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Regular Black Holes with Cosmological Constant*

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Abstract *We present a class of regular black holes with cosmological constant Λ in nonlinear electrodynamics. Instead of usual singularity behind black hole horizon, all fields and curvature invariants are regular everywhere for the regular black holes. Through gauge invariant approach, the linearly dynamical stability of the regular black hole is studied. In odd-parity sector, we find that the Λ term does not appear in the master equations of perturbations, which shows that the regular black hole is stable under odd-parity perturbations. On the other hand, for the even-parity sector, the master equations are more complicated than the case without the cosmological constant. We obtain the sufficient conditions for stability of the regular black hole. We also investigate the thermodynamic properties of the regular black hole, and find that those thermodynamic quantities do not satisfy the differential form of first law of black hole thermodynamics. The reason for violating the first law is revealed.*

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1 Introduction

Black hole physics is an interesting subject in modern theoretical physics. Usually black hole is always associated with a singularity inside the black hole. In the past few years, however, several interesting so-called regular black hole solutions[1−5] have been found by coupling general relativity to nonlinear electrodynamics (NED). For these black holes, instead of usual singularities at their center, all fields and curvature invariants are regular everywhere, and they asymptotically behave as ordinary charged Reissner–Nordström black hole solutions. The existence of these solutions does not contradict with the singularity theorems.

An important issue in general relativity is to analyze stability of a given static or stationary field configuration via perturbation. Often, the perturbation problem is divided into two parts, one is static background, which is solution of the coupled Einstein and electromagnetic field equations; the other is dynamic fluctuations (usually only linear sector involved) of gravitational and electromagnetic fields. Perturbations are small deviations from static background spacetime filled with electromagnetic field, and their motion equations are linearized ones derived from background equations. Investigation of stability usually generates two possible outcomes. i) Fluctuations' amplitudes grow in time. In that case the background is not physical since it is unstable. ii) Fluctuations decay in time, perturbed configuration eventually settles down to the background after possible emissions of gravitational and electromagnetic radiation; this case is just the so-called quasi-normal modes, which has been extensively studied (see comprehensive reviews,[6,7] and references therein).

Stability for regular black hole with NED source in the absence of cosmological constant has been studied by Moreno and Sarbach[8] recently. Their result is that for odd-parity perturbations, the solution is stable without any constraint, while for even parity, it is stable only if the NED source satisfies some conditions. These conditions are satisfied for the sources of the regular black holes found in Refs. [1] ∼ [3].

On the other hand, over the past years, there is much interest in black holes in asymptotically (anti-)de Sitter (AdS) spaces due to the AdS/CFT (conformal field theory) correspondence[9] and dS/CFT correspondence.[10] According to the AdS/CFT correspondence, one can not only describe the outside geometry of black holes in AdS spaces by using the boundary CFTs, but also the inside geometry. Therefore it is interesting to find the regular black holes in AdS spaces, in order to see the differences from the one corresponding to the AdS black holes with a singularity behind black hole horizon, from the viewpoint of dual CFTs.

In this paper, we will present a class of regular black holes in AdS spaces in NED, and further study dynamical stability of regular black holes by adopting gauge-invariant perturbation formalism introduced in

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Refs. [11] $\sim$ [14]. This formulism together with spherical harmonics tensor[15−18] and parity decoupling form an effective method to perturbation stability, which will be briefly reviewed in Sec. 2.

We will also study thermodynamics of black hole horizon. The mass, Hawking temperature and entropy are calculated, but we find that these thermodynamic quantities do not obey the differential form of first law of black hole thermodynamics. The reason will be discussed.

The regular black hole solution with Λ is presented in Sec. 2. In Sec. 3, we discuss the stability of the solution in odd parity, while even-parity perturbation will be addressed in Sec. 4. Thermodynamics of black hole horizon is investigated in Sec. 5.

The metric signature is $(-,+,+,+)$, and we use standard notations $2\omega_{(ab)} = \omega_{ab} + \omega_{ba}$ and $2\omega_{[ab]} = \omega_{ab} - \omega_{ba}$ for symmetrization and anti-symmetrization, respectively. Throughout this paper, Greek letters μ, ν denote spacetime indices, taking values in $0, 1, 2, 3$, while Roman letters a, b will denote 0, 1, and A, B denote 2, 3. In addition, for the spacetime manifold $M = \tilde{M} \times S^2$, the metric is decomposed as $g = \tilde{g} + r^2\hat{g}$, we use "$\tilde{}$" to denote quantities on $\tilde{M}$ while "$\hat{}$" to denote quantities on S^2, e.g. $\tilde{*}$, $\tilde{\triangle}$ denote the Hodge $*$ dual operator and Laplace operator with respect to $\tilde{g}$.

2 Regular Black Holes

The action of gravitational field coupled to a nonlinear electromagnetic field with a cosmological constant Λ can be written as

$$S(g, A) = \frac{1}{4\pi}\int \sqrt{-g}\,\mathrm{d}^4x\Big[\frac{1}{4}(R - 2\Lambda) - \mathcal{L}(F)\Big]\,, \tag{1}$$

where R denotes the scalar curvature of spacetime metric $g_{\mu\nu}$, and $\mathcal{L}(F)$ is an arbitrary function of $F \equiv F^{\mu\nu}F_{\mu\nu}/4$. In Einstein–Maxwell theory, $\mathcal{L}(F) = F$, while for regular black holes in NED, its form will be fixed later.

Varying the action with respect to metric and electromagnetic potential, one has the Einstein-NED equations and electromagnetic field equation and Bianchi identities in the so-called F-framework,

$$G_{\mu\nu} = 2\Big[\mathcal{L}_F F_\mu^\alpha F_{\nu\alpha} - g_{\mu\nu}\Big(\mathcal{L} + \frac{1}{2}\Lambda\Big)\Big]\,, \tag{2}$$

$$\nabla_\mu(\mathcal{L}_F F^{\mu\alpha}) = 0\,, \tag{3}$$

$$\nabla_\mu * F^{\mu\alpha} = 0\,. \tag{4}$$

By dual transformation,

$$g_{\mu\nu} \to g_{\mu\nu}\,, \quad F_{\mu\nu} \to *P_{\mu\nu}\,, \quad \mathcal{L}(F) \to -\mathcal{H}(P)\,,$$

where $P_{\mu\nu} \equiv \mathcal{L}_F F_{\mu\nu}$, the hamiltonian $\mathcal{H}$ is a function of $P \equiv P_{\mu\nu}P^{\mu\nu}/4 = (\mathcal{L}_F)^2 F$, and

$$\mathrm{d}\mathcal{H} = (\mathcal{L}_F)^{-1}\mathrm{d}((\mathcal{L}_F)^2F) = \mathcal{H}_P\,\mathrm{d}P\,.$$

Also we have Legendre transformation

$$\mathcal{H} = 2F\mathcal{L}_F - \mathcal{L}\,, \quad \mathcal{L} = 2P\mathcal{H}_P - \mathcal{H}\,, \tag{5}$$

and $*P^{\mu\alpha} * P^{\nu\alpha} = P^{\mu\alpha}P^{\nu\alpha} - 2\delta^\mu_\nu P$, then in the P-framework, corresponding equations to Eqs. (2) $\sim$ (4) are

$$G_\mu^{\ \nu} = 2\Big[\mathcal{H}_P P_{\mu\lambda}P^{\nu\lambda} - \delta_\mu^{\ \nu}\Big(2P\mathcal{H}_P - \mathcal{H} + \frac{1}{2}\Lambda\Big)\Big]\,, \tag{6}$$

$$\nabla_\mu P^{\alpha\mu} = 0\,, \tag{7}$$

$$\nabla_\mu(*\mathcal{H}_P P^{\alpha\mu}) = 0\,, \tag{8}$$

which are equivalent to the corresponding ones in the F-framework only if $\partial F/\partial P$ and $\partial P/\partial F$ are both nonzero. The above F-P duality is useful when transforming some results obtained under one framework into the other (for a more detailed analysis of duality of nonlinear electrodynamics, see Ref. [19], and so we will discuss our problem just in P-framework for convenience. In the meantime, Hamiltonian $\mathcal{H}$ is supposed negative and satisfies the weak field limits, i.e. $\mathcal{H}(P) \simeq P$ as $P \to 0$, see Ref. [1]), and $\mathcal{H}_P$ is assumed to be regular and positive for all $r > r_h$, where r_h denotes the horizon radius of black hole.

A spherically symmetric spacetime is defined as a manifold which admits SO(3) to be a group of isometries, with the group orbits of a space-like 2-surface of constant positive curvature and can locally be written as $\tilde{M} \times S^2$, where $\tilde{M}$ is a 2-dimensional pseudo-Riemannian manifold. The metric takes the form

$$g_{\mu\nu} = \tilde{g}_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b + r^2\hat{g}_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B\,. \tag{9}$$

We discuss a static electrically charged black hole solution so that one has $P_{\mu\nu} = 2\delta^\mu_{[t}\delta^\nu_{r]}D(r)$, from Eq. (7). Thus

$$P_{ab} = \frac{q}{r^2}\eta_{ab}\,, \quad P_{Ab} = P_{AB} = 0\,, \tag{10}$$

where η_{ab} is the volume element corresponding to $\tilde{g}$ and q is an integration constant that represents the electric charge as shown later. Then $P = -q^2/2r^4$ and equation (6) reduces to

$$G_{ab} = 2\tilde{g}_{ab}\Big(\mathcal{H} - \frac{1}{2}\Lambda\Big)\,, \quad G_{Ab} = 0\,,$$
$$G_{AB} = -2\hat{g}_{AB}\Big(\mathcal{L} + \frac{1}{2}\Lambda\Big)\,. \tag{11}$$

On the other hand, the components of Einstein tensor can be written as

$$G_{ab} = -\frac{2}{r}\tilde{\nabla}_a\tilde{\nabla}_b r + \frac{1}{r^2}(2r\tilde{\triangle}r + \tilde{g}(\mathrm{d}r, \mathrm{d}r) - 1)\tilde{g}_{ab}\,, \tag{12}$$

$$G_{Ab} = 0\,, \tag{13}$$

$$G_{AB} = \Big(r\tilde{\triangle}r - \frac{1}{2}r^2\tilde{R}\Big)\hat{g}_{AB}\,, \tag{14}$$

where $\tilde{R}$ is the Ricci scalar of $\tilde{g}$.

Since we are considering static solutions of Einstein-NED equations, where the Birkhoff theorem still holds, then the metric of $\tilde{M}$ can be written as $\tilde{g} = -N\mathrm{d}t^2 + N^{-1}\mathrm{d}r^2$, where $N = 1 - 2\mathbf{m}(r)/r$. Furthermore, by Eq. (11), one has $\partial_r\mathbf{m} = -r^2(\mathcal{H} - \Lambda/2)$, i.e.

$$\mathbf{m}(r) = \mathcal{M} + \int_r^\infty r^2\mathcal{H}\,\mathrm{d}r + \frac{\Lambda r^3}{6}\,, \tag{15}$$

where $\mathcal{M}$ is an integration constant. Thus given a function $\mathcal{H}$, one has an exact solution to equations of motion. Here we use the Hamiltonian $\mathcal{H}$ given in Ref. [1],

$$\mathcal{H}(P) = P\frac{(1-3\sqrt{-2\,q^2P})}{(1+\sqrt{-2\,q^2P})^3} - \frac{3}{2q^2s}\Big(\frac{\sqrt{-2q^2P}}{1+\sqrt{-2q^2P}}\Big)^{5/2}\,, \tag{16}$$

where $s = |q|/2m$, and m, q are parameters. The corresponding Lagrangian occurs to be

$$\mathcal{L} = P\frac{1-8\sqrt{-2q^2P}-6q^2P}{(1+\sqrt{-2\,q^2P})^4} - \frac{3}{4q^2s}\frac{(-2q^2P)^{5/4}(3-2\sqrt{-2q^2P})}{(1+\sqrt{-2q^2P})^{7/2}}\,. \tag{17}$$

It is easy to find that the Hamiltonian (16) reduces to Maxwell theory in the weak field limit, $\mathcal{H}\approx\mathcal{P}$, and obeys the weak energy conditions with $\mathcal{H}<0$ and $\mathcal{H}_{\mathcal{P}}>0$.

Substituting Eqs. (10) and (16) into Eq. (15), we arrive at

$$2\mathbf{m}(r) = 2\mathcal{M} - q^2\int_r^\infty\Big(\frac{(r^2-3q^2)r^2}{(r^2+q^2)^3} + \frac{6mr^2}{(r^2+q^2)^{5/2}}\Big)\mathrm{d}r + \frac{\Lambda r^3}{3}\,. \tag{18}$$

Note that the integrand above can be expressed as $\partial_r(\frac{2mr^3}{q^2(r^2+q^2)^{3/2}} - \frac{r^3}{(r^2+q^2)^2})$. Thus we find the solution

$$g = -\Big(1-\frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2} - \frac{\Lambda r^2}{3}\Big)\mathrm{d}t^2 + \Big(1-\frac{2mr^2}{(r^2+q^2)^{3/2}} + \frac{q^2r^2}{(r^2+q^2)^2} - \frac{\Lambda r^2}{3}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{19}$$

in which we have set the integration constant $\mathcal{M}=m$, otherwise there is still a singularity at $r=0$. The associated electric field strength E is given by

$$E = F_{\mathrm{tr}} = \mathcal{H}_P D = qr^4\Big(\frac{r^2-5q^2}{(r^2+q^2)^4} + \frac{15}{2}\frac{m}{(r^2+q^2)^{7/2}}\Big)\,. \tag{20}$$

When $\Lambda=0$, the solution reduces to the case discussed in Ref. [1], where it is shown that the field strength and all curvature invariants are regular everywhere. On the other hand, when $m=q=0$, the solution is just the dS ($\Lambda>0$ or AdS ($\Lambda<0$) space. It is easy to see that the appearance of cosmological constant does not destroy the regularity of the solution. Therefore the solution (19) describes a regular black hole in dS/AdS space. In the large r limit, this solution asymptotically approaches a Reissner–Nordstrom black hole in dS/AdS space depending on the sign of the cosmological constant Λ. Near the origin, however, the solution reduces to

$$g = -\Big(1-\frac{2mr^2}{q^3} + \frac{r^2}{q^2} - \frac{\Lambda r^2}{3}\Big)\mathrm{d}t^2 + \Big(1-\frac{2mr^2}{q^3} + \frac{r^2}{q^2} - \frac{\Lambda r^2}{3}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,. \tag{21}$$

Clearly, this solution is a dS or AdS space depending on the sign of the quantity $(6m/q^3 - 3/q^2 + \Lambda)$.

To analyze stability of the regular black hole, we first derive the linearized perturbation equations from Eqs. (6) and (7) in terms of $\delta g_{\mu\nu}$ and δA_μ as well as related small perturbations δP and $\delta P_{\mu\nu}$,

$$\delta G_{\mu\nu} = 2\Big\{(\mathcal{H}_{PP}P_\mu^\alpha P_{\nu\alpha} - g_{\mu\nu}\kappa\mathcal{H}_P)\delta P + \mathcal{H}_P(2P_{(\mu}^{\ \alpha}\delta P_{\nu)\alpha} - P_\mu^{\ \sigma}P_\nu^{\ \rho}\delta g_{\sigma\rho}) - \delta g_{\mu\nu}\Big(2\mathcal{H}_P P - \mathcal{H} + \frac{1}{2}\Lambda\Big)\Big\}\,,$$

$$\kappa = 1 + 2\mathcal{H}_P^{-1}\mathcal{H}_{PP}P\,, \tag{22}$$

$$\nabla^\mu\Big(\delta P_{\alpha\mu} - 2\delta g_{\sigma[\alpha}P_{\mu]}^\sigma + \frac{1}{2}g^{\sigma\rho}\delta g_{\sigma\rho}P_{\alpha\mu}\Big) = 0\,. \tag{23}$$

Before proceeding, let us introduce some basic concepts and steps of the method for the stability analysis.

(i) In the polar coordinate system, orthogonal decomposition based on spherical harmonics function Y_{lm} (scalar) is the standard method in many physical problems. As a generalization from local area to the whole manifold, the spherical harmonics tensor arises naturally (see for example, Ref. [15]). When an unknown object is tensor-like, while the metric and the background manifold are spherically symmetrical, we can use spherical harmonics tensor as the orthogonal bases of background manifold which can be viewed as a "global polar coordinate system", and express the tensor as a combination of these bases with some unknown scalar coefficients which are more convenient to deal with, especially for the gauge invariant approach. Here we just give some fundamental knowledge for (k,l)-tensor (for more details, see Ref. [8]).

- Base of (0, 0)-type, scalar function Ψ:

  standard spherical harmonics function Y_{lm}, even parity $(-1)^l$.

- Base of $(1,0)$ or $(0,1)$-type, vector or 1-form:

  $S^A = \eta^A_{\ B}\nabla^B\mathrm{Y}_{lm}$, $S_B = \eta^A_{\ B}\nabla_A\mathrm{Y}_{lm} = (\hat{*}\mathrm{d}\mathrm{Y}_{lm})_B$, odd parity $(-1)^{l+1}$; $S^A = \nabla^A\mathrm{Y}_{lm}$, $S_B = \nabla_B\mathrm{Y}_{lm} = \mathrm{d}\mathrm{Y}_{lm}$, $\mathrm{Y}_{lm}\mathrm{d}r$, even parity $(-1)^l$.

- Base of (2, 0) or (0, 2)-type, e.g. metric:

  $\nabla_{(A}S_{B)}$ (base for symmetric tensor), odd parity $(-1)^{l+1}$; $\nabla_A\nabla_B\mathrm{Y}_{lm}$, even parity $(-1)^l$.

The construction of these bases can be viewed as a multiplication of Y_{lm} and certain operators which transform as required tensor properties. According to parity, we can classify the operators into two cases: parity preserved type, e.g. d, ∇_A, and parity reversed type, e.g. $\eta^A_{\ B}, \hat{*}$.

(ii) An observation: under parity transformation $x \to -x$, a spherically symmetrical metric is invariant, i.e. the parity can be seen as a kind of integration constant of equations of motion. So we can divide the perturbation amplitudes into two types: odd parity and even parity. Notice that the $l = 0$ case means Y_{lm} is a constant, there is no gravitational perturbation and it is just the background solution. The $l = 1$ case is a little bit different from the $l > 1$ case, but this case is more easy to analyze. Hence we will mainly concentrate on the $l \geq 2$ case in this paper.

(iii) The next is to use the gauge freedom to simplify those unknown coefficients. Here we have 3 steps: First, make an infinitesimal coordinate transformation generated by a vector field X^μ, and observe the corresponding transformation of the coefficients in the expanded form for the perturbation amplitudes $\delta g_{\mu\nu}$ and δA_μ. Second, find some new amplitudes composed of the coefficients which are gauge-invariant, and then specify a gauge so that some coefficients of the amplitudes vanish, which simplifies the form of other perturbation amplitudes and equations. Third, obtain the perturbation equations in terms of gauge invariant amplitudes.

3 Odd-Parity Sector

On the sphere S^2 $(t, r = \text{const.})$, when one rotates the frame around the origin, the perturbation amplitudes (which have 14 independent components: 10 for $\delta g_{\mu\nu}$, and 4 for δA_μ) transform as a set of 3 scalars (δg_{ab}), 3 vectors (δg_{Ab}, δA_μ), and a 2-rank tensor (δg_{AB}). Hence the odd-parity perturbations of $\delta g_{\mu\nu}$ and δA_μ are parameterized in terms of two scalar fields k, ν, and a one-form vector $h = h_a \mathrm{d}x^a$ on the two-dimensional pseudo-Riemannian orbit space $\tilde{M} \equiv M/\mathrm{SO}(3)$,

$$\delta g_{ab} = 0\,, \quad \delta g_{Ab} = h_b S_A\,, \quad \delta g_{AB} = 2k\nabla_{(A}S_{B)}\,, \quad \delta A_\mu \mathrm{d}x^\mu = \nu S_B \mathrm{d}x^B\,. \tag{24}$$

Consider an infinitesimal coordinate transformation with odd parity, which is generated by a vector field X^μ. The vector must have the form[12]

$$X^a = 0\,, \quad X^A = \frac{f(x^a)}{r^2} g^{AB} S_B\,. \tag{25}$$

Under the transformations of $\delta g_{\mu\nu}$ and δA_μ by the vector, it is not difficult to find the following transformations:

$$h \mapsto h + r^2 \mathrm{d}\Big(\frac{f}{r^2}\Big)\,, \quad k \mapsto k + f\,, \quad \nu \mapsto \nu\,. \tag{26}$$

Thus the required gauge invariant amplitude is a 1-form,

$$\bar{h} \equiv h - r^2 \mathrm{d}\Big(\frac{k}{r^2}\Big)\,. \tag{27}$$

Then we can always fix a gauge so that $k = 0$, and $\bar{h} = h$ (usually called as Regge–Wheeler (RW) gauge). Now the form of related amplitudes and equations are simplified under the RW gauge.

For the electromagnetic field, we have $\delta P = 0$ since in the odd parity sector all scalar perturbations vanish, and from $F_{\mu\nu} = \mathrm{d}A = \partial_\mu A_\nu - \partial_\nu A_\mu$, we have

$$\begin{aligned}\delta P_{\mu\nu} &= \mathcal{H}_P^{-1}\delta \mathrm{d}A = \mathcal{H}_P^{-1}\mathrm{d}\delta A = \mathcal{H}_P^{-1}\mathrm{d}(\nu S_B \mathrm{d}x^B)_{\mu\nu} \\ &= \mathcal{H}_P^{-1}[\mathrm{d}\nu \wedge (S_B \mathrm{d}x^B) + \nu \mathrm{d}(S_B \mathrm{d}x^B)]_{\mu\nu}\,.\end{aligned}$$

Because

$$\begin{aligned}\mathrm{d}(S_B \mathrm{d}x^B) &= \mathrm{d}(\eta^A_B \bigtriangledown_A Y_{lm} \mathrm{d}x^B \\ &= \bigtriangledown_A(\eta^A_B \bigtriangledown_A Y_{lm})\mathrm{d}x^A \wedge \mathrm{d}x^B \\ &= \tilde{\triangle} Y_{lm} \mathrm{d}\Omega = -l(l+1) Y_{lm} \mathrm{d}\Omega\,,\end{aligned}$$

then one arrives at

$$\delta P_{\mu\nu} = \mathcal{H}_P^{-1}[\mathrm{d}\nu \wedge S_B \mathrm{d}x^B - l(l+1)\nu Y_{lm} \mathrm{d}\Omega]_{\mu\nu}\,. \tag{28}$$

Note the fact $g^{\alpha\beta}\delta g_{\alpha\beta} = 0$, and use the background equation (10) and the simplified form (28) of $\delta P_{\mu\nu}$, the linearized electromagnetic equation (23) turns to be

$$\tilde{\mathrm{d}}^\dagger(\mathcal{H}_P^{-1}\mathrm{d}\nu) + \frac{l(l+1)}{r^2}\mathcal{H}_P^{-1}\nu + q\tilde{*}\mathrm{d}\Big(\frac{h}{r^2}\Big) = 0\,, \tag{29}$$

where $\tilde{*}$ and $\tilde{\mathrm{d}}^\dagger \equiv \tilde{*}\mathrm{d}\,\tilde{*}$ denote the Hodge dual and co-differential, with respect to metric $\tilde{g}$. This equation is the same as the case without the cosmological constant.[8]

The right side of linearized Einstein-NED equation (22) can be obtained as

$$\begin{aligned}&\delta G_{Ab}\mathrm{d}x^b = 4(q(\tilde{*}\mathrm{d}\nu) - r^2\mathcal{L}h)\frac{S_A}{2r^2} - \Lambda h S_A\,, \\ &\delta G_{ab} = 0\,, \quad \delta G_{AB} = 0\,.\end{aligned} \tag{30}$$

The Λ-term appears only in δG_{Ab} component of Einstein-NED perturbation equation since $\delta g_{ab} = 0$ (scalar expansion) and $\delta g_{AB} = 0$ (RW gauge). On the other hand, the corresponding components of linearized Einstein tensor are

$$\begin{aligned}&\delta G_{Ab}\mathrm{d}x^b = \Big\{\tilde{\mathrm{d}}^\dagger\Big[r^4\mathrm{d}\Big(\frac{h}{r^2}\Big)\Big] + (l(l+1) - 2 + r^2 G^B_B)h\Big\}\frac{S_A}{2r^2}\,, \\ &\delta G_{ab} = 0\,, \quad \delta G_{AB} = -\tilde{\mathrm{d}}^\dagger h \nabla_{(A}S_{B)}\,.\end{aligned} \tag{31}$$

Comparing the above equations (30) and (31), and noting from Eq. (10) that $G^B_B = -4(\mathcal{L} + \Lambda/2)$, then the main linearized Einstein-NED equation in the RW gauge is

$$\tilde{\mathrm{d}}^\dagger\Big[r^4\mathrm{d}\Big(\frac{h}{r^2}\Big)\Big] + \lambda h - 4q\tilde{*}\mathrm{d}\nu = 0\,, \quad \lambda \equiv (l-1)(l+2)\,. \tag{32}$$

We find here that the Λ term disappears and thus the conclusion is the same as the case without the cosmological constant.

On the other hand, the equation concerning the component δG_{AB} of Einstein-NED equation deduces to

$$\tilde{\mathrm{d}}^\dagger h = 0\,, \tag{33}$$

which is void for $l = 1$ case, since when $l = 1$, $\nabla_{(A}S_{B)} = 0$. Equation (33) is called integrable condition.

Because $\bar{h}$ is gauge-invariant, we can replace h in the above equations with $\bar{h}$ to get the general perturbation equations under arbitrary gauge. Furthermore, under some reasonable assumption on the topology structure of $\tilde{M}$, due to Eq. (33), we can always find a potential Ψ so that

$$\bar{h} = \frac{1}{\sqrt{\lambda}} \tilde{*} \mathrm{d}(r\Psi) , \tag{34}$$

where introducing the factors λ and r turns out to be convenient later. In the meantime, we define $\Phi = \sqrt{4\mathcal{H}_P^{-1}}\nu$, and express the perturbation equations in terms of those two new amplitudes. Following Ref. [8], we obtain the following perturbation equations for the case $l \geq 1$:

$$\begin{aligned} &\mathrm{d}\Big\{r^3\Big[-\tilde{\triangle}\Psi + \Big(\frac{\lambda}{r^2} + r\tilde{\triangle}\Big(\frac{1}{r}\Big)\Big)\Psi - \frac{\sqrt{4\lambda\mathcal{H}_p}q}{r^3}\Phi\Big]\Big\} = 0 , \\ &-\tilde{\triangle}\Phi - \frac{\sqrt{4\lambda\mathcal{H}_p}q}{r^3}\Psi + \Big[\frac{l(l+1)}{r^2} + \mathcal{H}_P^{1/2}\tilde{\triangle}\mathcal{H}_P^{-1/2} \\ &\quad + \frac{4q^2}{r^4}\mathcal{H}_P\Big]\Phi = 0 , \end{aligned} \tag{35}$$

where $\tilde{\triangle} \equiv -\tilde{\mathrm{d}}^\dagger \mathrm{d} = \tilde{\nabla}^a\tilde{\nabla}_a$ is the Laplace operator with respect to metric $\tilde{g}$. The perturbation for the case $l = 1$ will produce a slowly rotating black hole. These two equations can be transformed to the form of a coupled system of wave equations with symmetric potential and then the stability of this regular black hole with cosmological constant can be directly deduced. Note that these two equations (35) can be cast into

$$(D + S)\begin{pmatrix}\Psi \\ \Phi\end{pmatrix} = 0 , \tag{36}$$

where

$$D = \begin{pmatrix} -r\tilde{\nabla}^a\frac{1}{r^2}\tilde{\nabla}_a r & 0 \\ 0 & -\mathcal{H}_p^{1/2}\tilde{\nabla}^a\mathcal{H}_p^{-1}\tilde{\nabla}_a\mathcal{H}_p^{1/2} \end{pmatrix} ,$$

$$S = \frac{1}{r^2}\begin{pmatrix} \lambda & -\frac{\sqrt{4\lambda\mathcal{H}_p}q}{r} \\ -\frac{\sqrt{4\lambda\mathcal{H}_p}q}{r} & l(l+1) + \frac{4q^2}{r^2}\mathcal{H}_p \end{pmatrix} .$$

4 Even-Parity Sector

Perturbation problem is more complicated in the even-parity sector than in the odd-parity case due to more unspecified coefficients involved. Luckily, the gauge-invariant approach is still effective.

Let us first recall the usual spherical harmonics tensor of even parity $(-1)^l$: Y_{lm} (scalar), $\nabla_A \mathrm{Y}_{lm}$, dY_{lm}, $\mathrm{Y}_{lm}\mathrm{d}r$ (vector, 1-form), and $\nabla_A\nabla_B \mathrm{Y}_{lm}$, $g_{AB}\mathrm{Y}_{lm}$ (0, 2)-type tensor. The perturbations of gravitational field and electromagnetic field can be expanded in this case as

$$\begin{aligned} &\delta g_{ab} = H_{ab}\mathrm{Y}_{lm} = \frac{1}{2}g_{ab}H\mathrm{Y}_{lm} + H^0_{ab}\mathrm{Y}_{lm} , \\ &\delta g_{aB} = Q_a\nabla_B\mathrm{Y}_{lm} , \\ &\delta g_{AB} = Kr^2 g_{AB}\mathrm{Y}_{lm} + Gr^2\Big(\nabla_A\nabla_B + \frac{1}{2}l(l+1)g_{AB}\Big)\mathrm{Y}_{lm} , \\ &\delta A_\mu \mathrm{d}x^\mu = \alpha\mathrm{Y}_{lm} + \mu\mathrm{dY}_{lm} + \mathrm{Y}_{lm}\mathrm{d}\mu , \\ &\delta F_{\mu\nu} = \partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu = [\,\mathrm{Y}_{lm}\mathrm{d}\alpha + \mathrm{dY}_{lm}\wedge\alpha]_{\mu\nu} , \end{aligned} \tag{37}$$

where H_{ab} is a symmetric tensor field with its trace part H and trace-free part H^0_{ab}, $Q = Q_b\mathrm{d}x^b$, $\alpha = \alpha_b\mathrm{d}x^b$ are two one-forms and K, G, μ are three scalar fields which are all defined on the orbits space $\tilde{M}$. Notice that here we have made a further decomposition for δg_{ab}, δg_{AB} (i.e. trace part + trace-free part), which is useful to the analysis below.

The related vector field X to generate the gauge transformation with even parity has the form[12]

$$X^a = \xi^a\mathrm{Y}_{lm} , \quad X^A = fg^{AB}\nabla_B\mathrm{Y}_{lm} , \tag{38}$$

where ξ^a is a vector field and f a function, both belonging to $\tilde{M}$. Correspondingly, the transformations of the perturbation coefficients behave as

$$\begin{aligned} &H_{ab} \mapsto H_{ab} + 2\nabla_{(a}\xi_{b)} , \quad Q_b \mapsto Q_b + \xi_b + r^2\nabla_b f , \\ &K \mapsto K + 2\frac{\xi_a\nabla^a r}{r} - l(l+1)f , \\ &G \mapsto G + 2f , \quad \alpha \mapsto \alpha + \mathcal{H}_P\frac{q}{r^2}\tilde{*}(\xi_a\mathrm{d}x^a) . \end{aligned} \tag{39}$$

Now by observing the above formulas, it is convenient to introduce a vector field $p_a = Q_a - (r^2\nabla_a G)/2$, which changes as $p_a \mapsto p_a + \xi_a$. The gauge invariants for even parity can easily be constructed as[14]

$$\begin{aligned} &\bar{H}_{ab} = H_{ab} - 2\nabla_{(a}p_{b)} , \\ &\bar{K} = K - 2\frac{p^a\nabla_a r}{r} + \frac{1}{2}l(l+1)G , \\ &\bar{\alpha} = \alpha - \mathcal{H}_P\frac{q}{r^2}\tilde{*}(p_a\mathrm{d}x^a) . \end{aligned} \tag{40}$$

As the case of the odd-parity sector, we can fix a gauge (RW gauge for even parity) to make $p_a = 0$, $Q_b = 0$, $G = 0$, and the simplified perturbation amplitudes become

$$\begin{aligned} &\delta g_{ab} = H_{ab}\mathrm{Y}_{lm} = \frac{1}{2}g_{ab}H\mathrm{Y}_{lm} + H^0_{ab}\mathrm{Y}_{lm} , \\ &\delta g_{Ab} = 0 , \quad \delta g_{AB} = Kr^2 g_{AB}\mathrm{Y}_{lm} . \end{aligned} \tag{41}$$

Next we can obtain the components of linearized Einstein tensor by the above perturbations,

$$\begin{aligned} &g^{ab}\delta G_{ab} = T\,\mathrm{Y}_{lm} , \quad \delta G^0_{ab}\nabla^a r\mathrm{d}x^b = V\,\mathrm{Y}_{lm} , \\ &\delta G_{Ab}\mathrm{d}x^b = \frac{1}{2}U\nabla_A\mathrm{Y}_{lm} , \\ &\delta G^0_{AB} = S\Big(\nabla_A\nabla_B + \frac{1}{2}l(l+1)g_{AB}\Big)\mathrm{Y}_{lm} , \end{aligned} \tag{42}$$

where two scalars S, T and two 1-forms U, V can be expressed by H^0_{ab}, H, K, and $N(= 1 - 2\mathbf{m}(r)/r)$. Since these functions are not explicitly related to the cosmological constant, we conclude that the concrete forms of them are the same as those in the case without the cosmological constant (see Eqs. (35) ∼ (38) in Ref. [8]). The cosmological constant appears in the Einstein-NED equations (for the corresponding equations without the cosmological constant see Eqs. (39) ∼ (41) in Ref. [8]),

$$T = -4\mathcal{H}_P\frac{q}{r^2}\pi - H\Lambda , \quad \pi \equiv -\frac{\mathcal{H}_P^{-1}}{\kappa}\tilde{*}\mathrm{d}\alpha + \frac{\kappa-1}{2\kappa}\frac{q}{r^2}H , \quad U = 4\mathcal{H}_P\frac{q}{r^2}\mathrm{d}\phi - 2\Lambda Q_b\mathrm{d}x^b ,$$

$$\tilde{*}\mathrm{d}\phi \equiv -\mathcal{H}_P^{-1}\alpha\,, \quad V = \Big(\mathcal{H} - \frac{1}{2}\Lambda\Big) r\,\mathrm{d}K\,. \tag{43}$$

From the fixed gauge $Q_b = 0$ and the background equation of G_{AB}, as well as the simplified form of δg_{AB}, we find that δG_{AB} is proportional to g_{AB}, i.e.

$$\delta G_{AB} = \Big(r\tilde{\triangle}r - \frac{1}{2}r^2\tilde{R}\Big)\delta g_{AB} = K\Big(r\tilde{\triangle}r - \frac{1}{2}r^2\tilde{R}\Big) g_{AB}\,\mathrm{Y}_{lm}\,. \tag{44}$$

Since the trace-free part of δG_{AB} vanishes, one has $S = 0$. By some straightforward calculations we get $S = -H/2$.[8] Thus we then have $H = 0$.

Following Ref. [8], after a tedious calculation, we get the perturbation equation of gravitational field,

$$2r\tilde{g}(\mathrm{d}\varsigma, \mathrm{d}r) + l(l+1)\varsigma + \Big[r\lambda + 6m + 2r^3\Big(\mathcal{H} - \frac{1}{2}\Lambda\Big)\Big]K - 8N\mathcal{H}_P\frac{q}{r}\phi = 0\,, \tag{45}$$

where K can be eliminated by

$$\tilde{\mathrm{d}}^\dagger Z - \frac{2}{r}\tilde{g}(Z, \mathrm{d}r) - \Big(\frac{\lambda}{r} + 4\mathcal{H}_P\frac{q^2}{r^3}\Big)K + 4\Big[-\tilde{\mathrm{d}}^\dagger\Big(\mathcal{H}_P\frac{q}{r^2}\mathrm{d}r\Big) + 2N\mathcal{H}_P\frac{q}{r^3} - \frac{l(l+1)}{r^2}\frac{q}{r}\mathcal{H}_P\Big]\phi = 0\,, \tag{46}$$

and $Z = C - r\,\mathrm{d}K + 4\mathcal{H}_p\phi(q/r^2)\mathrm{d}r = \mathrm{d}\varsigma$. On the other hand, we have the linearized equation for the electromagnetic field,

$$\mathcal{H}_P^{-1}\tilde{\mathrm{d}}^\dagger(\mathcal{H}_P\,\mathrm{d}\phi) + \kappa\frac{l(l+1)}{r^2}\phi + \frac{q}{2r^2}(2\kappa K - H) = 0\,, \tag{47}$$

where $\tilde{*}\mathrm{d}\phi = -\mathcal{H}_P^{-1}\alpha$ as mentioned above. By introducing the following two functions ($a \equiv 6\mathbf{m}(r)/r + 2r^2(\mathcal{H} - \Lambda/2)$):

$$\Psi = \sqrt{\lambda}\frac{\varsigma}{a+\lambda}\,, \quad \Phi = \sqrt{4\mathcal{H}_P}\Big(\phi - \frac{q}{r}\frac{\varsigma}{a+\lambda}\Big)\,, \tag{48}$$

we can cast the perturbation equations of gravitational field and electromagnetic field into

$$(D+S)\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = 0\,, \tag{49}$$

where

$$D = \begin{pmatrix} -\frac{1}{r}\tilde{\nabla}^a r^2\tilde{\nabla}_a\frac{1}{r} & 0 \\ 0 & -\mathcal{H}_P^{-1/2}\tilde{\nabla}^a\mathcal{H}_P\tilde{\nabla}_a\mathcal{H}_P^{-1/2}\end{pmatrix}, \quad S = \begin{pmatrix} \frac{\lambda}{r^2(a+\lambda)}[c_1 + \frac{2Nb}{a+\lambda}] & -\frac{\sqrt{4\lambda\mathcal{H}_P}q}{r^3(a+\lambda)}[\omega + \frac{2Nb}{a+\lambda}] \\ -\frac{\sqrt{4\lambda\mathcal{H}_P}q}{r^3(a+\lambda)}[\omega + \frac{2Nb}{a+\lambda}] & \kappa\frac{l(l+1)}{r^2} + \frac{4\mathcal{H}_P q^2}{r^4(a+\lambda)}[c_2 + \frac{2Nb}{a+\lambda}]\end{pmatrix}, \tag{50}$$

with

$$b = \lambda + 4\mathcal{H}_P\frac{q^2}{r^2}\,, \quad c_1 = \lambda + 1 - N - 2r^2\mathcal{H} + \Lambda r^2\,, \quad c_2 = c_1 + 4N\kappa\,, \quad \omega = c_1 + 2N\kappa\,. \tag{51}$$

Before getting into the main issue to discuss stability of the regular black hole, we would like to recall some general concepts and methods involved in the stability analysis. The usual approach to analyze the linear stability problem is to look for solutions growing exponentially in time. For example, consider a hyperbolic equation with "mode solution", $\partial^2\Psi/\partial t^2 = \triangle\Psi - V\Psi$, $\Psi(t,x) = \Phi(x)\,\mathrm{e}^{kt}$, then we have the eigenvalue equation $D\Phi = -k^2\Phi$, $D = -\triangle + V$. If k is real and positive, we call the system is unstable. Further, we introduce the concept of stable structure which is the analog of causal structure in general relativity. If the system has no exponentially growing solution, we call it 1-stable or stable (conversely if that kind of solution exists, called unstable); if the system has no linearly growing solution, it is 2-stable or linearly stable (conversely if it has no exponentially growing solution but has linearly growing solution, called linearly instable); if all solutions with acceptable initial data remain bounded with time, it is 3-stable or bounded stable; while if all solutions are bounded and some solutions are still stable after a certain perturbation (e.g. change some coefficients by adding a constant or give a small perturbation on initial data etc.), it is 4-stable or strong stable. One tool of analyzing stability problem is the spectral theory (see for example Ref. [20]). Here for our purpose we just introduce a theorem.

Theorem For a wave-like equation with symmetric potential of the form

$$(-P\tilde{\nabla}^a P^{-2}\tilde{\nabla}_a P + S)u = 0\,, \ \text{ or } \ (\partial_t^2 - P\partial_{r_*}P^{-2}\partial_{r_*}P + NS)\begin{pmatrix}\Psi\\ \Phi\end{pmatrix} = 0\,, \tag{52}$$

where the second equation above is under Schwarzschild coordinates and $\partial_{r_*} = N\partial_r$ denotes the derivative with respect to the tortoise coordinate and P is a positive-definite symmetric matrix and S is a symmetric matrix. If the potential matrix S is strictly positive for $\forall r > r_h$, where r_h is the horizon radius of the black hole, while initial data is smooth and has compact support, then the system is bounded stable (i.e. $\exists$ bound $C < \infty$ such that $|u(t, r_*)| \le C$ for all $t \ge 0$ and $r_* \in R$). If S is just nonnegative when $r > r_h$, the system is linearly instable (i.e. exists linearly growing perturbation).

Now let us return to our problem. It is not difficult to see that equations (36) and (49) just belong to this type of equations described in the theorem. So we can obtain the condition from the theorem for the bounded stability of regular black holes with Λ term. At first let us consider the odd-parity case (36). It is clear that the matrix S is positive-definite, therefore the regular black hole is stable under the odd-parity perturbations even when a cosmological constant is present. In the even-parity case, the stability of solution requires the matrix S to be strictly positive. From Eq. (50), we see that the positive-definiteness of the diagonal elements of the matrix requires

$$c_1 + \frac{2Nb}{a+\lambda} > 0\,, \quad \kappa\frac{l(l+1)}{r^2} + \frac{4\mathcal{H}_p q^2}{r^4(a+\lambda)}\Big(c_2 + \frac{2Nb}{a+\lambda}\Big) > 0\,, \tag{53}$$

and $\det S > 0$ yields

$$\kappa l(l+1)[(a+\lambda)c_1 + 2\lambda N] + 8N\kappa\mathcal{H}_P\frac{q}{r^2}[l(l+1) - 2N\kappa] > 0\,. \tag{54}$$

We can see that due to the appearance of the cosmological constant, the stability conditions turn out to be more complicated than the case without the cosmological constant.

(i) The case of a negative cosmological constant, namely the solution is asymptotically anti-de Sitter. In this case, $N > 0$ outside the black hole horizon. If $c_1 > 0$, $\mathcal{H} < 0$, $\mathcal{H}_p > 0$ and $\kappa > 0$ outside the black hole horizon, the condition (53) can be naturally satisfied, while the condition (54) certainly be obeyed if $0 < 2N\kappa \le l(l+1)$. Note that $a > 0$ for $r > r_h$ always holds. Its proof is as follows.

Proof Recalling $N = 1 - (2\mathbf{m}(r)/r)$, $\mathbf{m}(r_h) = r_h/2$, we have

$$\partial_r N = \frac{2\mathbf{m} - 2\mathbf{m}'r}{r^2} = \frac{2\mathbf{m} + 2r^3(\mathcal{H} - \Lambda/2)}{r^2}\,, \quad 0 \le r\tilde{\triangle}r|_{r=r_h} = r_h\partial_r N(r_h) = 1 + 2r_h^2\Big(\mathcal{H} - \frac{\Lambda}{2}\Big)\,.$$

In addition, we have

$$a(r_h) = \frac{6\mathbf{m}(r_h)}{r_h} + 2r_h^2\Big(\mathcal{H} - \frac{\Lambda}{2}\Big) = 3 + 2r_h^2\Big(\mathcal{H} - \frac{\Lambda}{2}\Big) = 2 + 1 + 2r_h^2\Big(\mathcal{H} - \frac{\Lambda}{2}\Big) \ge 2\,.$$

On the other hand, we can show

$$\partial_r(ra) = \partial_r\Big[6\mathbf{m} + 2r^3\Big(\mathcal{H} - \frac{\Lambda}{2}\Big)\Big] = -6r^2\Big(\mathcal{H} - \frac{\Lambda}{2}\Big) + 6r^2\Big(\mathcal{H} - \frac{\Lambda}{2}\Big) + 2r^3\mathcal{H}_P P_r = 4\frac{\mathcal{H}_P q^2}{r^2} > 0\,.$$

This finishes the proof of the required result.

(ii) In the case of a positive cosmological constant. The solution is asymptotically de Sitter. In that case, a cosmological horizon r_c will appear; outside the cosmological horizon, $N < 0$, while $N > 0$ between the black hole horizon r_h and cosmological horizon r_c. We see from Eq. (51) that within the same condition above, $c_1 > 0$ is easier to be satisfied than the case with a negative cosmological constant between the black hole horizon and cosmological horizon. Because of the complex of the expressions of Eqs. (53) and (54), it seems difficult to further derive some qualitative conclusion on the stability of the regular black holes (19). But anyway we have obtained the sufficient condition (53) and (54) of stability for the regular black hole solution (19).

5 Thermodynamics of Regular Black Holes

An important point of the black hole is that it has thermodynamic properties like an ordinary thermodynamic system. In Einstein gravity theory, a black hole has entropy proportional to its horizon area and Hawking temperature proportional to its surface gravity on the horizon, and in particular, these thermodynamic quantities obey the first law of thermodynamics.[21] In this section, we study thermodynamic properties of regular black hole (19) by calculating the mass, electric charge, electric potential, temperature and entropy of the black hole. Then we check whether the first law of black hole thermodynamics still holds or not for the regular black holes.

Using (20), the black hole charge Q_{bh} can be calculated by the following formula on a large radius surface,

$$Q_{bh} = \frac{1}{4\pi}\int_{r\to\infty} F_{\rm tr} r^2 \sin\theta\, {\rm d}\theta {\rm d}\varphi = q\,. \tag{55}$$

On the other hand, since the solution (19) asymptotically approaches a Reissner–Nordström-dS/AdS solution, considering the dS/AdS space as the background, we find that the parameter m is just the black hole mass M, i.e. $M = m$. Note that the black hole horizon is determined by the equation $N(r)|_{r=r_h} = 0$. The black hole mass therefore can be expressed as a function of the horizon radius r_h and charge q,

$$M = \frac{(r_h^2 + q^2)^{3/2}}{2}\Big(\frac{1}{r_h^2} - \frac{\Lambda}{3} + \frac{q^2}{(r_h^2 + q^2)^2}\Big)\,. \tag{56}$$

According to electrodynamics, the electric potential can be calculated in the usual way

$$\begin{aligned}\phi &= \int_r^\infty E\,{\rm d}r \\ &= q\int_r^\infty \Big[\frac{15}{2}\frac{my^4}{(y^2+q^2)^{7/2}} + \frac{y^4(y^2 - 5q^2)}{(y^2+q^2)^4}\Big]{\rm d}y\,. \end{aligned}\tag{57}$$

Note that the integrand above can be expressed as $\partial_y[(3/2)my^5/q^2(y^2+q^2)^{5/2} - y^5/(y^2+q^2)^3]$. Thus one arrives at

$$\phi = \frac{3m}{2q} - \frac{3mr^5}{2q(r^2+q^2)^{5/2}} + \frac{qr^5}{(r^2+q^2)^3}\,. \tag{58}$$

The Hawking temperature on the event horizon for any stationary black holes is always given by the formula $T = \kappa/2\pi$, where κ is the surface gravity on the horizon. For the regular black hole (19), we get

$$\begin{aligned}T &= \frac{1}{4\pi}(-g_{tt})'|_{r=r_h} \\ &= \frac{1}{2\pi}\Big(\frac{q^2 r_h(q^2 - r_h^2)}{(r_h^2+q^2)^3} - \frac{mr_h(2q^2 - r_h^2)}{(r_h^2+q^2)^{5/2}} - \frac{\Lambda r_h}{3}\Big)\,. \end{aligned}\tag{59}$$

Entropy of this black hole still obeys the so-called horizon area formula, $S = A/4 = \pi r_h^2$. It is quite remarkable to note that these thermodynamic quantities do not satisfy the first law of charged static black hole thermodynamics, $\mathrm{d}M = T\mathrm{d}S + \phi\mathrm{d}q$. And $\mathrm{d}M - T\mathrm{d}S - \phi\mathrm{d}q$ also cannot be expressed as a differential form. These conclusions remain valid even when the cosmological constant is absent. Furthermore, we find that the first law of black hole thermodynamics also does not hold for all regular black holes found in Refs. [1] $\sim$ [5]. However, it is interesting to note that the first law of black hole thermodynamics has been shown to hold for any NED theory.[22] The question naturally arises why the first law of black hole thermodynamics does not hold for these regular black holes in the NED theory. In the usual black hole solutions, the black hole mass and charge are two integration constants of equations of motion derived from a certain Lagrangian, the mass and charge are two parameters labelling the family of solution of the theory under consideration. However, for the so-called regular black hole solutions like Eq. (19), the mass m and charge q appear as two parameters in the Lagrangian like Eq. (17). Thus different m and q represent different black hole solutions in different theories. This leads to the invalidness of the first law of black hole thermodynamics, which relates different integration constants in the same family of solutions.

6 Summary

In this paper, an exact solution of regular black hole with cosmological constant in general relativity coupled to a kind of NED sources has been presented, which asymptotically behaves as an RN-dS/AdS black hole in the large radius limit, and as a dS/AdS space near the origin. We have studied linear stability of the solution. For the odd-parity perturbations, we have found that the cosmological constant does not appear in the master equations, which shows that the regular black hole is stable for the odd-parity perturbations. For the even-parity, we have also obtained sufficient conditions for stability of the regular black holes. In this case, the master equations are more complicated than the case without the cosmological constant.

We have also discussed thermodynamic properties of the regular black hole. The mass, electric charge, Hawking temperature and entropy have been calculated. In particular, we have checked out that the differential form of first law of black hole thermodynamics does not hold for so-called regular black holes. The reason has been pointed out. The invalidness of first law is due to the appearance of mass and electric charge of black hole solution in the Lagrangian of NED.

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