

Vacuum Black Hole Mass Formula Is a Vanishing Noether Charge*

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Abstract *The Noether current and its variation relation with respect to diffeomorphism invariance of gravitational theories have been derived from the horizontal variation and vertical-horizontal bi-variation of the Lagrangian, respectively. For Einstein's GR in the stationary, axisymmetric black holes, the mass formula in vacuum can be derived from this Noether current although it definitely vanishes. This indicates that the mass formula of black holes is a vanishing Noether charge in this case. The first law of black hole thermodynamics can also be derived from the variation relation of this vanishing Noether current.*

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1 Introduction

It is well known that Noether's theorem links the conservation law of certain quantity with certain symmetry. From 1970's, one of the most exciting discoveries in GR is the black hole mechanics that relates among others the geometric properties of black holes with the thermodynamic laws. In thermodynamics, the first law exhibits the relation among the changes of energy, entropy and other macroscopical quantities. On the other hand, in either classical mechanics or field theories the energy conservation law is directly related with the time translation invariance. Furthermore, its covariant form is closely related with the re-parametrization invariance of spacetime coordinates, i.e. the diffeomorphism invariance of the theory. Therefore, it is significant to explore whether the diffeomorphism invariance of the gravitational theories should be behind the mass formula and its differential one, i.e. the first law of black hole thermodynamics.

During last decade, Wald and his collaborators as well as other authors (see, for example, Refs. [1]–[5]) have studied this problem and found certain link between the first law of black hole thermodynamics and the diffeomorphism invariance of the gravitational theories. They have also claimed that the black hole entropy is the Noether charge.^[2–4] But, some problems are still open even for the classical black hole thermodynamics such as how to relate the formula for mass itself with the conservation current with respect to the diffeomorphism invariance, in what sense the black hole entropy is a Noether charge and so on so forth.

In this paper we explore these problems by the variational approach based on Wald's approach. We mainly work out the relation between diffeomorphism invariance and the mass formula as well as the first law of black hole thermodynamics in a manifest way. We first sketch the derivation for the Noether current and its variation relation with respect to the diffeomorphism invariance of the Lagrangian 4-form by taking horizontal variation and vertical-horizontal bi-variation of the Lagrangian 4-form of the gravitational theories on $(\mathcal{M}^{1,3}, g)$, respectively. For the vacuum gravitational fields in GR, we show that this current vanishes definitely. However, for the vacuum stationary, axisymmetric black holes, this vanishing Noether charge of the vanishing current leads to the mass formula for the black holes. This indicates that the entropy of the black hole itself cannot be regarded as an entire Noether charge at least for the vacuum cases instead the mass formula, as a whole, should be regarded as the vanishing Noether charge in a certain sense. Further, we derive the first law of black hole thermodynamics^[7] from the variation relation of the Noether current. Finally, we end with some concluding remarks.

2 Diffeomorphism Invariance and Its Noether Currents

Let L be a diffeomorphism-invariant Lagrangian 4-form for gravitational field metric theory on the spacetime manifold $(\mathcal{M}^{1,3}, g)$ with metric g of signature -2 . This means that if f is a diffeomorphism map on $\mathcal{M}^{1,3}$, the Lagrangian 4-form satisfies

$$f^*(L(g_{ab})) = L(f^*(g_{ab})), \quad (1)$$

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where f^* is the induced map by f .

The horizontal variation of \mathbf{L} with respect to the diffeomorphism is defined as

$$\hat{\delta}\mathbf{L} = \frac{d}{d\lambda}\mathbf{L}(f_\lambda^*g_{ab})\Big|_{\lambda=0} = \mathcal{L}_\xi\mathbf{L}(g_{ab}), \quad (2)$$

where f_λ^* denotes the induced map of a one-parameter diffeomorphism group f_λ generated by a vector field ξ^a on $\mathcal{M}^{1,3}$, \mathcal{L}_ξ is the Lie derivative with respect to ξ .

It is easy to show the following formal equation (see, for example, Ref. [1])

$$\hat{\delta}\mathbf{L} = \mathbf{E}\hat{\delta}g + d\Theta(\hat{\delta}g), \quad (3)$$

where $\mathbf{E} = 0$ gives rise to the Euler–Lagrange equation for the gravitational field theory and Θ the symplectic potential. In addition, from the diffeomorphism invariance of the Lagrangian it follows

$$\hat{\delta}\mathbf{L} = \mathcal{L}_\xi\mathbf{L} = d(\xi \cdot \mathbf{L}). \quad (4)$$

Combination of Eqs. (3) and (4) directly leads to a conservation equation

$$d * \mathbf{j} + \mathbf{E}\hat{\delta}g = 0, \quad (5)$$

where \mathbf{j} is the Noether current with respect to the diffeomorphism invariance

$$\mathbf{j} := *(\Theta(\hat{\delta}g) - \xi \cdot \mathbf{L}), \quad \text{mod } (*d\alpha). \quad (6)$$

It is conserved if and only if the field equation $\mathbf{E} = 0$ holds. This current (6) derived from the horizontal variation of the Lagrangian 4-form with respect to the diffeomorphism invariance is in the same form as the one defined by Wald *et al.* Note that if ξ is a Killing vector, equation (4) vanishes and the Noether current becomes

$$\mathbf{j} := *\Theta(\hat{\delta}g), \quad \text{mod } (*d\beta). \quad (7)$$

In order to derive the differential formula of mass, i.e. the first law of the black hole thermodynamics, it is needed the variation relation of this Noether current. To this end, it is natural to calculate the following bi-variation with respect to vertical and horizontal variation δ and $\hat{\delta}$ of \mathbf{L} ,

$$\delta\hat{\delta}\mathbf{L} = \delta(\mathbf{E}\hat{\delta}g + d\Theta(\hat{\delta}g)) = \delta d(\xi \cdot \mathbf{L}). \quad (8)$$

Therefore, from Eqs. (6) and (8), it follows the variation relation of this Noether current

$$0 = \delta(\mathbf{E}\hat{\delta}g) + d\delta * \mathbf{j}, \quad (9)$$

where we have used the commutative property between the vertical variation operator and the differential operator, i.e. $\delta d = d\delta$. Thus, for the conservation of variation of the Noether current (6), i.e.

$$d\delta * \mathbf{j} = 0, \quad (10)$$

the necessary and sufficient condition is

$$\delta(\mathbf{E}\hat{\delta}g) = 0. \quad (11)$$

3 Diffeomorphism in General Relativity

3.1 Black Hole Mass Formula Is a Noether Charge

Let us now consider the vacuum gravitational fields in GR. The Hilbert–Einstein action in natural units on $(\mathcal{M}^{1,3}, g)$ reads

$$\mathbf{L} = \frac{1}{16\pi}R\epsilon, \quad (12)$$

where R is the scalar curvature and ϵ the volume element determined by g_{ab} .

According to Eq. (3), the horizontal variation of the Hilbert–Einstein action induced by f_λ can be calculated

$$\hat{\delta}\mathbf{L} = \frac{1}{16\pi}[G_{ab}\mathcal{L}_\xi g^{ab} + \nabla_a(\Theta^a(\hat{\delta}g))]\epsilon, \quad (13)$$

where G_{ab} is the Einstein tensor and

$$\Theta^a(\hat{\delta}g) = \frac{1}{16\pi}[2\nabla^a\nabla_b\xi^b - \nabla_b\nabla^a\xi^b - \nabla_b\nabla^b\xi^a].$$

On the other hand, the diffeomorphism invariance of the Lagrangian directly leads to

$$\hat{\delta}\mathbf{L} = \frac{1}{16\pi}\mathcal{L}_\xi(R\epsilon) = \frac{1}{16\pi}\nabla_c(R\xi^c)\epsilon. \quad (14)$$

From the above two equations, it follows the covariant conservation law for the Noether current Eq. (6) with respect to the diffeomorphism invariance for the vacuum gravitational fields in GR

$$0 = \frac{1}{16\pi}G_{ab}\hat{\delta}g^{ab} + \nabla^a j_a, \quad (15)$$

where the Noether current is given by

$$\mathbf{j}_a = \frac{1}{8\pi}G_{ab}\xi^b + \frac{1}{16\pi}[\nabla^b\nabla_a\xi_b - \nabla^b\nabla_b\xi_a]. \quad (16)$$

In this paper, we focus on the case of stationary, axisymmetric black holes and with ξ being the Killing vector

$$\xi^a = t^a + \Omega_H\phi^a, \quad (17)$$

where t^a and ϕ^a is the time-like and space-like Killing vector of the space-time, respectively. Since ξ is a Killing vector, the corresponding conserved current \mathbf{j} is given by Eq. (7) or Eq. (16). The current (16) becomes

$$\mathbf{j}_a = \frac{1}{8\pi}R_{ab}\xi^b + \frac{1}{8\pi}G_{ab}\xi^b, \quad (18)$$

where the following equation has been used

$$R_{eb}\xi^b = \nabla^b\nabla_{[e}\xi_{b]} = -\frac{1}{4}\epsilon_{efab}\nabla^f\epsilon^{abcd}\nabla_c\xi_d. \quad (19)$$

It is now very obvious but important to note that this Noether current and consequently its charge on an entire Cauchy surface Σ ,

$$Q := \int_\Sigma * \mathbf{j} \quad (20)$$

definitely vanish for the vacuum gravitational fields with the Killing vector (17) due to the Einstein equation. It

is significant to emphasize, however, that although this Noether charge vanishes, it still leads to the mass formula.

Let us now show how this vanishing Noether charge leads to the mass formula. From the definition (20), this vanishing Noether charge is given by

$$0 \equiv Q = \frac{1}{8\pi} \int_{\Sigma} [\nabla^b \nabla_{[a} \xi_{b]} + G_{ab} \xi^b] d\sigma^a, \quad (21)$$

where $d\sigma^a$ is the surface element on Σ . The first term is a total divergence, whose integral may reduce to the surface one on the boundary of Σ . The second integral vanishes due to the Einstein equation. Note that the entire Cauchy surface Σ has two boundaries. One is the spatial infinity S_{∞} and the other is the bifurcate surface $S_H^{(-)}$ of the event horizon, where the upper index $(-)$ denotes the opposite orientation. Thus, the vanishing Noether charge is composed of two parts \mathcal{Q}_H and \mathcal{Q}_{∞} at $S_H^{(-)}$ and S_{∞} , respectively. Namely,

$$Q := \mathcal{Q}_{\infty} - \mathcal{Q}_H = -\frac{1}{16\pi} \int_{S_{\infty}} *d\xi + \frac{1}{16\pi} \int_{S_H} *d\xi. \quad (22)$$

By definition, the Komar mass and the angular momentum are given by

$$M := -\frac{1}{8\pi} \int_{S_{\infty}} *d\xi, \quad J := \frac{1}{16\pi} \int_{S_{\infty}} *d\xi, \quad (23)$$

respectively. For the case of Killing vector, the Komar mass M is the same as the ADM mass of the black hole configurations. Thus it follows the expression for \mathcal{Q}_{∞}

$$\mathcal{Q}_{\infty} := -\frac{1}{16\pi} \int_{S_{\infty}} *d\xi = \frac{1}{2} M - \Omega_H J. \quad (24)$$

On the other hand, for the Killing vector (17),

$$\nabla^a \xi^b = \kappa \epsilon^{ab} \quad (25)$$

on S_H , where κ is the surface gravity and ϵ^{ab} the bi-normal to S_H . Thus it is easy to get the expression for \mathcal{Q}_H

$$\mathcal{Q}_H := -\frac{1}{16\pi} \int_{S_H} *d\xi = \frac{\kappa}{8\pi} A, \quad (26)$$

where A is the area of the bifurcate surface of the event horizon. It is \mathcal{Q}_H that is called as the Noether charge in Ref. [2].

Therefore, it has been shown that the vanishing Noether charge, as a whole, with respect to the diffeomorphism invariance leads to the mass formula for the vacuum gravitational fields of stationary, axisymmetric black holes in GR^[6,7]

$$Q = \frac{1}{2} \left(M - 2\Omega_H J - \frac{\kappa}{4\pi} A \right) = 0. \quad (27)$$

3.2 The Differential Mass Formula

Let us now derive, from the variation relation of this vanishing Noether charge, the differential formula for mass, i.e. the first law of black hole thermodynamics, for

the vacuum stationary, axisymmetric black holes in GR by the variational approach.

For the vacuum gravitational fields in GR, the variation relation of this vanishing Noether current Eq. (9) becomes

$$0 = \frac{1}{16\pi} \delta(G_{ab} \hat{\delta} g^{ab}) + \delta[\nabla^a \mathbf{j}_a], \quad (28)$$

where \mathbf{j}_a is given by Eq. (16). As was mentioned before, it is obvious that the conserved current \mathbf{j} vanishes for the Killing vector field ξ^a (17) if the vacuum Einstein equation holds. Further, if the variation or the perturbation δg is restricted in such a way that both g and $g + \delta g$ are stationary, axisymmetric black hole configurations and ξ^a is the Killing vector (17), the vanishing Noether current now is Eq. (18) and the variation of the conserved current should also vanish, i.e. $\delta \mathbf{j} = 0$ as well.

Thus it is straightforward to get

$$0 = \frac{1}{8\pi} \int_{\Sigma} \delta[R_{ab} \xi^b d\sigma^a] = \delta \left[M - \frac{\kappa}{4\pi} A - 2\Omega_H J \right], \quad (29)$$

and equation (11) is also satisfied. Here Σ is the Cauchy surface.

It should be noticed that equation (29) is just the start point of Bardeen, Carter and Hawking's calculation for the first law of the black hole mechanics.^[8] As was required in Ref. [8], under the above perturbations, the positions of event horizon and the two Killing vector fields in Eq. (17) are unchanged. Consequently, equation (29) definitely gives rise to the differential formula for mass, i.e., the first law of black hole thermodynamics, among the stationary, axisymmetric black hole configurations^[6,7]

$$\delta M - \frac{\kappa}{8\pi} \delta A - \Omega_H \delta J = 0. \quad (30)$$

Thus both the mass formula (27) and its differential formula (32), i.e., the first law of black hole thermodynamics for the stationary, axisymmetric black hole configurations are all derived from the diffeomorphism invariance of the Lagrangian by the variational approach. Especially, they are in fact the vanishing Noether charge and its perturbation among the stationary, axisymmetric black hole configurations.

4 Concluding Remarks

Finally, a few remarks are in order:

1) From the derivation via the variational approach in this paper, it can be seen that neither the entropy of black holes nor the total energy of the gravitational fields is the entire Noether charge of the conserved current with respect to the diffeomorphism invariance on the Cauchy surface at least for the vacuum gravitational fields. In fact, the entire mass formula (27) itself, as a whole, should be viewed as a Noether charge of the current Eq. (16) on the Cauchy surface and this charge definitely vanishes.

2) At first glance it seems to be intricate why the conserved current \mathbf{j} (18) for the stationary, axisymmetric black hole configurations vanishes. In fact, the Bianchi identity may shed light on this point from another angle of view, since the Bianchi identity may be viewed as a consequence of the diffeomorphism invariance. Using the Bianchi identity and the vacuum Einstein equation, it is easy to show

$$\nabla^a(G_{ab}\xi^b) = 0, \quad \forall \xi. \quad (31)$$

This means that the conserved current \mathbf{j} can be gotten from the Bianchi identity up to a constant factor. Therefore, at least for the stationary, axisymmetric black holes, it would have to also be shown that the mass formula and the first law of black hole thermodynamics are the consequences of the Bianchi identity and are its perturbation as well. In addition, this also indicates why the Noether charge of the conserved current \mathbf{j} should vanish.

On the other hand, it also seems to be a consequence of the equivalence between gravitational mass and inertial mass. In ordinary local field theories including the gravitational field, the charge, as an integral over a Cauchy surface of the time-like component of the conserved current with respect to the diffeomorphism invariance under some orthogonal normal tetrad, is the local energy density. But the above equivalence indicates that there should be no local energy density for the gravitational field in general relativity. Otherwise, if there were local mass for

the gravitational fields, not only this equivalence could no longer be correct but GR even Newton's theory could be reformulated to fit the observations. The vanishing Noether current and its vanishing charge with respect to the diffeomorphism invariance just reflect this fundamental property for the gravitational fields.

3) It should be noticed that the requirement on ξ^a being a Killing vector might not be necessary. In fact, for the vector ξ^a what are needed possibly are its asymptotic behavior near the horizon and that at the spatial infinity. If it is required some appropriate quasi-local horizon condition^[9] such as isolated horizon, the mass formula Eq. (27) could also be gotten. In other words, the derivation of the mass formula might be generalized into some non-stationary space-time configurations, which have been considered by other authors from different motivation.^[9]

4) In this paper what have been dealt with are the vacuum gravitational fields of 4-dimensions. In principle, all these results may be formulated for the gravitational fields with sources and of other dimensions. This topic will be published elsewhere.

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