

Anisotropic Harmonic Oscillator in a Static Electromagnetic Field*

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Abstract A nonrelativistic charged particle moving in an anisotropic harmonic oscillator potential plus a homogeneous static electromagnetic field is studied. Several configurations of the electromagnetic field are considered. The Schrödinger equation is solved analytically in most of the cases. The energy levels and wave functions are obtained explicitly. In some of the cases, the ground state obtained is not a minimum wave packet, though it is of the Gaussian type. Coherent and squeezed states and their time evolution are discussed in detail.

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1 Introduction

The anisotropic harmonic oscillator potential is of physical interest in quantum mechanics, since it may describe the motion of, say, an electron in an anisotropic metal lattice. With external electromagnetic fields imposed on, such models may also be useful in semiconductor physics.^[1,2] For an isotropic harmonic oscillator potential and a homogeneous static magnetic field the Schrödinger equation can be easily solved in the cylindrical coordinates. If the harmonic oscillator potential is anisotropic, the problem is not so easy. Nevertheless, it can be solved analytically because the Hamiltonian is quadratic in the canonical coordinates and momenta. There exist quite general formalisms for solving systems with such quadratic Hamiltonians,^[3] and the above problem represents a typical example.^[4] In Ref. [4] only the isotropic case is considered as an example for illustrating the general method. The anisotropic case with a homogeneous static magnetic field was studied in Refs. [5] and [6]. In these works the magnetic field was arranged to point to one of the rectangular axis directions. In this paper we will solve the problem in a somewhat easier formalism. A homogeneous static electric field is also included, though its effect is trivial. We will also deal with a case where the magnetic field has a more general direction. This seems not having been considered before. Coherent and squeezed states of these systems will be discussed in some detail. In terms of these concepts the time evolution of some wave packets can be discussed very conveniently.

In the next section we consider a charged harmonic oscillator with arbitrary frequencies ω_x , ω_y and ω_z in three rectangular axis directions. A homogeneous static mag-

netic field in the z direction and a homogeneous static electric field in an arbitrary direction are imposed to the system. The problem can be easily reduced to one equivalent to the case without an electric field by making some coordinate transformation. In Sec. 3 we diagonalize the reduced Hamiltonian and give the energy eigenvalues. Although this has been previously done, the method used here is different and seems more convenient, especially for the subsequent discussions of coherent and squeezed states. The wave functions are explicitly worked out in Sec. 4. In Sec. 5 we discuss some special cases, some of which may need different handlings, or even cannot be solved. In Sec. 6 we consider a somewhat different case, where $\omega_z = 0$ and $E_z = 0$ (E_z is the z component of the electric field), but the magnetic field has an x or y component, in addition to the z component. To our knowledge, this was not considered before. It is solved analytically by the formalism of Sec. 3. In Sec. 7 we discuss the coherent and squeezed states and their time evolution in detail. A brief summary is given in Sec. 8.

2 Reduction of the Hamiltonian

Consider a charged particle with charge q and mass M , moving in an anisotropic harmonic oscillator potential, a homogeneous static magnetic field $\mathbf{B} = B\mathbf{e}_z$, and a homogeneous static electric field $\mathbf{E} = E_x\mathbf{e}_x + E_y\mathbf{e}_y + E_z\mathbf{e}_z$, where \mathbf{e}_x , \mathbf{e}_y and \mathbf{e}_z are unit vectors of the rectangular coordinate system, and B , E_x , E_y and E_z are all constants. The stationary Schrödinger equation is

$$H^T\Psi(\mathbf{x}) = \mathcal{E}^T\Psi(\mathbf{x}), \quad (1)$$

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where the Hamiltonian is

$$H^T = \frac{1}{2M} \left(\mathbf{p} - \frac{q}{c} \mathbf{A} \right)^2 + \frac{1}{2} M (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) - q \mathbf{E} \cdot \mathbf{x}, \quad (2)$$

where $\mathbf{E} \cdot \mathbf{x} = E_x x + E_y y + E_z z$. Since the magnetic field points to the z direction, one may take A_x and A_y independent of z , and $A_z = 0$. Then the Hamiltonian can be decomposed as $H^T = H^{xy} + H^z$, where

$$H^{xy} = \frac{1}{2M} \left[\left(p_x - \frac{q}{c} A_x \right)^2 + \left(p_y - \frac{q}{c} A_y \right)^2 \right] + \frac{1}{2} M (\omega_x^2 x^2 + \omega_y^2 y^2) - q (E_x x + E_y y), \quad (3)$$

and

$$H^z = \frac{1}{2M} p_z^2 + \frac{1}{2} M \omega_z^2 z^2 - q E_z z. \quad (4)$$

Therefore the total wave function can be factorized as $\Psi(\mathbf{x}) = \psi(x, y) Z(z)$, where $\psi(x, y)$ and $Z(z)$ satisfy the following equations

$$H^{xy} \psi(x, y) = \mathcal{E}^{xy} \psi(x, y), \quad (5)$$

$$H^z Z(z) = \mathcal{E}^z Z(z) \quad (6)$$

with $\mathcal{E}^{xy} + \mathcal{E}^z = \mathcal{E}^T$. Obviously, H_z can be recast in the form

$$H^z = \frac{1}{2M} p_z^2 + \frac{1}{2} M \omega_z^2 (z - z_0)^2 - \frac{q^2 E_z^2}{2M \omega_z^2}, \quad (7)$$

where $z_0 = q E_z / M \omega_z^2$, so that equation (6) can be easily solved with the following results,

$$\mathcal{E}_{n_3}^z = \left(n_3 + \frac{1}{2} \right) \hbar \omega_z - \frac{q^2 E_z^2}{2M \omega_z^2}, \quad n_3 = 0, 1, 2, \dots, \quad (8)$$

$$Z_{n_3}(z) = \psi_{n_3}^{\omega_z}(z - z_0), \quad n_3 = 0, 1, 2, \dots, \quad (9)$$

where we have quoted the standard wave functions for harmonic oscillators

$$\psi_n^\omega(x) = \left(\frac{\sqrt{M\omega}}{2^n n! \sqrt{\pi \hbar}} \right)^{1/2} \exp\left(-\frac{M\omega}{2\hbar} x^2\right) H_n\left(\sqrt{\frac{M\omega}{\hbar}} x\right), \quad n = 0, 1, 2, \dots \quad (10)$$

Therefore our main task is to solve Eq. (5). We take the gauge

$$A_x = -\frac{1}{2} B \tilde{y}, \quad A_y = \frac{1}{2} B \tilde{x}, \quad (11)$$

where $\tilde{x} = x - x_0$, $\tilde{y} = y - y_0$, and $x_0 = q E_x / M \omega_x^2$, $y_0 = q E_y / M \omega_y^2$, then H^{xy} can be written as

$$H^{xy} = \frac{1}{2M} (p_x^2 + p_y^2) + \frac{1}{2} M \omega_1^2 \tilde{x}^2 + \frac{1}{2} M \omega_2^2 \tilde{y}^2 - \omega_B \tilde{L}_z - \frac{q^2 E_x^2}{2M \omega_x^2} - \frac{q^2 E_y^2}{2M \omega_y^2}, \quad (12)$$

where $\tilde{L}_z = \tilde{x} p_y - \tilde{y} p_x$, $\omega_B = qB/2Mc$, and

$$\omega_1 = \sqrt{\omega_x^2 + \omega_B^2}, \quad \omega_2 = \sqrt{\omega_y^2 + \omega_B^2}. \quad (13)$$

Note that ω_B may be either positive or negative, depending on the signs of q and B . Now equation (5) can be written as

$$H\psi(x, y) = \mathcal{E}\psi(x, y), \quad (14a)$$

where

$$H = \frac{1}{2M} (p_x^2 + p_y^2) + \frac{1}{2} M \omega_1^2 \tilde{x}^2 + \frac{1}{2} M \omega_2^2 \tilde{y}^2 - \omega_B \tilde{L}_z, \quad (14b)$$

$$\mathcal{E} = \mathcal{E}^{xy} + \frac{q^2 E_x^2}{2M \omega_x^2} + \frac{q^2 E_y^2}{2M \omega_y^2}. \quad (15)$$

Equation (14) has the same form as that for an anisotropic harmonic oscillator moving on the xy plane under the influence of a homogeneous static magnetic field perpendicular to the plane, except that x, y are replaced by \tilde{x}, \tilde{y} . The main feature is that the reduced Hamiltonian is quadratic in the canonical variables. Though this equation has been studied by some authors, we will solve it in a somewhat different and simpler way in the next two sections. Here we point out that the calculations below will be simpler if one takes the gauge $A_x = -B\tilde{y}$, $A_y = 0$ or $A_x = 0$, $A_y = B\tilde{x}$. However, we prefer the above gauge (11), since in this gauge it would be convenient to compare the results with known ones when $\omega_x = \omega_y$, or $\omega_1 = \omega_2$.

3 Diagonalization of the Reduced Hamiltonian

If $\omega_x = \omega_y$, equation (14) can be easily solved in the cylindrical coordinates. In the general case this does not work, so other methods are necessary. We define a column vector

$$X = (\tilde{x}, p_x, \tilde{y}, p_y)^\tau, \quad (16)$$

where τ denotes matrix transposition, then the reduced Hamiltonian (14b) can be written as

$$H = \frac{1}{2} X^\tau \mathcal{H} X = \frac{1}{2} X^\dagger \mathcal{H} X, \quad (17)$$

where \mathcal{H} is a c-number matrix which we take to be symmetric. We do not write down \mathcal{H} here since this is easy. If one can diagonalize \mathcal{H} then the Hamiltonian would become a sum of two one-dimensional Hamiltonians for harmonic oscillators or something similar (repulsive harmonic oscillators or free particles). However, a crucial point here is to preserve the commutation relation

$$[X_\alpha, X_\beta] = -\hbar (\Sigma_y)_{\alpha\beta}, \quad \alpha, \beta = 1, 2, 3, 4, \quad (18)$$

after the linear transformation that diagonalizes \mathcal{H} , where $\Sigma_y = \text{diag}(\sigma_y, \sigma_y)$, and σ_y is the second Pauli matrix. Therefore the needed transformations are the so-called symplectic ones.^[3,4] Here we will transform the Hamiltonian into a form expressed in terms of raising and lowering operators, so the formalism is somewhat different. Let us proceed as follows.

Because H is quadratic in the canonical variables X , it is easy to show that

$$[iH, X] = \hbar\Omega X, \quad (19)$$

where

$$\Omega = i\Sigma_y \mathcal{H} \quad (20)$$

is another c-number matrix. As both \mathcal{H} and $i\Sigma_y$ are real, so is Ω . Our first step is to diagonalize Ω , so we write down it here,

$$\Omega = \begin{pmatrix} 0 & 1/M & \omega_B & 0 \\ -M\omega_1^2 & 0 & 0 & \omega_B \\ -\omega_B & 0 & 0 & 1/M \\ 0 & -\omega_B & -M\omega_2^2 & 0 \end{pmatrix}. \quad (21)$$

The characteristic polynomial for this matrix is

$$\det(\lambda I - \Omega) = \lambda^4 + b\lambda^2 + c, \quad (22a)$$

where

$$b = \omega_x^2 + \omega_y^2 + 4\omega_B^2, \quad c = \omega_x^2\omega_y^2. \quad (22b)$$

Since

$$\Delta \equiv b^2 - 4c = (\omega_x^2 - \omega_y^2)^2 + 8\omega_B^2(\omega_x^2 + \omega_y^2 + 2\omega_B^2) \geq 0, \quad (23)$$

we have two real roots for λ^2 . If $\omega_x = 0$ or $\omega_y = 0$, one of the two roots is zero, and the following discussions are not valid. We will return to this case latter. Currently we assume that both ω_x and ω_y are nonzero, then $c > 0$ and $\sqrt{\Delta} < b$, and both roots for λ^2 are negative. Therefore the above characteristic polynomial has four pure imaginary roots

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{-i\sigma_1, i\sigma_1, -i\sigma_2, i\sigma_2\}, \quad (24)$$

where

$$\sigma_1 = \left(\frac{b + \sqrt{\Delta}}{2}\right)^{1/2}, \quad \sigma_2 = \left(\frac{b - \sqrt{\Delta}}{2}\right)^{1/2}, \quad (25)$$

and $\sigma_1 \geq \sigma_2 > 0$. The equal sign appears when $\omega_B = 0$ and $\omega_x = \omega_y$.

Because Ω is not symmetric, right eigenvectors and left ones are different. We define two left eigenvectors (row vectors) u_i corresponding to the eigenvalues $-i\sigma_i$ ($i = 1, 2$) by

$$u_i \Omega = -i\sigma_i u_i, \quad i = 1, 2, \quad (26)$$

then the other two are u_i^* , corresponding to the eigenvalues $i\sigma_i$ ($i = 1, 2$). Similarly, the right eigenvectors (column vectors) are v_i and v_i^* ($i = 1, 2$), satisfying the equations

$$\Omega v_i = -i\sigma_i v_i, \quad i = 1, 2, \quad (27)$$

and the complex conjugate ones. From these eigenvalue equations it can be shown that

$$u_i^* v_j = u_i v_j^* = 0, \quad \forall i, j = 1, 2, \quad (28a)$$

and $u_i v_j = u_i^* v_j^* = 0$ for $\sigma_i \neq \sigma_j$ (this should be checked individually when $\sigma_1 = \sigma_2$, see below). By appropriately choosing the normalization constants we have

$$u_i v_j = u_i^* v_j^* = \delta_{ij}, \quad \forall i, j = 1, 2. \quad (28b)$$

The specific expressions for the above eigenvectors are not necessary in this section. However, a relation between the left and right eigenvectors is very useful in the following. Using Eqs. (20), (26), and (27), it is easy to show that $v_i \propto \Sigma_y u_i^\dagger$, so we choose

$$v_i = -\Sigma_y u_i^\dagger, \quad i = 1, 2. \quad (29)$$

Note that the condition (28b) only determines the product of the normalization constants in u_i and v_i . With this relation both the constants are fixed (up to a phase factor). Also note that equations (28b) and (29) lead to $-u_i \Sigma_y u_i^\dagger = 1$ (no summation over i on the left-hand side). The left-hand side of this equation is real (which can be easily shown), but not necessarily positive. Actually it can be shown by using the eigenvalue equation that $-u_i \Sigma_y u_i^\dagger = (u_i \Sigma_y) \mathcal{H} (u_i \Sigma_y)^\dagger / \sigma_i$. Thus the sign of this quantity depends on the matrix \mathcal{H} , and in general equation (29) should be replaced by $v_i = \epsilon_i \Sigma_y u_i^\dagger$, where $\epsilon_i = \pm 1$. For the present case, however, equation (29) is sound.

Now we define a 4×4 matrix Q by arranging the column vectors in the following order

$$Q = (v_1, v_1^*, v_2, v_2^*). \quad (30)$$

Using Eq. (28) it is easy to show that

$$Q^{-1} = (u_1^\tau, u_1^{*\tau}, u_2^\tau, u_2^{*\tau})^\tau. \quad (31)$$

With the help of the eigenvalue equations (26) and (27), we have

$$Q^{-1} \Omega Q = \text{diag}(-i\sigma_1, i\sigma_1, -i\sigma_2, i\sigma_2). \quad (32)$$

Therefore the matrix Ω is diagonalized. Using the relation (29), it is not difficult to show that

$$Q^\dagger = -\Sigma_z Q^{-1} \Sigma_y, \quad (33)$$

where $\Sigma_z = \text{diag}(\sigma_z, \sigma_z)$, and σ_z is the third Pauli matrix. This relation is very useful in the following.

Next we will diagonalize the Hamiltonian. With the above results this is easy. We define two operators

$$a_i = u_i X / \sqrt{\hbar}, \quad i = 1, 2. \quad (34a)$$

Their hermitian conjugates are

$$a_i^\dagger = u_i^* X / \sqrt{\hbar}, \quad i = 1, 2. \quad (34b)$$

Using Eqs. (18), (28), and (29) it is easy to show that the nonvanishing commutators among these operators are

$$[a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2. \quad (35)$$

Therefore they are similar to the raising and lowering operators for harmonic oscillators. We define a column vector

$$A = (a_1, a_1^\dagger, a_2, a_2^\dagger)^\tau, \quad (36)$$

then equation (34) can be written in the matrix form

$$A = Q^{-1}X/\sqrt{\hbar}. \quad (34')$$

The inverse is

$$X = \sqrt{\hbar}QA, \quad (37)$$

and thus

$$X^\tau = X^\dagger = \sqrt{\hbar}A^\dagger Q^\dagger. \quad (38)$$

Substituting these into Eq. (17), and using Eqs. (20), (32), and (33), it is rather easy to show that

$$H = \frac{1}{2}\hbar A^\dagger \Sigma A, \quad (39)$$

where $\Sigma = \text{diag}(\sigma_1, \sigma_1, \sigma_2, \sigma_2)$. Using the commutation relations (35) this becomes

$$H = \hbar\sigma_1\left(a_1^\dagger a_1 + \frac{1}{2}\right) + \hbar\sigma_2\left(a_2^\dagger a_2 + \frac{1}{2}\right). \quad (40)$$

Therefore the Hamiltonian is diagonalized. More specifically, it becomes the sum of two one-dimensional harmonic oscillator Hamiltonians. The energy levels are readily available

$$\begin{aligned} \mathcal{E}_{n_1 n_2} &= \hbar\sigma_1\left(n_1 + \frac{1}{2}\right) + \hbar\sigma_2\left(n_2 + \frac{1}{2}\right), \\ n_1, n_2 &= 0, 1, 2, \dots \end{aligned} \quad (41)$$

When $\omega_B = 0$, we have $\sigma_1 = \max(\omega_x, \omega_y)$ and $\sigma_2 = \min(\omega_x, \omega_y)$, and these energy levels reduce to known results, as expected. Substituting these into Eq. (15) we obtain the energy levels for the motion on the xy plane

$$\begin{aligned} \mathcal{E}_{n_1 n_2}^{xy} &= \hbar\sigma_1\left(n_1 + \frac{1}{2}\right) + \hbar\sigma_2\left(n_2 + \frac{1}{2}\right) \\ &\quad - \frac{q^2 E_x^2}{2M\omega_x^2} - \frac{q^2 E_y^2}{2M\omega_y^2}, \\ n_1, n_2 &= 0, 1, 2, \dots \end{aligned} \quad (42)$$

The total energy is the sum of $\mathcal{E}_{n_1 n_2}^{xy}$ and $\mathcal{E}_{n_3}^z$ given by Eq. (8). The wave functions on the xy plane will be worked out in the next section.

The method employed in this section is similar to that used in Ref. [7], which dealt with a charged particle moving in a rotating magnetic field (also studied in Refs. [8] and [9]). However, we present here a relation between the right and left eigenvectors, which leads to the very useful relation (33), so that the calculation here seems more straightforward and easier. The method can be easily extended to systems with more degrees of freedom.

4 The Wave Functions in the Coordinate Representation

The eigenstates of the Hamiltonian (40) corresponding to the eigenvalues (41) are

$$\begin{aligned} |n_1 n_2\rangle &= \frac{1}{\sqrt{n_1! n_2!}} (a_1^\dagger)^{n_1} (a_2^\dagger)^{n_2} |00\rangle, \\ n_1, n_2 &= 0, 1, 2, \dots, \end{aligned} \quad (43)$$

where $|00\rangle$ is the ground state satisfying

$$a_1|00\rangle = a_2|00\rangle = 0. \quad (44)$$

The task of this section is to work out these states in the coordinate representation, that is, the wave functions

$$\psi_{n_1 n_2}(x, y) = \langle xy | n_1 n_2 \rangle. \quad (45)$$

For this purpose the left eigenvectors are necessary. These are given by

$$\begin{aligned} u_i &= K_i(-iM\sigma_i(\sigma_i^2 - \omega_y^2 - 2\omega_B^2), \sigma_i^2 - \omega_y^2, \\ &\quad M\omega_B(\sigma_i^2 + \omega_y^2), i2\omega_B\sigma_i), \quad i = 1, 2, \end{aligned} \quad (46a)$$

and their complex conjugates, where the normalization constants are

$$K_i = \{2M\sigma_i[(\sigma_i^2 - \omega_y^2)^2 + 4\omega_B^2\omega_y^2]\}^{-1/2}. \quad (46b)$$

When $\omega_B = 0$, $\{\sigma_i | i = 1, 2\} = \{\omega_x, \omega_y\}$. For ω_x , the last two components of u_i vanish. For ω_y , the first two components of u_i vanish. Thus the orthogonality is ensured. If the further condition $\omega_x = \omega_y$ holds (which yields $\sigma_1 = \sigma_2$), one may take the limit $\omega_y \rightarrow \omega_x$ after everything is worked out.

First we should work out the ground state. In the coordinate representation, equation (44) takes the form

$$(\xi_{ij}\tilde{x}_j - i\hbar\eta_{ij}\partial_j)\psi_{00}(x, y) = 0, \quad i = 1, 2, \quad (47)$$

where $x_1 = x$, $x_2 = y$ and similarly for \tilde{x}_1 and \tilde{x}_2 , $\partial_j = \partial/\partial x_j = \partial/\partial \tilde{x}_j$, and we have defined two 2×2 matrices

$$\begin{aligned} \xi &= (\xi_{ij}) = \begin{pmatrix} u_{11} & u_{13} \\ u_{21} & u_{23} \end{pmatrix}, \\ \eta &= (\eta_{ij}) = \begin{pmatrix} u_{12} & u_{14} \\ u_{22} & u_{24} \end{pmatrix}, \end{aligned} \quad (48)$$

where $u_{i\beta}$ ($i = 1, 2$, $\beta = 1, 2, 3, 4$) is the β -th component of u_i . Suppose that

$$\begin{aligned} \psi_{00}(x, y) &= N_0 \exp[-s(x, y)], \\ s(x, y) &= \tilde{x}_i \Lambda_{ij} \tilde{x}_j / 2\hbar, \end{aligned} \quad (49)$$

where S is a 2×2 symmetric matrix whose elements are complex numbers, and N_0 is a normalization constant. Substituting this into Eq. (47) we obtain

$$\Lambda = i\eta^{-1}\xi. \quad (50)$$

This can be easily worked out. The elements are

$$\Lambda_{11} = \frac{M\omega_x(\sigma_1 + \sigma_2)}{\omega_x + \omega_y} \equiv \hbar\lambda_x^2, \quad (51a)$$

$$\Lambda_{22} = \frac{M\omega_y(\sigma_1 + \sigma_2)}{\omega_x + \omega_y} \equiv \hbar\lambda_y^2, \quad (51b)$$

$$\Lambda_{12} = \Lambda_{21} = \frac{iM\omega_B(\omega_x - \omega_y)}{\omega_x + \omega_y} \equiv i\hbar\lambda_{xy}, \quad (51c)$$

where the relation $\sigma_1\sigma_2 = \omega_x\omega_y$ has been used. Therefore the ground-state wave function is

$$\psi_{00}(x, y) = N_0 \exp\left(-\frac{1}{2}\lambda_x^2\tilde{x}^2 - \frac{1}{2}\lambda_y^2\tilde{y}^2 - i\lambda_{xy}\tilde{x}\tilde{y}\right), \quad (52a)$$

where the normalization constant is

$$N_0 = \sqrt{\frac{\lambda_x\lambda_y}{\pi}} = \left[\frac{M(\sigma_1 + \sigma_2)\sqrt{\omega_x\omega_y}}{\pi\hbar(\omega_x + \omega_y)}\right]^{1/2}. \quad (52b)$$

When $\omega_B = 0$, we have $\lambda_x = \sqrt{M\omega_x/\hbar}$, $\lambda_y = \sqrt{M\omega_y/\hbar}$, and $\lambda_{xy} = 0$. On the other hand, if $\omega_x = \omega_y$, we have $\omega_1 = \omega_2$, $\sigma_1 = \omega_1 + |\omega_B|$, $\sigma_2 = \omega_1 - |\omega_B|$, and $\lambda_x = \lambda_y = \sqrt{M\omega_1/\hbar}$, $\lambda_{xy} = 0$. These are all expected

results.

It is remarkable that the uncertainty relation for the above ground state is

$$\Delta x\Delta p_x = \Delta y\Delta p_y = \frac{\hbar}{2}\left(1 + \frac{\lambda_{xy}^2}{\lambda_x^2\lambda_y^2}\right)^{1/2} \geq \frac{\hbar}{2}. \quad (53)$$

The equal sign holds only when $B = 0$ or $\omega_x = \omega_y$. Therefore the ground state is in general not a minimum wave packet, though it is of the Gaussian type. It seems that this is not noticed in the previous literature.

For any function $F(x, y)$, it is easy to show that

$$(\xi_{ij}^*\tilde{x}_j - i\hbar\eta_{ij}^*\partial_j)F(x, y) = -i\hbar \exp[s^*(x, y)]\eta_{ij}^*\partial_j\{\exp[-s^*(x, y)]F(x, y)\}, \quad i = 1, 2. \quad (54)$$

According to Eq. (43), and using the above relation we obtain

$$\begin{aligned} \psi_{n_1 n_2}(x, y) &= \frac{(-i\sqrt{\hbar})^{n_1+n_2}}{\sqrt{n_1!n_2!}} N_0 K_1^{n_1} K_2^{n_2} \exp\left(\frac{1}{2}\lambda_x^2\tilde{x}^2 + \frac{1}{2}\lambda_y^2\tilde{y}^2 - i\lambda_{xy}\tilde{x}\tilde{y}\right) \\ &\times [(\sigma_1^2 - \omega_y^2)\partial_x - i2\omega_B\sigma_1\partial_y]^{n_1} [(\sigma_2^2 - \omega_x^2)\partial_x - i2\omega_B\sigma_2\partial_y]^{n_2} \exp(-\lambda_x^2\tilde{x}^2 - \lambda_y^2\tilde{y}^2). \end{aligned} \quad (55)$$

5 Some Special Cases

In this section we discuss some special cases where some of the parameters are zero.

(i) If $\omega_z = 0$ and $E_z = 0$, the motion in the z direction is free, and $\mathcal{E}_z \geq 0$ is continuous.

(ii) If $\omega_z = 0$ but $E_z \neq 0$, the motion in the z direction is that in a homogeneous electric field, which is also well known, and \mathcal{E}_z is also continuous, and can take any real value.^[10]

(iii) In the preceding sections we have assumed that both ω_x and ω_y are nonzero. In this case, if any one of E_x or E_y is zero, it can be substituted into the above results directly and no modification to the formalism is needed. However, if one of ω_x and ω_y is zero, the situation is different. We would deal with $\omega_y = 0$ in the following. The other case could be discussed in a similar way.

(iv) If $\omega_y = 0$ but $E_y \neq 0$, the Hamiltonian could not be reduced to a quadratic form, and to our knowledge the problem could not be solved analytically.

(v) If $\omega_y = 0$ and $E_y = 0$, we choose the gauge $A_x = 0$, $A_y = Bx$, and let

$$\psi(x, y) = \frac{1}{\sqrt{2\pi}} \exp(iky)\phi(x), \quad (56)$$

then $\phi(x)$ satisfies the equation

$$\left[\frac{1}{2M}p_x^2 + \frac{1}{2}M\tilde{\omega}_1^2(x - x_k)^2\right]\phi(x) = \mathcal{E}^x\phi(x), \quad (57)$$

where

$$\tilde{\omega}_1 = \sqrt{\omega_x^2 + 4\omega_B^2}, \quad x_k = \frac{qE_x + 2k\hbar\omega_B}{M\tilde{\omega}_1^2}, \quad (58)$$

and

$$\mathcal{E}^x = \mathcal{E}^{xy} + \frac{(qE_x + 2k\hbar\omega_B)^2}{2M\tilde{\omega}_1^2} - \frac{\hbar^2 k^2}{2M}. \quad (59)$$

The energy levels and wave functions for Eq. (57) are readily available, so the final results are

$$\begin{aligned} \mathcal{E}_{n_1 k}^{xy} &= \hbar\tilde{\omega}_1\left(n_1 + \frac{1}{2}\right) - \frac{(qE_x + 2k\hbar\omega_B)^2}{2M\tilde{\omega}_1^2} + \frac{\hbar^2 k^2}{2M}, \\ n_1 &= 0, 1, 2, \dots, \quad k \in (-\infty, +\infty), \end{aligned} \quad (60)$$

and the corresponding wave functions are

$$\psi_{n_1 k}(x, y) = \frac{1}{\sqrt{2\pi}} \exp(iky)\psi_{n_1}^{\tilde{\omega}_1}(x - x_k). \quad (61)$$

If further ω_x or E_x or both are zero, the results can be obtained from the above ones directly.

(vi) Finally, if both ω_x and ω_y are zero, one can appropriately choose the coordinate axes of the xy plane such that $E_y = 0$. Thus it is a special case of the above case (v).

6 A More General Magnetic Field

In this section we set $\omega_z = 0$ and $E_z = 0$, and consider a magnetic field with an x or y component (but not both) in addition to the z component. To our knowledge this was not considered previously. Without loss of generality we take

$$\mathbf{B} = B'e_x + Be_z. \quad (62)$$

The gauge convenient for the present case might be

$$A_x = 0, \quad A_y = B\tilde{x} = B(x - x_0), \quad A_z = B'y, \quad (63)$$

where x_0 is the same as defined before. In this gauge the total Hamiltonian takes the form

$$\begin{aligned} H^T &= \frac{1}{2M}(p_x^2 + p_y^2) + \frac{1}{2M}(p_z - 2M\omega'_B y)^2 + \frac{1}{2}M\tilde{\omega}_1^2\tilde{x}^2 \\ &+ \frac{1}{2}M\omega_y^2 y^2 - qE_y y - 2\omega_B\tilde{x}p_y - \frac{q^2 E_x^2}{2M\omega_x^2}, \end{aligned} \quad (64)$$

where ω_B and $\tilde{\omega}_1$ are the same as defined before, and $\omega'_B = qB'/2Mc$. p_z is obviously a conserved quantity, so we make the factorization

$$\Psi(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \exp(ikz)\psi(x, y). \quad (65)$$

The Schrödinger equation (1) is then reduced to the form (14a) where now

$$H = \frac{1}{2M}(p_x^2 + p_y^2) + \frac{1}{2}M\tilde{\omega}_1^2\tilde{x}^2 + \frac{1}{2}M\tilde{\omega}_2^2\tilde{y}^2 - 2\omega_B\tilde{x}p_y, \quad (66)$$

and

$$\mathcal{E} = \mathcal{E}^T + \frac{q^2 E_x^2}{2M\omega_x^2} + \frac{(qE_y + 2k\hbar\omega'_B)^2}{2M\tilde{\omega}_2^2} - \frac{\hbar^2 k^2}{2M}. \quad (67)$$

In these equations $\tilde{y} = y - y_k$, which is different from the previous one, and

$$\tilde{\omega}_2 = \sqrt{\omega_y^2 + 4\omega_B'^2}, \quad y_k = \frac{qE_y + 2k\hbar\omega'_B}{M\tilde{\omega}_2^2}. \quad (68)$$

Now the reduced Hamiltonian H is quadratic, and different from Eq. (14b) only in the last term. Thus the problem can be solved in much the same way as before. We only give the results here. The energy levels are

$$\begin{aligned} \mathcal{E}_{n_1 n_2 k}^T &= \hbar\sigma_1\left(n_1 + \frac{1}{2}\right) + \hbar\sigma_2\left(n_2 + \frac{1}{2}\right) \\ &\quad - \frac{q^2 E_x^2}{2M\omega_x^2} - \frac{(qE_y + 2k\hbar\omega'_B)^2}{2M\tilde{\omega}_2^2} + \frac{\hbar^2 k^2}{2M}, \\ n_1, n_2 &= 0, 1, 2, \dots, \quad k \in (-\infty, +\infty), \end{aligned} \quad (69)$$

where σ_1 and σ_2 are given by Eqs. (25), (22b), and (23), but with ω_y replaced by $\tilde{\omega}_2$. The wave functions are given by

$$\Psi_{n_1 n_2 k}(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \exp(ikz)\psi_{n_1 n_2 k}(x, y), \quad (70)$$

where $\psi_{n_1 n_2 k}(x, y)$ has the same form as Eq. (55) with ω_y replaced by $\tilde{\omega}_2$ everywhere, including in the expressions for λ_x , λ_y , K_1 and K_2 , but here

$$\lambda_{xy} = -\frac{2M\omega_B\tilde{\omega}_2}{\hbar(\omega_x + \tilde{\omega}_2)}. \quad (71)$$

Also note that \tilde{y} depends on k , which is the reason why $\psi_{n_1 n_2 k}(x, y)$ has the subscript k . The orthonormal relation is

$$\int \Psi_{n'_1 n'_2 k'}^*(\mathbf{x})\Psi_{n_1 n_2 k}(\mathbf{x}) d\mathbf{x} = \delta(k - k')\delta_{n_1 n'_1}\delta_{n_2 n'_2}. \quad (72)$$

By the way, the wave functions (61) satisfy a similar orthonormal relation.

As before, the ground state obtained here, though being of the Gaussian type, is not a minimum wave packet, except when $B = 0$ (then $\mathbf{B} = B'\mathbf{e}_x$).

7 Coherent and Squeezed States

Coherent and squeezed states are useful objects in quantum mechanics and quantum optics. These are widely discussed in the literature. In the presence of magnetic fields, some discussions can be found in Ref. [11].

There are also examples on other applications of such states.^[12] However, such states for anisotropic harmonic oscillators in the presence of magnetic fields are not discussed previously, to the best of our knowledge. In this section we will discuss the definition, the properties and the time evolution of such states for the case treated in Secs. 3 and 4. The discussions can be easily extended to the case of Sec. 6.

Since the motion in the z direction is separable and is essentially that of a usual harmonic oscillator, we only discuss the coherent and squeezed states on the $\tilde{x}\tilde{y}$ plane (which is a simple translation of the xy plane), and their time evolution is governed by the Hamiltonian H given by Eq. (14a) or (40) (the additional constants in H^{xy} has only trivial consequence for the time evolution). The definitions used below are natural generalizations of those for a single harmonic oscillator.^[13–15]

We define a unitary displacement operator $D(\alpha_1, \alpha_2) = D_1(\alpha_1)D_2(\alpha_2)$, where

$$D_i(\alpha_i) = \exp(\alpha_i a_i^\dagger - \alpha_i^* a_i), \quad i = 1, 2, \quad (73)$$

and the α_i 's are complex numbers. For an arbitrary state $|\varphi\rangle$ one can define a corresponding displaced state

$$|\varphi, \alpha_1 \alpha_2\rangle_D = D(\alpha_1, \alpha_2)|\varphi\rangle = D_1(\alpha_1)D_2(\alpha_2)|\varphi\rangle. \quad (74)$$

It is easy to show that

$$D^\dagger(\alpha_1, \alpha_2)a_i D(\alpha_1, \alpha_2) = a_i + \alpha_i, \quad i = 1, 2, \quad (75)$$

so if $|\varphi\rangle = |00\rangle$ is the ground state we have

$$\alpha_i |00, \alpha_1 \alpha_2\rangle_D = \alpha_i |00, \alpha_1 \alpha_2\rangle_D, \quad i = 1, 2. \quad (76)$$

Thus the coherent states may be defined as the displaced ground state $|00, \alpha_1 \alpha_2\rangle_D$. In terms of the original variables the displacement operator takes the form

$$\begin{aligned} D(\alpha_1, \alpha_2) &= \exp\left(-\frac{i}{2\hbar}x_i^D p_i^D\right) \exp\left(\frac{i}{\hbar}p_i^D \tilde{x}_i\right) \\ &\quad \times \exp\left(-\frac{i}{\hbar}x_i^D p_i\right), \end{aligned} \quad (77)$$

where

$$\begin{aligned} x_i^D &= i\sqrt{\hbar}(\alpha_j \eta_{ji}^* - \text{c.c.}), \\ p_i^D &= -i\sqrt{\hbar}(\alpha_j \xi_{ji}^* - \text{c.c.}). \end{aligned} \quad (78)$$

If the wave function for the state $|\varphi\rangle$ is $\varphi(\tilde{x}, \tilde{y})$, then the one for the displaced state $|\varphi, \alpha_1 \alpha_2\rangle_D$ is

$$\begin{aligned} \varphi_{\alpha_1 \alpha_2}^D(\tilde{x}, \tilde{y}) &= \exp\left(-\frac{i}{2\hbar}x_i^D p_i^D\right) \exp\left(\frac{i}{\hbar}p_i^D \tilde{x}_i\right) \\ &\quad \times \varphi(\tilde{x} - x^D, \tilde{y} - y^D). \end{aligned} \quad (79)$$

Thus we see the reason why the operator $D(\alpha_1, \alpha_2)$ is called a displacement operator. From this and Eq. (52) it is easy to obtain the wave function for the coherent states. Obviously the displaced state and the original one have the same shape in the configuration space, thus the

uncertainty ΔX_α in $|\varphi, \alpha_1 \alpha_2\rangle_D$ is the same as that in $|\varphi\rangle$. This can also be easily shown by using Eqs. (37) and (75).

Now consider the time evolution of the displaced states. It is easy to show that

$$e^{-iHt/\hbar} a_i e^{iHt/\hbar} = \exp(i\sigma_i t) a_i, \quad i = 1, 2. \quad (80)$$

If the state at the initial time $t = 0$ is $|\psi(0)\rangle = |\varphi, \alpha_1 \alpha_2\rangle_D$, then the state at the time t is

$$|\psi(t)\rangle = |\varphi(t), \alpha_{1t} \alpha_{2t}\rangle_D = D_1(\alpha_{1t}) D_2(\alpha_{2t}) |\varphi(t)\rangle, \quad (81)$$

where $\alpha_{it} = \exp(-i\sigma_i t) \alpha_i$ and $|\varphi(t)\rangle = e^{-iHt/\hbar} |\varphi\rangle$. Therefore if $|\varphi(t)\rangle$ is known, $|\psi(t)\rangle$ can be obtained by a time-dependent displacement. A simple special case is $|\varphi\rangle = |n_1 n_2\rangle$, that is

$$|\psi(0)\rangle = |n_1 n_2, \alpha_1 \alpha_2\rangle_D, \quad (82)$$

a displaced number state. In this case

$$|\varphi(t)\rangle = \exp\left[-i\left(n_1 + \frac{1}{2}\right)\sigma_1 t - i\left(n_2 + \frac{1}{2}\right)\sigma_2 t\right] |n_1 n_2\rangle,$$

and

$$|\psi(t)\rangle = \exp\left[-i\left(n_1 + \frac{1}{2}\right)\sigma_1 t - i\left(n_2 + \frac{1}{2}\right)\sigma_2 t\right] \times |n_1 n_2, \alpha_{1t} \alpha_{2t}\rangle_D. \quad (83)$$

This means that $|\psi(t)\rangle$ is also a displaced number state, except that the displacement parameters are time-dependent, and a time-dependent phase factor is gained. The position and velocity of the wave packet $\psi(t, \tilde{x}, \tilde{y}) = \langle \tilde{x} \tilde{y} | \psi(t) \rangle$ is characterized by the expectation values $x_\alpha^c(t) = \langle \psi(t) | X_\alpha | \psi(t) \rangle$. Using Eq. (37) it is easy to show that

$$x_\alpha^c(t) = \sqrt{\hbar} [\exp(-i\sigma_j t) \alpha_j v_{j\alpha} + \text{c.c.}]. \quad (84)$$

By straightforward calculations one can show that they satisfy the following equation

$$\dot{x}_\alpha^c(t) = \Omega_{\alpha\beta} x_\beta^c(t), \quad (85)$$

where the eigenvalue equation (27) has been used, and at $t = 0$ they give the same results as those obtained by using Eqs. (78) and (79). On the other hand, the equations of motion for X_α governed by the Hamiltonian H according to classical mechanics are

$$\dot{X}_\alpha = \{X_\alpha, H\}_{\text{PB}} = \Omega_{\alpha\beta} X_\beta, \quad (86)$$

where the Poisson bracket $\{X_\alpha, H\}_{\text{PB}}$ turns out to be of the same form as the commutator $[X_\alpha, H]/i\hbar$ because H is quadratic in X_α . The above two equations mean that the center of the wave packet moves like a classical particle. In fact, it can be shown that this holds for any wave packet in any quadratic system. For the displaced number states, some more specific properties should be emphasized. First, the shape of the wave packet keep unchanged with time. Second, the center of the wave packet is oscillating, but the motion is in general not periodic except when σ_2/σ_1 is a rational number.

Next we discuss the squeezed states. We define a unitary squeeze operator $S(\zeta_1, \zeta_2) = S_1(\zeta_1) S_2(\zeta_2)$, where

$$S_i(\zeta_i) = \exp\left(\frac{1}{2}\zeta_i a_i^\dagger a_i^\dagger - \frac{1}{2}\zeta_i^* a_i a_i\right), \quad i = 1, 2, \quad (87)$$

and the ζ_i 's are complex numbers. For an arbitrary state $|\varphi\rangle$ the corresponding squeezed state may be defined as

$$|\varphi, \zeta_1 \zeta_2\rangle_s = S(\zeta_1, \zeta_2) |\varphi\rangle = S_1(\zeta_1) S_2(\zeta_2) |\varphi\rangle. \quad (88)$$

The following equation is useful for subsequent calculations

$$S^\dagger(\zeta_1, \zeta_2) a_i S(\zeta_1, \zeta_2) = a_i \cosh \rho_i + a_i^\dagger e^{i\phi_i} \sinh \rho_i, \quad (89)$$

$$i = 1, 2,$$

where $\rho_i = |\zeta_i|$ and $\phi_i = \arg \zeta_i$.

Unlike the displaced state, it is difficult to obtain the explicit wave function for the squeezed state in terms of the wave function for the original one. However, in the squeezed number state $|n_1 n_2, \zeta_1 \zeta_2\rangle_s$, it can be shown by using Eq. (89) that

$$\langle X_\alpha \rangle = 0, \quad (90)$$

$$\Delta X_\alpha = \{\hbar(2n_j + 1)[|v_{j\alpha}|^2 \cosh 2\rho_j + \text{Re}(v_{j\alpha}^2 e^{i\phi_j}) \sinh 2\rho_j]\}^{1/2}. \quad (91)$$

Comparing with the results for $\zeta_1 = \zeta_2 = 0$, we see that the center of the squeezed state is the same as that of the original one, but the uncertainties are different. Thus the states are indeed "squeezed".

Now consider the time evolution of the squeezed states. If the state at the initial time $t = 0$ is $|\psi(0)\rangle = |\varphi, \zeta_1 \zeta_2\rangle_s$, then the state at the time t is

$$|\psi(t)\rangle = |\varphi(t), \zeta_{1t} \zeta_{2t}\rangle_s = S_1(\zeta_{1t}) S_2(\zeta_{2t}) |\varphi(t)\rangle, \quad (92)$$

where $\zeta_{it} = \exp(-i2\sigma_i t) \zeta_i$ and $|\varphi(t)\rangle = \exp(-iHt/\hbar) |\varphi\rangle$. Therefore if $|\varphi(t)\rangle$ is known, $|\psi(t)\rangle$ can be obtained by a time-dependent squeeze. A simple special case is $|\varphi\rangle = |n_1 n_2\rangle$, that is

$$|\psi(0)\rangle = |n_1 n_2, \zeta_1 \zeta_2\rangle_s, \quad (93)$$

a squeezed number state. In this case

$$|\varphi(t)\rangle = \exp\left[-i\left(n_1 + \frac{1}{2}\right)\sigma_1 t - i\left(n_2 + \frac{1}{2}\right)\sigma_2 t\right] |n_1 n_2\rangle,$$

and

$$|\psi(t)\rangle = \exp\left[-i\left(n_1 + \frac{1}{2}\right)\sigma_1 t - i\left(n_2 + \frac{1}{2}\right)\sigma_2 t\right] \times |n_1 n_2, \zeta_{1t} \zeta_{2t}\rangle_s. \quad (94)$$

This means that $|\psi(t)\rangle$ is also a squeezed number state, except that the squeeze parameters are time-dependent, and a time-dependent phase factor is gained. Though it is difficult to obtain the wave function $\psi(t, \tilde{x}, \tilde{y}) = \langle \tilde{x} \tilde{y} | \psi(t) \rangle$ explicitly, in these states it can be shown that

$$\langle X_\alpha \rangle_t = 0, \quad (95)$$

and

$$\Delta_t X_\alpha = \{\hbar(2n_j + 1)[|v_{j\alpha}|^2 \cosh 2\rho_j + \operatorname{Re}(v_{j\alpha}^2 e^{i\phi_j - i2\sigma_j t}) \sinh 2\rho_j]\}^{1/2}. \quad (96)$$

These results mean that the center of the wave packet is at rest, but the uncertainties are oscillating. Thus the shape of the wave packet changes with time apparently. As before, the motion is in general not periodic except when σ_2/σ_1 is a rational number.

8 Summary

In this paper we have studied a charged anisotropic harmonic oscillator moving in a homogeneous static electromagnetic field. Several configurations of the electromagnetic field are considered. One of these configurations has been studied in the literature. However, the formalism used here seems more convenient. We have studied the coherent and squeezed states of these systems in some detail. In terms of these concepts the time evolution of some wave packets can be discussed very conveniently.

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