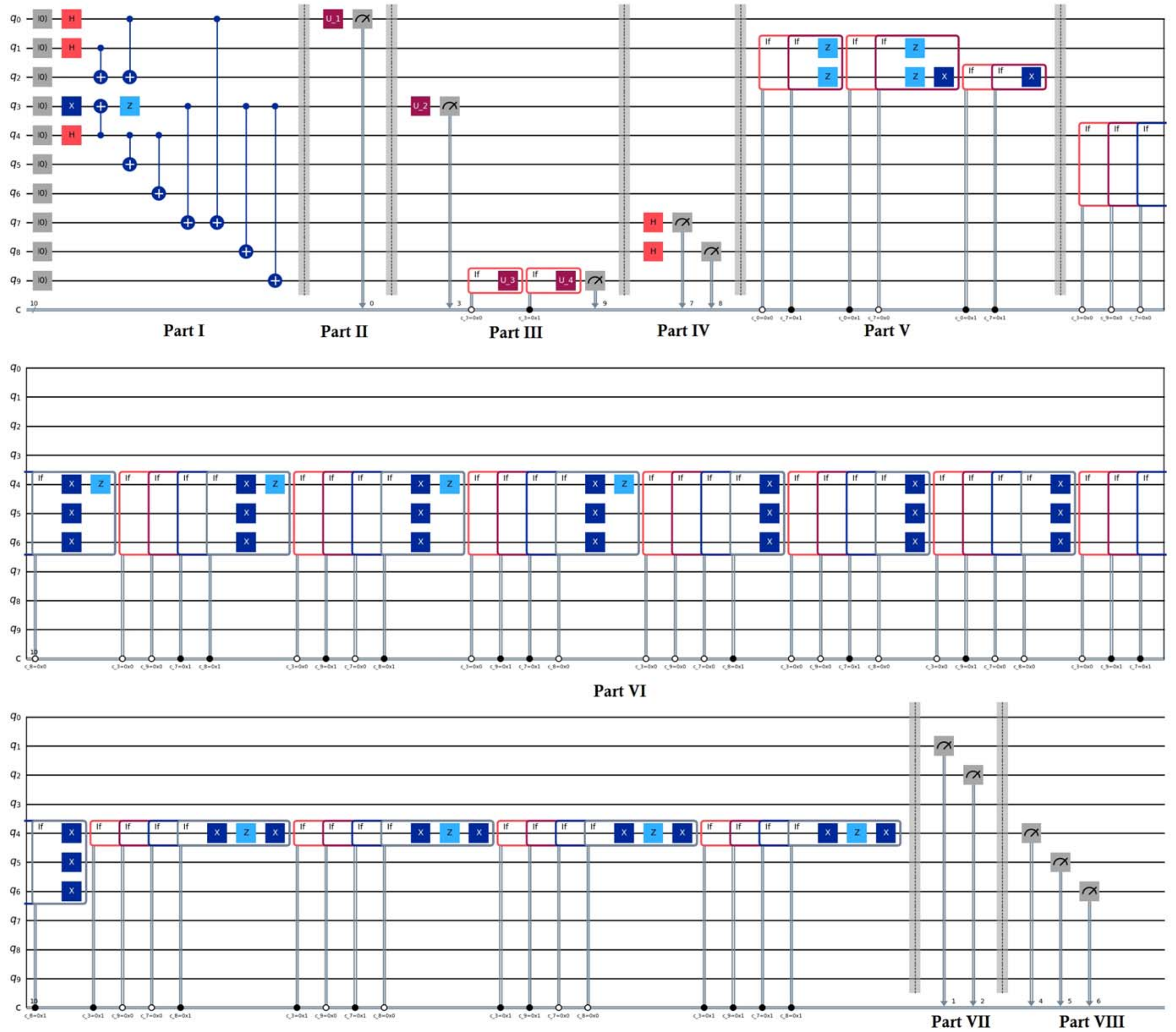


Figure 5. Quantum circuit for the protocol. Symbols utilized are conventional and the steps performed herein are described in section 5.

Eve knows only the value of θ, which is $\theta = 0$. Then, we run the quantum circuit by taking $a = \frac{1}{\sqrt{3}}$, $b = \frac{1}{\sqrt{6}}$, $x_0 = \frac{1}{\sqrt{2}}$, $y_0 = \frac{1}{\sqrt{2}}$ and $\theta = 0$. The output state of qubits (q_1, q_2) and (q_4, q_5, q_6) after the experiment are given in figure 6. From figure 6, it can be seen that our protocol is feasible in a quantum simulation platform.

Experiment 2: If Alice possesses knowledge of the state $|\phi\rangle = \frac{1}{2}(|00\rangle + |11\rangle) + \frac{1}{2}(|01\rangle + |10\rangle)$, and Bob and Eve are jointly aware of the state $|\Psi\rangle = \sqrt{\frac{2}{3}}|000\rangle + \sqrt{\frac{1}{3}}|111\rangle$ in that Bob has knowledge that $x_0 = \sqrt{\frac{2}{3}}$, $y_0 = \sqrt{\frac{1}{3}}$ while $\theta = 0$ is known only to Eve. We execute the quantum circuit with parameters set as $a = \frac{1}{2}$, $b = \frac{1}{2}$, $x_0 = \sqrt{\frac{2}{3}}$, $y_0 = \sqrt{\frac{1}{3}}$, and $\theta = 0$. The resulting output states of qubits (q_1, q_2) and (q_4, q_5, q_6) after the experiment are depicted in figure 7. It is evident from figure 7 that our protocol demonstrates feasibility within a quantum simulation framework.

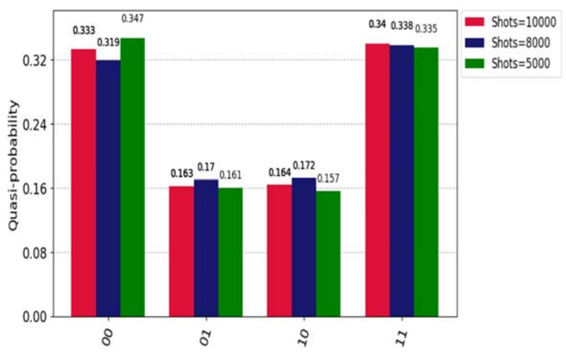
6. Efficiency of the protocol

In this section, we explore the efficiency of our protocol. For quantum communication protocols, such as teleportation, RSP and JRSP, efficiency is defined as $\eta = \frac{q_a}{q_b + q_c}$. Here, q_a represents the number of qubits to be teleported or remotely prepared, q_b indicates the number of resource qubits used as a quantum channel and q_c accounts for the classical bits required for the entire protocol, which is the classical communication cost.

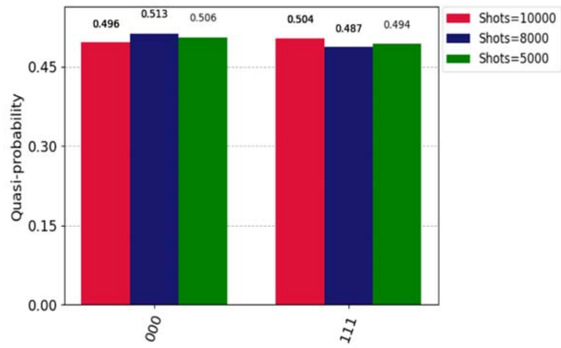
In our proposed protocol, we have $q_a = 5$, $q_b = 10$ and $q_c = 5$. Consequently, the efficiency of our protocol is given by:

$$\eta = \frac{5}{10 + 5} \approx 33.33\%.$$

In the following, we present a comparison of the performance of our protocol with some similar works. This comparison is with respect to efficiency η given above.

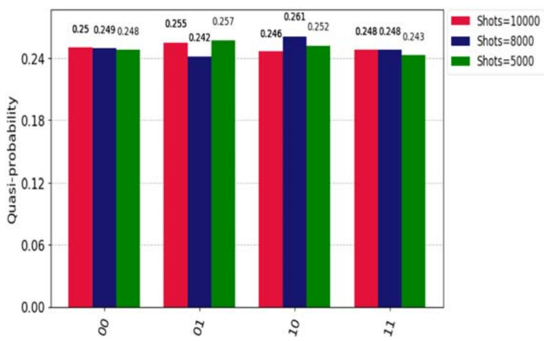


(a) Probability of the state of the qubits q_1 and q_2 for experiment 1.

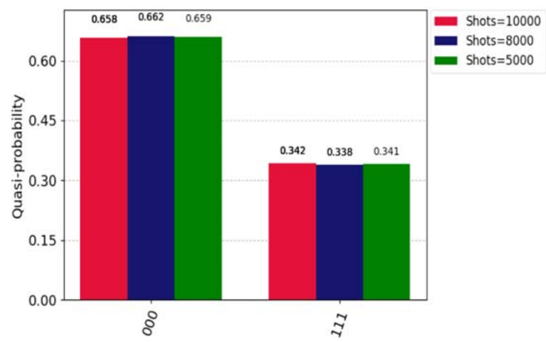


(b) Probability of the state of the qubits q_4, q_5 and q_6 for experiment 1.

Figure 6. Histogram of the experimental results for experiment 1.



(a) Probability of the state of the qubits q_1 and q_2 for experiment 2.



(b) Probability of the state of the qubits q_4, q_5 and q_6 for experiment 2.

Figure 7. Histogram of the experimental results for experiment 2.

Table 3. Comparison of the efficiency of our present protocol with other works.

References	Type	q_a	q_b	Classical cost (q_c)	Noise analysis	Efficiency
[46]	BRSP	$(1+2)$	10	6-bit	No	≈18.75%
[47]	HBQCP	$(1+1)$	5	3-bit	No	≈25%
[48]	HBQCP	$(1+1)$	7	6-bit	No	≈15.38%
[49]	HBQCP	$(2+1)$	8	7-bit	No	≈20%
[50]	HBQCP	$(1+1)$	5	4-bit	Yes	≈22.22%
Our Protocol	HBQCP	$(2+3)$	10	5-bit	Yes	≈33.33%

Consideration of the above table shows that our protocol performs better in terms of efficiency η amongst those considered here.

7. Discussion and conclusion

Hierarchical quantum communication protocols are needed to satisfy the practical demands arising out of the fact that often in real-life situations we need to differentiate between tasks on the basis of their importance. In this paper, with a view to such a situation, we consider the different levels of control of the supervisors on different parties. What is important is the choice of an appropriate quantum channel, which, in our case, is a ten-qubit entangled state. Although the creation and use of multipartite entanglement is difficult, its involvement is an unavoidable feature in this situation. Another difficulty that may arise due to the fragile nature of entanglement, is that it makes them prone to noise. We have considered quantum noise modeled through Kraus operators. The particular cases of amplitude-damping, bit-flip, phase-flip noises have been considered. This list is not exhaustive, as other kinds of noises that have not been considered may also be present. There are some interesting features in the fidelity analysis. For instance, the states $|\phi\rangle = \frac{1}{2}(|00\rangle + |11\rangle) + \frac{1}{2}(|01\rangle + |10\rangle)$ and $|\psi\rangle = \frac{1}{\sqrt{2}}|000\rangle + \frac{1}{\sqrt{2}}|111\rangle$ remain unaffected by bit-flip noise, as can be seen in figures 3(d) and (e). In all the cases of fidelity analysis, it can be seen that the fidelity tends to unit value as the noise parameter tends to zero.

We also note that the efficiency of the protocol is better than many other similar protocols, as shown in table 3.

Acknowledgments

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Conflict of interest

On behalf of all authors, the corresponding author states that there are no conflicts of interest.

Appendix A. Expressions of $|R_i\rangle$

Table A1. Values of $|R_i\rangle$.

$\vert R_i\rangle$	Expression
$\vert R_1\rangle$	$= ax_0\vert 00000\rangle + b(1-q)x_0\vert 00001\rangle + b(1-q)x_0\vert 00010\rangle + a \times (1-q)x_0\vert 00011\rangle + a(1-q)^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-q)^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-q)^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-q)^3y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_2\rangle$	$= a(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\vert 11100\rangle + b(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\vert 11101\rangle + b(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\vert 11110\rangle + a(1-q)^{\frac{5}{2}}\sqrt{q}\,y_0\vert 11111\rangle$
$\vert R_3\rangle$	$= a(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-q)^{\frac{5}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_4\rangle$	$= a(1-q)qy_0\vert 11100\rangle + b(1-q)qy_0\vert 11101\rangle + b(1-q)qy_0\vert 11110\rangle + a(1-q)^2qy_0\vert 11111\rangle$
$\vert R_5\rangle$	$= b\sqrt{(1-q)}\sqrt{q}\,x_0\vert 00001\rangle + b\sqrt{(1-q)}\sqrt{q}\,x_0\vert 00010\rangle + a(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + a(1-q)^{\frac{5}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_6\rangle$	$= a(1-q)qy_0\vert 11100\rangle + a(1-q)^2qy_0\vert 11111\rangle$
$\vert R_7\rangle$	$= a(1-q)qy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + a(1-q)^2qy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_8\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{3}{2}}y_0\vert 11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0\vert 11111\rangle$
$\vert R_9\rangle$	$= a(1-q)^{\frac{3}{2}}\sqrt{q}\,x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-q)^{\frac{3}{2}}\sqrt{q}\,x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-q)^{\frac{3}{2}}\sqrt{q}\,x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-q)^{\frac{5}{2}}\sqrt{q}\,x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_{10}\rangle$	$= a(1-q)qx_0\vert 11100\rangle + b(1-q)qx_0\vert 11101\rangle + b(1-q)qx_0\vert 11110\rangle + a(1-q)^2qx_0\vert 11111\rangle$
$\vert R_{11}\rangle$	$= a(1-q)qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-q)qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-q)qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-q)^2qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_{12}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0\vert 11100\rangle + b\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0\vert 11101\rangle + b\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0\vert 11110\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0\vert 11111\rangle$
$\vert R_{13}\rangle$	$= a(1-q)qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + a(1-q)^2qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_{14}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0\vert 11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0\vert 11111\rangle$
$\vert R_{15}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert R_{16}\rangle$	$= aq^2x_0\vert 11100\rangle + a(1-q)q^2x_0\vert 11111\rangle$
$\vert R_{17}\rangle$	$= b\sqrt{(1-q)}\sqrt{q}\,x_0\vert 00000\rangle + a\sqrt{(1-q)}\sqrt{q}\,x_0\vert 00010\rangle + b(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + a(1-q)^{\frac{5}{2}}\sqrt{q}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle$
$\vert R_{18}\rangle$	$= b(1-q)qy_0\vert 11100\rangle + a(1-q)^2qy_0\vert 11110\rangle$
$\vert R_{19}\rangle$	$= b(1-q)qy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + a(1-q)^2qy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle$
$\vert R_{20}\rangle$	$= b\sqrt{(1-q)}\,q^{\frac{3}{2}}y_0\vert 11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0\vert 11110\rangle$
$\vert R_{21}\rangle$	$= bqx_0\vert 00000\rangle + a(1-q)^2qy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle$
$\vert R_{22}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0\vert 11110\rangle$
$\vert R_{23}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle$
$\vert R_{24}\rangle$	$= a(1-q)q^2y_0\vert 11110\rangle$
$\vert R_{25}\rangle$	$= b(1-q)qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + a(1-q)^2qx_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle$

Table A1. (Continued.)

$\|R_i\rangle$	Expression
$\|R_{26}\rangle$	$= b\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0\|11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0\|11110\rangle$
$\|R_{27}\rangle$	$= b\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0 e^{i\theta}\|11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0 e^{i\theta}\|11110\rangle$
$\|R_{28}\rangle$	$= bq^2x_0\|11100\rangle + a(1-q)q^2x_0\|11110\rangle$
$\|R_{29}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0 e^{i\theta}\|11110\rangle$
$\|R_{30}\rangle$	$= a(1-q)q^2x_0\|11110\rangle$
$\|R_{31}\rangle$	$= a(1-q)q^2x_0 e^{i\theta}\|11110\rangle$
$\|R_{32}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{5}{2}}x_0\|11110\rangle$
$\|R_{33}\rangle$	$= b\sqrt{(1-q)}\,\sqrt{q}\,x_0\|00000\rangle + a\sqrt{(1-q)}\,\sqrt{q}\,x_0\|00001\rangle$ $+ b(1-q)^{\frac{3}{2}}\sqrt{q}\,y_0 e^{i\theta}\|11100\rangle + a(1-q)^{\frac{5}{2}}\sqrt{q}\,y_0 e^{i\theta}\|11101\rangle$
$\|R_{34}\rangle$	$= b(1-q)qy_0\|11100\rangle + a(1-q)^2qy_0\|11101\rangle$
$\|R_{35}\rangle$	$= b(1-q)qy_0 e^{i\theta}\|11100\rangle + a(1-q)^2qy_0 e^{i\theta}\|11101\rangle$
$\|R_{36}\rangle$	$= b\sqrt{(1-q)}\,q^{\frac{3}{2}}y_0\|11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0\|11101\rangle$
$\|R_{37}\rangle$	$= bqx_0\|00000\rangle + a(1-q)^2qy_0 e^{i\theta}\|11101\rangle$
$\|R_{38}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0\|11101\rangle$
$\|R_{39}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0 e^{i\theta}\|11101\rangle$
$\|R_{40}\rangle$	$= a(1-q)q^2y_0\|11101\rangle$
$\|R_{41}\rangle$	$= b(1-q)qx_0 e^{i\theta}\|11100\rangle + a(1-q)^2qx_0 e^{i\theta}\|11101\rangle$
$\|R_{42}\rangle$	$= b\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0\|11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0\|11101\rangle$
$\|R_{43}\rangle$	$= b\sqrt{(1-q)}\,q^{\frac{3}{2}}x_0 e^{i\theta}\|11100\rangle + a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0 e^{i\theta}\|11101\rangle$
$\|R_{44}\rangle$	$= bq^2x_0\|11100\rangle + a(1-q)q^2x_0\|11101\rangle$
$\|R_{45}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0 e^{i\theta}\|11101\rangle$
$\|R_{46}\rangle$	$= a(1-q)q^2x_0\|11101\rangle$
$\|R_{47}\rangle$	$= a(1-q)q^2x_0 e^{i\theta}\|11101\rangle$
$\|R_{48}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{5}{2}}x_0\|11101\rangle$
$\|R_{49}\rangle$	$= aqx_0\|00000\rangle + a(1-q)^2qy_0 e^{i\theta}\|11100\rangle$
$\|R_{50}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0\|11100\rangle$
$\|R_{51}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0 e^{i\theta}\|11100\rangle$
$\|R_{52}\rangle$	$= a(1-q)q^2y_0\|11100\rangle$
$\|R_{53}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}y_0 e^{i\theta}\|11100\rangle$
$\|R_{54}\rangle$	$= a(1-q)q^2y_0\|11100\rangle$
$\|R_{55}\rangle$	$= a(1-q)q^2y_0 e^{i\theta}\|11100\rangle$
$\|R_{56}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{5}{2}}y_0\|11100\rangle$
$\|R_{57}\rangle$	$= a(1-q)^{\frac{3}{2}}q^{\frac{3}{2}}x_0 e^{i\theta}\|11100\rangle$
$\|R_{58}\rangle$	$= a(1-q)q^2x_0\|11100\rangle$
$\|R_{59}\rangle$	$= a(1-q)q^2x_0 e^{i\theta}\|11100\rangle$
$\|R_{60}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{5}{2}}x_0\|11100\rangle$
$\|R_{61}\rangle$	$= a(1-q)q^2x_0 e^{i\theta}\|11100\rangle$
$\|R_{62}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{5}{2}}x_0\|11100\rangle$
$\|R_{63}\rangle$	$= a\sqrt{(1-q)}\,q^{\frac{5}{2}}x_0 e^{i\theta}\|11100\rangle$
$\|R_{64}\rangle$	$= aq^3x_0\|11100\rangle$

Appendix B. Expressions of $|H_i\rangle$

Table B1. Values of $|H_i\rangle$.

$\lvert H_i\rangle$	Expression
$\lvert H_1\rangle$	$= a(1-s)^3x_0\lvert 00000\rangle + b(1-s)^3x_0\lvert 00001\rangle + b(1-s)^3x_0\lvert 00010\rangle + a(1-s)^3x_0\lvert 00011\rangle + a(1-s)^3y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-s)^3 \times y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b(1-s)^3y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-s)^3y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_2\rangle$	$= a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 11100\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 11101\rangle + b \times (1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 11110\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 11111\rangle$
$\lvert H_3\rangle$	$= a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00000\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00001\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00010\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00011\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b \times (1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_4\rangle$	$= a(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + b(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + b(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + a(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + a(1-s)^2sy_0\lvert 11100\rangle + b(1-s)^2sy_0\lvert 11101\rangle + b(1-s)^2sy_0\lvert 11110\rangle + a(1-s)^2sy_0\lvert 11111\rangle$
$\lvert H_5\rangle$	$= a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00000\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00001\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00010\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\lvert 00011\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b \times (1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_6\rangle$	$= a(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + b(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + b(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + a(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + a(1-s)^2sy_0\lvert 11100\rangle + b(1-s)^2sy_0\lvert 11101\rangle + b(1-s)^2sy_0\lvert 11110\rangle + a(1-s)^2sy_0\lvert 11111\rangle$
$\lvert H_7\rangle$	$= a(1-s)^2sx_0\lvert 00000\rangle + b(1-s)^2sx_0\lvert 00001\rangle + b(1-s)^2sx_0\lvert 00010\rangle + a(1-s)^2sx_0\lvert 00011\rangle + a(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b \times (1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-s)^2 \times sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_8\rangle$	$= a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 11101\rangle + b(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}}y_0\lvert 11110\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 11111\rangle$
$\lvert H_9\rangle$	$= a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 00000\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 00001\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 00010\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0\lvert 00011\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b \times (1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_{10}\rangle$	$= a(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + b(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + b(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + a(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + a(1-s)^2sx_0\lvert 11100\rangle + b(1-s)^2sx_0\lvert 11101\rangle + b(1-s)^2sx_0\lvert 11110\rangle + a(1-s)^2sx_0\lvert 11111\rangle$

Table B1. (Continued.)

$\vert H_i\rangle$	Expression
$\vert H_{11}\rangle =$	$a(1-s)^2sy_0\vert 00000\rangle + b(1-s)^2sy_0\vert 00001\rangle + b(1-s)^2sy_0 \vert 00010\rangle + a(1-s)^2sy_0\vert 00011\rangle + a(1-s)^2sx_0e^{i\theta}\vert 11100\rangle + b \times (1-s)^2sx_0e^{i\theta}\vert 11101\rangle + b(1-s)^2sx_0e^{i\theta}\vert 11110\rangle + a(1-s)^2 sx_0e^{i\theta}\vert 11111\rangle$
$\vert H_{12}\rangle =$	$a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 11101\rangle + b(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}}x_0\vert 11110\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 11111\rangle$
$\vert H_{13}\rangle =$	$a(1-s)^2sy_0\vert 00000\rangle + b(1-s)^2sy_0\vert 00001\rangle + b(1-s)^2sy_0 \vert 00010\rangle + a(1-s)^2sy_0\vert 00011\rangle + a(1-s)^2sx_0e^{i\theta}\vert 11100\rangle + b \times (1-s)^2sx_0e^{i\theta}\vert 11101\rangle + b(1-s)^2sx_0e^{i\theta}\vert 11110\rangle + a(1-s)^2 sx_0e^{i\theta}\vert 11111\rangle$
$\vert H_{14}\rangle =$	$a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0e^{i\theta}\vert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 11101\rangle + b(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}}x_0\vert 11110\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 11111\rangle$
$\vert H_{15}\rangle =$	$a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0 \vert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 11101\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 11110\rangle + a \times (1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 11111\rangle$
$\vert H_{16}\rangle =$	$a(1-s)s^2y_0e^{i\theta}\vert 00000\rangle + b(1-s)s^2y_0e^{i\theta}\vert 00001\rangle + b(1-s)s^2y_0e^{i\theta}\vert 00010\rangle + a(1-s)s^2y_0e^{i\theta}\vert 00011\rangle + a(1-s)s^2x_0\vert 11100\rangle + b(1-s)s^2x_0\vert 11101\rangle + b(1-s)s^2x_0\vert 11110\rangle + a(1-s)s^2x_0\vert 11111\rangle$
$\vert H_{17}\rangle =$	$b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\vert 00000\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\vert 00001\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}x_0\vert 00010\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}x_0\vert 00011\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}y_0e^{i\theta}\vert 11100\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}y_0e^{i\theta}\vert 11101\rangle + a \times (1-s\ ^{\frac{5}{2}}\sqrt{s}y_0e^{i\theta}\ 11110\rangle + b\ 1-s\ ^{\frac{5}{2}}\sqrt{s}y_0e^{i\theta}\ 11111\rangle$
$\vert H_{18}\rangle =$	$b(1-s)^2sx_0e^{i\theta}\vert 00000\rangle + a(1-s)^2sx_0e^{i\theta}\vert 00001\rangle + a(1-s)^2sx_0e^{i\theta}\vert 00010\rangle + b(1-s)^2sx_0e^{i\theta}\vert 00011\rangle + b(1-s)^2sy_0\vert 11100\rangle + a(1-s)^2sy_0\vert 11101\rangle + a(1-s)^2sy_0 \vert 11110\rangle + b(1-s)^2sy_0\vert 11111\rangle$
$\vert H_{19}\rangle =$	$b(1-s)^2sx_0\vert 00000\rangle + a(1-s)^2sx_0\vert 00001\rangle + a(1-s)^2sx_0 \vert 00010\rangle + b(1-s)^2sx_0\vert 00011\rangle + b(1-s)^2sy_0e^{i\theta}\vert 11100\rangle + a \times (1-s)^2sy_0e^{i\theta}\vert 11101\rangle + a(1-s)^2sy_0e^{i\theta}\vert 11110\rangle + b(1-s)^2 sy_0e^{i\theta}\vert 11111\rangle$
$\vert H_{20}\rangle =$	$b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 00000\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 00001\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 00010\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0e^{i\theta}\vert 00011\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 11100\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 11101\rangle + a(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}}y_0\vert 11110\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 11111\rangle$
$\vert H_{21}\rangle =$	$b(1-s)^2sx_0\vert 00000\rangle + a(1-s)^2sx_0\vert 00001\rangle + a(1-s)^2sx_0 \vert 00010\rangle + b(1-s)^2sx_0\vert 00011\rangle + b(1-s)^2sy_0e^{i\theta}\vert 11100\rangle + a \times (1-s)^2sy_0e^{i\theta}\vert 11101\rangle + a(1-s)^2sy_0e^{i\theta}\vert 11110\rangle + b(1-s)^2 \times sy_0e^{i\theta}\vert 11111\rangle$

Table B1. (Continued.)

$\lvert H_i\rangle$	Expression
$\lvert H_{22}\rangle =$	$b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 11100\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 11101\rangle + a(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}}y_0\lvert 11110\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 11111\rangle$
$\lvert H_{23}\rangle =$	$b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\lvert 00000\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\lvert 00001\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0 \lvert 00010\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\lvert 00011\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + b \times (1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_{24}\rangle =$	$b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + b(1-s)s^2y_0\lvert 11100\rangle + a(1-s)s^2y_0\lvert 11101\rangle + a(1-s)s^2y_0\lvert 11110\rangle + b(1-s)s^2y_0\lvert 11111\rangle$
$\lvert H_{25}\rangle =$	$b(1-s)^2sy_0\lvert 00000\rangle + a(1-s)^2sy_0\lvert 00001\rangle + a(1-s)^2sy_0 \lvert 00010\rangle + b(1-s)^2sy_0\lvert 00011\rangle + b(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + a \times (1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + a(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + b(1-s)^2 \times sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_{26}\rangle =$	$b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\lvert 11100\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\lvert 11101\rangle + a(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}}x_0\lvert 11110\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\lvert 11111\rangle$
$\lvert H_{27}\rangle =$	$b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 00000\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 00001\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0 \lvert 00010\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 00011\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + b \times (1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_{28}\rangle =$	$b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + b(1-s)s^2x_0\lvert 11100\rangle + a(1-s)s^2x_0\lvert 11101\rangle + a(1-s)s^2x_0\lvert 11110\rangle + b(1-s)s^2x_0\lvert 11111\rangle$
$\lvert H_{29}\rangle =$	$b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 00000\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 00001\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0 \lvert 00010\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\lvert 00011\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + b \times (1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_{30}\rangle =$	$b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + b(1-s)s^2x_0\lvert 11100\rangle + a(1-s)s^2x_0\lvert 11101\rangle + a(1-s)s^2x_0\lvert 11110\rangle + b(1-s)s^2x_0\lvert 11111\rangle$
$\lvert H_{31}\rangle =$	$b(1-s)s^2y_0\lvert 00000\rangle + a(1-s)s^2y_0\lvert 00001\rangle + a(1-s)s^2y_0\lvert 00010\rangle + b(1-s)s^2y_0\lvert 00011\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_{32}\rangle =$	$b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\lvert 11100\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\lvert 11101\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}} \times x_0\lvert 11110\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\lvert 11111\rangle$
$\lvert H_{33}\rangle =$	$b(1-s)^{\frac{5}{2}}\sqrt{s}\,x_0\lvert 00000\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}\,x_0\lvert 00001\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}\,x_0\lvert 00010\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}\,x_0\lvert 00011\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}\,y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + a(1-s)^{\frac{5}{2}}\sqrt{s}\,y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + a \times (1-s)^{\frac{5}{2}}\sqrt{s}\,y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + b(1-s)^{\frac{5}{2}}\sqrt{s}\,y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert H_{34}\rangle =$	$b(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00000\rangle + a(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00001\rangle + a(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00010\rangle + b(1-s)^2sx_0\mathrm{e}^{\mathrm{i}\theta}\lvert 00011\rangle + b(1-s)^2sy_0\lvert 11100\rangle + a(1-s)^2sy_0\lvert 11101\rangle + a(1-s)^2sy_0 \lvert 11110\rangle + b(1-s)^2sy_0\lvert 11111\rangle$

Table B1. (Continued.)

$\vert H_i\rangle$	Expression
$\vert H_{35}\rangle =$	$b(1-s)^2 s x_0\vert 00000\rangle + a(1-s)^2 s x_0\vert 00001\rangle + a(1-s)^2 s x_0 \vert 00010\rangle + b(1-s)^2 s x_0\vert 00011\rangle + b(1-s)^2 s y_0 e^{i\theta}\vert 11100\rangle + a \times (1-s)^2 s y_0 e^{i\theta}\vert 11101\rangle + a(1-s)^2 s y_0 e^{i\theta}\vert 11110\rangle + b(1-s)^2 \times s y_0 e^{i\theta}\vert 11111\rangle$
$\vert H_{36}\rangle =$	$b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00000\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00001\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00010\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00011\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 11100\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 11101\rangle + a(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}} y_0\vert 11110\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 11111\rangle$
$\vert H_{37}\rangle =$	$b(1-s)^2 s x_0\vert 00000\rangle + a(1-s)^2 s x_0\vert 00001\rangle + a(1-s)^2 s x_0 \vert 00010\rangle + b(1-s)^2 s x_0\vert 00011\rangle + b(1-s)^2 s y_0 e^{i\theta}\vert 11100\rangle + a \times (1-s)^2 s y_0 e^{i\theta}\vert 11101\rangle + a(1-s)^2 s y_0 e^{i\theta}\vert 11110\rangle + b(1-s)^2 \times s y_0 e^{i\theta}\vert 11111\rangle$
$\vert H_{38}\rangle =$	$b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00000\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00001\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00010\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 00011\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 11100\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 11101\rangle + a(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}} y_0\vert 11110\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 11111\rangle$
$\vert H_{39}\rangle =$	$b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0\vert 00000\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0\vert 00001\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 \vert 00010\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0\vert 00011\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 11100\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 11101\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 11110\rangle + b \times (1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 11111\rangle$
$\vert H_{40}\rangle =$	$b(1-s)s^2 x_0 e^{i\theta}\vert 00000\rangle + a(1-s)s^2 x_0 e^{i\theta}\vert 00001\rangle + a(1-s)s^2 x_0 e^{i\theta}\vert 00010\rangle + b(1-s)s^2 x_0 e^{i\theta}\vert 00011\rangle + b(1-s)s^2 y_0\vert 11100\rangle + a(1-s)s^2 y_0\vert 11101\rangle + a(1-s)s^2 y_0\vert 11110\rangle + b(1-s)s^2 y_0\vert 11111\rangle$
$\vert H_{41}\rangle =$	$74b(1-s)^2 s y_0\vert 00000\rangle + a(1-s)^2 s y_0\vert 00001\rangle + a(1-s)^2 s y_0 \vert 00010\rangle + b(1-s)^2 s y_0\vert 00011\rangle + b(1-s)^2 s x_0 e^{i\theta}\vert 11100\rangle + a \times (1-s)^2 s x_0 e^{i\theta}\vert 11101\rangle + a(1-s)^2 s x_0 e^{i\theta}\vert 11110\rangle + b(1-s)^2 \times s x_0 e^{i\theta}\vert 11111\rangle$
$\vert H_{42}\rangle =$	$b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 00000\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 00001\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 00010\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 e^{i\theta}\vert 00011\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0\vert 11100\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0\vert 11101\rangle + a(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}} x_0\vert 11110\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0\vert 11111\rangle$
$\vert H_{43}\rangle =$	$b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 00000\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 00001\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 \vert 00010\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 00011\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11100\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11101\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11110\rangle + b \times (1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11111\rangle$
$\vert H_{44}\rangle =$	$b(1-s)s^2 y_0 e^{i\theta}\vert 00000\rangle + a(1-s)s^2 y_0 e^{i\theta}\vert 00001\rangle + a(1-s)s^2 y_0 e^{i\theta}\vert 00010\rangle + b(1-s)s^2 y_0 e^{i\theta}\vert 00011\rangle + b(1-s)s^2 x_0\vert 11100\rangle + a(1-s)s^2 x_0\vert 11101\rangle + a(1-s)s^2 x_0\vert 11110\rangle + b(1-s)s^2 x_0\vert 11111\rangle$
$\vert H_{45}\rangle =$	$b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 00000\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 00001\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0 \vert 00010\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} y_0\vert 00011\rangle + b(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11100\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11101\rangle + a(1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11110\rangle + b \times (1-s)^{\frac{3}{2}} s^{\frac{3}{2}} x_0 e^{i\theta}\vert 11111\rangle$
$\vert H_{46}\rangle =$	$b(1-s)s^2 y_0 e^{i\theta}\vert 00000\rangle + a(1-s)s^2 y_0 e^{i\theta}\vert 00001\rangle + a(1-s)s^2 y_0 e^{i\theta}\vert 00010\rangle + b(1-s)s^2 y_0 e^{i\theta}\vert 00011\rangle + b(1-s)s^2 x_0\vert 11100\rangle + a(1-s)s^2 x_0\vert 11101\rangle + a(1-s)s^2 x_0\vert 11110\rangle + b(1-s)s^2 x_0\vert 11111\rangle$
$\vert H_{47}\rangle =$	$b(1-s)s^2 y_0\vert 00000\rangle + a(1-s)s^2 y_0\vert 00001\rangle + a(1-s)s^2 y_0\vert 00010\rangle + b(1-s)s^2 y_0\vert 00011\rangle + b(1-s)s^2 x_0 e^{i\theta}\vert 11100\rangle + a(1-s)s^2 x_0 e^{i\theta}\vert 11101\rangle + a(1-s)s^2 x_0 e^{i\theta}\vert 11110\rangle + b(1-s)s^2 x_0 e^{i\theta}\vert 11111\rangle$

Table B1. (Continued.)

$\vert H_i\rangle$	Expression
$\vert H_{48}\rangle =$	$b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00000\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00001\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00010\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00011\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\vert 11100\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\vert 11101\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}} \times x_0\vert 11110\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\vert 11111\rangle$
$\vert H_{49}\rangle =$	$a(1-s)^2sx_0\vert 00000\rangle + b(1-s)^2sx_0\vert 00001\rangle + b(1-s)^2sx_0 \vert 00010\rangle + a(1-s)^2sx_0\vert 00011\rangle + a(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b \times (1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-s)^2sy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-s)^2 \times sy_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert H_{50}\rangle =$	$a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 11101\rangle + b(1-s)^{\frac{3}{2}} \times s^{\frac{3}{2}}y_0\vert 11110\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 11111\rangle$
$\vert H_{51}\rangle =$	$a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0 \vert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a \times (1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert H_{52}\rangle =$	$a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00000\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00001\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00010\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00011\rangle + a(1-s)s^2y_0\vert 11100\rangle + b(1-s)s^2y_0\vert 11101\rangle + b(1-s)s^2y_0\vert 11110\rangle + a(1-s)s^2y_0\vert 11111\rangle$
$\vert H_{53}\rangle =$	$a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0 \vert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\vert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a \times (1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert H_{54}\rangle =$	$a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00000\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00001\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00010\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00011\rangle + a(1-s)s^2y_0\vert 11100\rangle + b(1-s)s^2y_0\vert 11101\rangle + b(1-s)s^2y_0\vert 11110\rangle + a(1-s)s^2y_0\vert 11111\rangle$
$\vert H_{55}\rangle =$	$a(1-s)s^2x_0\vert 00000\rangle + b(1-s)s^2x_0\vert 00001\rangle + b(1-s)s^2x_0\vert 00010\rangle + a(1-s)s^2x_0\vert 00011\rangle + a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert H_{56}\rangle =$	$a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00000\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00001\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00010\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 00011\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\vert 11100\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\vert 11101\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}} y_0\vert 11110\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\vert 11111\rangle$
$\vert H_{57}\rangle =$	$a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 00000\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 00001\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0 \vert 00010\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}y_0\vert 00011\rangle + a(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a \times (1-s)^{\frac{3}{2}}s^{\frac{3}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert H_{58}\rangle =$	$a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00000\rangle + b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00001\rangle + b(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00010\rangle + a(1-s)s^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00011\rangle + a(1-s)s^2x_0\vert 11100\rangle + b(1-s)s^2x_0\vert 11101\rangle + b(1-s)s^2x_0\vert 11110\rangle + a(1-s)s^2x_0\vert 11111\rangle$
$\vert H_{59}\rangle =$	$a(1-s)s^2y_0\vert 00000\rangle + b(1-s)s^2y_0\vert 00001\rangle + b(1-s)s^2y_0\vert 00010\rangle + a(1-s)s^2y_0\vert 00011\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert H_{60}\rangle =$	$a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00000\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00001\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00010\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 00011\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\vert 11100\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\vert 11101\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}} \times x_0\vert 11110\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\vert 11111\rangle$
$\vert H_{61}\rangle =$	$a(1-s)s^2y_0\vert 00000\rangle + b(1-s)s^2y_0\vert 00001\rangle + b(1-s)s^2y_0\vert 00010\rangle + a(1-s)s^2y_0\vert 00011\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-s)s^2x_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$

Table B1. (Continued.)

$\|H_i\rangle$	Expression
$\|H_{62}\rangle =$	$a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\|00000\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\|00001\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\|00010\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\|00011\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\|11100\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\|11101\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}} \times x_0\|11110\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\|11111\rangle$
$\|H_{63}\rangle =$	$a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\|00000\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0\|00001\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}y_0 \|00010\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}y_0\|00011\rangle + a\sqrt{1-s}\,s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\|11100\rangle + b \times \sqrt{1-s}\,s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\|11101\rangle + b\sqrt{1-s}\,s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\|11110\rangle + a\sqrt{1-s} \times s^{\frac{5}{2}}x_0\mathrm{e}^{\mathrm{i}\theta}\|11111\rangle$
$\|H_{64}\rangle =$	$as^3y_0\mathrm{e}^{\mathrm{i}\theta}\|00000\rangle + bs^3y_0\mathrm{e}^{\mathrm{i}\theta}\|00001\rangle + bs^3y_0\mathrm{e}^{\mathrm{i}\theta}\|00010\rangle + as^3y_0\mathrm{e}^{\mathrm{i}\theta}\|00011\rangle + as^3x_0\|11100\rangle + bs^3x_0\|11101\rangle + bs^3x_0\|11110\rangle + as^3x_0\|11111\rangle$

Appendix C. Expressions of $|G_i\rangle$

Table C1. Values of $|G_i\rangle$.

$\|G_i\rangle$	Expression
$\|G_1\rangle=$	$a(1-p)^3x_0\|00000\rangle + b(1-p)^3x_0\|00001\rangle + b(1-p)^3x_0\|00010\rangle + a(1-p)^3x_0\|00011\rangle + a(1-p)^3y_0\mathrm{e}^{\mathrm{i}\theta}\|11100\rangle + b(1-p)^3 \times y_0\mathrm{e}^{\mathrm{i}\theta}\|11101\rangle + b(1-p)^3y_0\mathrm{e}^{\mathrm{i}\theta}\|11110\rangle + a(1-p)^3y_0\mathrm{e}^{\mathrm{i}\theta}\|11111\rangle$
$\|G_2\rangle=$	$a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00000\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00001\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00010\rangle + a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00011\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11100\rangle - b(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11101\rangle - b \times (1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11110\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11111\rangle$
$\|G_3\rangle=$	$a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00000\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00001\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00010\rangle + a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00011\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11100\rangle - b(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11101\rangle - b \times (1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11110\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11111\rangle$
$\|G_4\rangle=$	$a(1-p)^2px_0\|00000\rangle + b(1-p)^2px_0\|00001\rangle + b(1-p)^2px_0 \|00010\rangle + a(1-p)^2px_0\|00011\rangle + a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11100\rangle + b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11101\rangle + b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11111\rangle$
$\|G_5\rangle=$	$a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00000\rangle - b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00001\rangle - b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00010\rangle + a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\|00011\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11100\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11101\rangle + b \times (1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11110\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\|11111\rangle$
$\|G_6\rangle=$	$a(1-p)^2px_0\|00000\rangle - b(1-p)^2px_0\|00001\rangle - b(1-p)^2px_0 \|00010\rangle + a(1-p)^2px_0\|00011\rangle + a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11100\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11101\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\|11111\rangle$

Table C1. (Continued.)

$\vert G_i\rangle$	Expression
$\vert G_7\rangle=$	$a(1-p)^2px_0\vert 00000\rangle - b(1-p)^2px_0\vert 00001\rangle - b(1-p)^2px_0\vert 00010\rangle + a(1-p)^2px_0\vert 00011\rangle + a(1-p)^2py_0e^{i\theta}\vert 11100\rangle - b(1-p)^2py_0e^{i\theta}\vert 11101\rangle - b(1-p)^2py_0e^{i\theta}\vert 11110\rangle + a \times (1-p)^2py_0e^{i\theta}\vert 11111\rangle$
$\vert G_8\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11111\rangle$
$\vert G_9\rangle=$	$a(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00000\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00001\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00010\rangle + a(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00011\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11100\rangle - b(1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11101\rangle - b \times (1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11110\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11111\rangle$
$\vert G_{10}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle + b(1-p)^2px_0\vert 00001\rangle + b(1-p)^2px_0\vert 00010\rangle + a(1-p)^2px_0\vert 00011\rangle + a(1-p)^2py_0e^{i\theta}\vert 11100\rangle + b(1-p)^2py_0e^{i\theta}\vert 11101\rangle + b(1-p)^2py_0e^{i\theta}\vert 11110\rangle + a \times (1-p)^2py_0e^{i\theta}\vert 11111\rangle$
$\vert G_{11}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle + b(1-p)^2px_0\vert 00001\rangle + b(1-p)^2px_0\vert 00010\rangle + a(1-p)^2px_0\vert 00011\rangle + a(1-p)^2py_0e^{i\theta}\vert 11100\rangle + b(1-p)^2py_0e^{i\theta}\vert 11101\rangle + b(1-p)^2py_0e^{i\theta}\vert 11110\rangle + a \times (1-p)^2py_0e^{i\theta}\vert 11111\rangle$
$\vert G_{12}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11100\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11101\rangle - b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11111\rangle$
$\vert G_{13}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle - b(1-p)^2px_0\vert 00001\rangle - b(1-p)^2px_0\vert 00010\rangle + a(1-p)^2px_0\vert 00011\rangle + a(1-p)^2py_0e^{i\theta}\vert 11100\rangle - b(1-p)^2py_0e^{i\theta}\vert 11101\rangle - b(1-p)^2py_0e^{i\theta}\vert 11110\rangle + a \times (1-p)^2py_0e^{i\theta}\vert 11111\rangle$
$\vert G_{14}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11111\rangle$
$\vert G_{15}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\vert 11111\rangle$
$\vert G_{16}\rangle=$	$a(1-p)p^2x_0\vert 00000\rangle - b(1-p)p^2x_0\vert 00001\rangle - b(1-p)p^2x_0\vert 00010\rangle + a(1-p)p^2x_0\vert 00011\rangle + a(1-p)p^2y_0e^{i\theta}\vert 11100\rangle - b(1-p)p^2y_0e^{i\theta}\vert 11101\rangle - b(1-p)p^2y_0e^{i\theta}\vert 11110\rangle + a(1-p)p^2y_0e^{i\theta}\vert 11111\rangle$
$\vert G_{17}\rangle=$	$a(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00000\rangle - b(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00001\rangle + b(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00010\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}x_0\vert 00011\rangle + a(1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11100\rangle - b(1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11101\rangle + b \times (1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11110\rangle - a(1-p)^{\frac{5}{2}}\sqrt{p}y_0e^{i\theta}\vert 11111\rangle$

Table C1. (Continued.)

$\vert G_i\rangle$	Expression
$\vert G_{18}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle - b(1-p)^2px_0\vert 00001\rangle + b(1-p)^2px_0\vert 00010\rangle - a(1-p)^2px_0\vert 00011\rangle - a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{19}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle - b(1-p)^2px_0\vert 00001\rangle + b(1-p)^2px_0\vert 00010\rangle - a(1-p)^2px_0\vert 00011\rangle - a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{20}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{21}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle + b(1-p)^2px_0\vert 00001\rangle - b(1-p)^2px_0\vert 00010\rangle - a(1-p)^2px_0\vert 00011\rangle - a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{22}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle - b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{23}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle - b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{24}\rangle=$	$a(1-p)p^2x_0\vert 00000\rangle + b(1-p)p^2x_0\vert 00001\rangle - b(1-p)p^2x_0\vert 00010\rangle - a(1-p)p^2x_0\vert 00011\rangle - a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle - b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{25}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle - b(1-p)^2px_0\vert 00001\rangle + b(1-p)^2px_0\vert 00010\rangle - a(1-p)^2px_0\vert 00011\rangle - a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{26}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{27}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{28}\rangle=$	$a(1-p)p^2x_0\vert 00000\rangle - b(1-p)p^2x_0\vert 00001\rangle + b(1-p)p^2x_0\vert 00010\rangle - a(1-p)p^2x_0\vert 00011\rangle - a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle + b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle - b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle + a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$

Table C1. (Continued.)

$\vert G_i\rangle$	Expression
$\vert G_{29}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle+a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle-b\times(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{30}\rangle=$	$a(1-p)p^2x_0\vert 00000\rangle+b(1-p)p^2x_0\vert 00001\rangle-b(1-p)p^2x_0\vert 00010\rangle-a(1-p)p^2x_0\vert 00011\rangle-a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle-b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle+b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle+a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{31}\rangle=$	$a(1-p)p^2x_0\vert 00000\rangle+b(1-p)p^2x_0\vert 00001\rangle-b(1-p)p^2x_0\vert 00010\rangle-a(1-p)p^2x_0\vert 00011\rangle-a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle-b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle+b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle+a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{32}\rangle=$	$a\sqrt{1-p}\,p^{\frac{5}{2}}x_0\vert 00000\rangle+b\sqrt{1-p}\,p^{\frac{5}{2}}x_0\vert 00001\rangle-b\sqrt{1-p}\,p^{\frac{5}{2}}x_0\vert 00010\rangle-a\sqrt{1-p}\,p^{\frac{5}{2}}x_0\vert 00011\rangle+a\sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle+b\sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle-b\sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle-a\times\sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{33}\rangle=$	$a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\vert 00000\rangle+b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\vert 00001\rangle-b(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\vert 00010\rangle-a(1-p)^{\frac{5}{2}}\sqrt{p}\,x_0\vert 00011\rangle+a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle+b(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle-b\times(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle-a(1-p)^{\frac{5}{2}}\sqrt{p}\,y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{34}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle+b(1-p)^2px_0\vert 00001\rangle-b(1-p)^2px_0\vert 00010\rangle-a(1-p)^2px_0\vert 00011\rangle-a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle-b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle+b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle+a\times(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{35}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle+b(1-p)^2px_0\vert 00001\rangle-b(1-p)^2px_0\vert 00010\rangle-a(1-p)^2px_0\vert 00011\rangle-a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle-b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle+b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle+a\times(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{36}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle+a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle-b\times(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{37}\rangle=$	$a(1-p)^2px_0\vert 00000\rangle-b(1-p)^2px_0\vert 00001\rangle+b(1-p)^2px_0\vert 00010\rangle-a(1-p)^2px_0\vert 00011\rangle-a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle+b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle-b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle+a\times(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{38}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle+a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle+b\times(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{39}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00000\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00001\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00010\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\vert 00011\rangle+a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle+b\times(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$
$\vert G_{40}\rangle=$	$a(1-p)p^2x_0\vert 00000\rangle-b(1-p)p^2x_0\vert 00001\rangle+b(1-p)p^2x_0\vert 00010\rangle-a(1-p)p^2x_0\vert 00011\rangle-a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11100\rangle+b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11101\rangle-b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11110\rangle+a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\vert 11111\rangle$

Table C1. (Continued.)

$\lvert G_i\rangle$	Expression
$\lvert G_{41}\rangle=$	$a(1-p)^2px_0\lvert 00000\rangle + b(1-p)^2px_0\lvert 00001\rangle - b(1-p)^2px_0 \lvert 00010\rangle - a(1-p)^2px_0\lvert 00011\rangle - a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{42}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00000\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle - b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{43}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00000\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle - b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{44}\rangle=$	$a(1-p)p^2x_0\lvert 00000\rangle + b(1-p)p^2x_0\lvert 00001\rangle - b(1-p)p^2x_0\lvert 00010\rangle - a(1-p)p^2x_0\lvert 00011\rangle - a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle - b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{45}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00001\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00010\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00011\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{46}\rangle=$	$a(1-p)p^2x_0\lvert 00000\rangle - b(1-p)p^2x_0\lvert 00001\rangle + b(1-p)p^2x_0\lvert 00010\rangle - a(1-p)p^2x_0\lvert 00011\rangle - a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle - b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{47}\rangle=$	$a(1-p)p^2x_0\lvert 00000\rangle - b(1-p)p^2x_0\lvert 00001\rangle + b(1-p)p^2x_0\lvert 00010\rangle - a(1-p)p^2x_0\lvert 00011\rangle - a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle - b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{48}\rangle=$	$a\sqrt{1-p}\,p^{\frac{5}{2}}x_0\lvert 00000\rangle - b\sqrt{1-p}\,p^{\frac{5}{2}}x_0\lvert 00001\rangle + b\sqrt{1-p}\,p^{\frac{5}{2}}x_0 \lvert 00010\rangle - a\sqrt{1-p}\,p^{\frac{5}{2}}x_0\lvert 00011\rangle + a\sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle - b\sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b\sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle - a \times \sqrt{1-p}\,p^{\frac{5}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{49}\rangle=$	$a(1-p)^2px_0\lvert 00000\rangle - b(1-p)^2px_0\lvert 00001\rangle - b(1-p)^2px_0 \lvert 00010\rangle + a(1-p)^2px_0\lvert 00011\rangle + a(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle - b(1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a \times (1-p)^2py_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{50}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00010\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00011\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{51}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00000\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00001\rangle - b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00010\rangle + a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\lvert 00011\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle + b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle + b \times (1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle - a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$
$\lvert G_{52}\rangle=$	$a(1-p)p^2x_0\lvert 00000\rangle - b(1-p)p^2x_0\lvert 00001\rangle - b(1-p)p^2x_0\lvert 00010\rangle + a(1-p)p^2x_0\lvert 00011\rangle + a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11100\rangle - b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11101\rangle - b(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11110\rangle + a(1-p)p^2y_0\mathrm{e}^{\mathrm{i}\theta}\lvert 11111\rangle$

Table C1. (Continued.)

$\|G_i\rangle$	Expression
$\|G_{53}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00000\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00001\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00010\rangle+a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00011\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11100\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11101\rangle-b\times(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11110\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11111\rangle$
$\|G_{54}\rangle=$	$a(1-p)p^2x_0\|00000\rangle+b(1-p)p^2x_0\|00001\rangle+b(1-p)p^2x_0\|00010\rangle+a(1-p)p^2x_0\|00011\rangle+a(1-p)p^2y_0e^{i\theta}\|11100\rangle+b(1-p)p^2y_0e^{i\theta}\|11101\rangle+b(1-p)p^2y_0e^{i\theta}\|11110\rangle+a(1-p)p^2y_0e^{i\theta}\|11111\rangle$
$\|G_{55}\rangle=$	$a(1-p)p^2x_0\|00000\rangle+b(1-p)p^2x_0\|00001\rangle+b(1-p)p^2x_0\|00010\rangle+a(1-p)p^2x_0\|00011\rangle+a(1-p)p^2y_0e^{i\theta}\|11100\rangle+b(1-p)p^2y_0e^{i\theta}\|11101\rangle+b(1-p)p^2y_0e^{i\theta}\|11110\rangle+a(1-p)p^2y_0e^{i\theta}\|11111\rangle$
$\|G_{56}\rangle=$	$a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00000\rangle+b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00001\rangle+b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00010\rangle+a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00011\rangle-a\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11100\rangle-b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11101\rangle-b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11110\rangle-a\times\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11111\rangle$
$\|G_{57}\rangle=$	$a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00000\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00001\rangle-b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00010\rangle+a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}x_0\|00011\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11100\rangle+b(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11101\rangle+b\times(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11110\rangle-a(1-p)^{\frac{3}{2}}p^{\frac{3}{2}}y_0e^{i\theta}\|11111\rangle$
$\|G_{58}\rangle=$	$a(1-p)p^2x_0\|00000\rangle-b(1-p)p^2x_0\|00001\rangle-b(1-p)p^2x_0\|00010\rangle+a(1-p)p^2x_0\|00011\rangle+a(1-p)p^2y_0e^{i\theta}\|11100\rangle-b(1-p)p^2y_0e^{i\theta}\|11101\rangle-b(1-p)p^2y_0e^{i\theta}\|11110\rangle+a(1-p)p^2y_0e^{i\theta}\|11111\rangle$
$\|G_{59}\rangle=$	$a(1-p)p^2x_0\|00000\rangle-b(1-p)p^2x_0\|00001\rangle-b(1-p)p^2x_0\|00010\rangle+a(1-p)p^2x_0\|00011\rangle+a(1-p)p^2y_0e^{i\theta}\|11100\rangle-b(1-p)p^2y_0e^{i\theta}\|11101\rangle-b(1-p)p^2y_0e^{i\theta}\|11110\rangle+a(1-p)p^2y_0e^{i\theta}\|11111\rangle$
$\|G_{60}\rangle=$	$a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00000\rangle-b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00001\rangle-b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00010\rangle+a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00011\rangle-a\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11100\rangle+b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11101\rangle+b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11110\rangle-a\times\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11111\rangle$
$\|G_{61}\rangle=$	$a(1-p)p^2x_0\|00000\rangle+b(1-p)p^2x_0\|00001\rangle+b(1-p)p^2x_0\|00010\rangle+a(1-p)p^2x_0\|00011\rangle+a(1-p)p^2y_0e^{i\theta}\|11100\rangle+b(1-p)p^2y_0e^{i\theta}\|11101\rangle+b(1-p)p^2y_0e^{i\theta}\|11110\rangle+a(1-p)p^2y_0e^{i\theta}\|11111\rangle$
$\|G_{62}\rangle=$	$a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00000\rangle+b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00001\rangle+b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00010\rangle+a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00011\rangle-a\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11100\rangle-b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11101\rangle-b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11110\rangle-a\times\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11111\rangle$
$\|G_{63}\rangle=$	$a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00000\rangle+b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00001\rangle+b\sqrt{1-p}p^{\frac{5}{2}}x_0\|00010\rangle+a\sqrt{1-p}p^{\frac{5}{2}}x_0\|00011\rangle-a\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11100\rangle-b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11101\rangle-b\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11110\rangle-a\times\sqrt{1-p}p^{\frac{5}{2}}y_0e^{i\theta}\|11111\rangle$
$\|G_{64}\rangle=$	$ap^3x_0\|00000\rangle+bp^3x_0\|00001\rangle+bp^3x_0\|00010\rangle+ap^3x_0\|00011\rangle+ap^3y_0e^{i\theta}\|11100\rangle+bp^3y_0e^{i\theta}\|11101\rangle+bp^3y_0e^{i\theta}\|11110\rangle+ap^3y_0e^{i\theta}\|11111\rangle$

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