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Quantum Corrections to the Radiation of Schwarzschild-anti-de Sitter Black Hole with Topological Defect

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Abstract *We extend Zhang and Zhao's recent work to the Schwarzschild-anti-de Sitter black hole with topological defect, whose Arnowitt–Deser–Misner (ADM) mass is no longer identical to its mass parameter. The behavior of the tunneling massive particle is investigated and the emission rate is calculated. The result satisfies an underlying unitary theory and takes the same functional form as that of the mass-less particle.*

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In 1975, Stephen Hawking first proved the existence of black hole radiation[1] described as a tunneling process triggered by vacuum fluctuations near the horizon. But, actual derivation of Hawking radiation did not proceed in this way. Recently, a new method, which describes the particle in dynamic geometry treatment, was developed by Kraus and Wilczek[2] and elaborated upon by Parikh and Wilczek.[3,4] Taking the self-gravitation interaction and the unfixed background space-time into account, they gave a leading correction to the black hole. There are two key points in their work. Firstly, they pointed out there is no pre-existing potential barrier. Instead, the barrier is defined by the radiation particle itself. Secondly, Painlevé coordinate that well behaves at the horizon was introduced. Following the method, people have investigated the Hawking radiation of various space-times.[5–8] However, in all these investigations, the ADM mass of the black hole is identical to their mass parameter, and the investigations are limited to the mass-less particles. Recently, Zhang and Zhao[9,10] extended Parikh and Wilczek's work to the case of charged massive particle and made a great deal of success. However, the Hawking radiation of black holes with topological defect has not been studied and is rather meaningful for us to study. In this paper, we discuss the Hawking radiation of the massive particle from Schwarzschild-anti-de Sitter black hole with a topological defect, whose ADM mass is no longer identical to its parameter, by two methods. Both of the results are consistent with Parikh and Wilczek's and give a correction to the Hawking radiation of the black hole. Throughout the paper, the geometrized units ($G \equiv c \equiv \hbar = 1$) are used.

The line element of Schwarzschild-anti-de Sitter black hole with a topological defect can be written as

$$\begin{aligned} \mathrm{d}s^2 = &-\Big(1-\eta^2-\frac{2M'}{r'}-\frac{\lambda}{3}r'^2\Big)\mathrm{d}t'^2 \\ &+\Big(1-\eta^2-\frac{2M'}{r'}-\frac{\lambda}{3}r'^2\Big)^{-1}\mathrm{d}r'^2 \\ &+r'^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2)\,, \end{aligned} \tag{1}$$

where M', η, and λ are the black hole mass, the parameter related to the scale of symmetry breaking, and the cosmological constant, respectively. For a typical grand unification scale $\eta \approx 10^{-6}$, we have $1-\eta^2 \approx 1$. According to Ref. [11], we introduce the following transformation:

$$\begin{aligned} &t' = (1-\eta^2)^{-1/2}t\,, \quad r' = (1-\eta^2)^{1/2}r\,, \\ &M' = (1-\eta^2)^{3/2}m\,, \end{aligned} \tag{2}$$

and get

$$\begin{aligned} \mathrm{d}s^2 = &-\Big(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\Big)\mathrm{d}t^2+\Big(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\Big)^{-1}\mathrm{d}r^2 \\ &+r^2(1-\eta^2)(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2)\,. \end{aligned} \tag{3}$$

From the line element (3), we can easily find that the space-time around the scale of a gauge-symmetry breaking has a deficit solid angle. Thus the space-time has a topological defect. According to the null hyper-surface,[12,13] the event horizon r_h, and the other two negative roots r_+ and r_- can be obtained as

$$\begin{aligned} &r_+ = 2\sqrt{\frac{1}{\lambda}}\cos\frac{\varphi}{3}\,, \quad r_h = -2\sqrt{\frac{1}{\lambda}}\cos\frac{\varphi+\pi}{3}\,, \\ &r_- = -2\sqrt{\frac{1}{\lambda}}\cos\frac{\varphi-\pi}{3}\,, \end{aligned} \tag{4}$$

where φ satisfies $\cos\varphi = -3(1-\eta^2)m\sqrt{\lambda}$. Due to coordinate singularities in the line element (5), it is not convenient to discuss the Hawking radiation of the black hole, and an effect approach should be in Painlevé coordinate system.[14] Letting $t' = t + f(r)$ and considering the constant-time slice is flat Euclidean in radial, we can get

$$\begin{aligned} \mathrm{d}s^2 &= -\Big(1-\frac{2m}{r}-\frac{\lambda}{3}r^2\Big)\mathrm{d}t^2+2\sqrt{\frac{2m}{r}+\frac{\lambda}{3}r^2}\,\mathrm{d}t\,\mathrm{d}r+\mathrm{d}r^2+r^2(1-\eta^2)(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2) \\ &= g_{00}\,\mathrm{d}t^2+2g_{01}\,\mathrm{d}t\,\mathrm{d}r+\mathrm{d}r^2+r^2(1-\eta^2)(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2)\,. \end{aligned} \tag{5}$$

Now there is no coordinate singularity in the line element and the true characteristic of the space-time, as being stationary but not static, is manifest. Moreover, the metric satisfies Landau's theory of the coordinate clock synchronization.[15] These properties are helpful for us to discuss the Hawking radiation.

According to Refs. [9] and [16], the radial geodesics of the massive particles, which is different from that of the uncharged mass-less particles, is the phase velocity of the particle. From the line element (5), the phase velocity equation of the outgoing particle is

$$\dot{r} = -\frac{1}{2}\frac{g_{00}}{g_{01}} = \frac{1}{2r}\sqrt{\frac{\lambda r}{3}}\frac{(r^3 - 3r/\lambda + 6m/\lambda)}{\sqrt{r^3 + 6m/\lambda}} \,. \tag{6}$$

If there is not scale of a gauge-symmetry breaking in the space-time, the line element (3) will reduce to the Schwarzschild space-time, whose ADM mass is identical to its mass parameter m. Due to the existence of the scale of gauge-symmetry breaking, however, the ADM mass of the space-time should be $M' = (1-\eta^2)m$. Let us calculate the emission rate. Fixing the total ADM mass and allowing the black hole mass to fluctuate, when a particle with energy ω tunnels out, the mass parameter of the black hole should change. According to Ref. [17], the particle's energy measured at infinite distance is $\omega' = (1-\eta^2)\omega$. Since the infinite blue shift near the horizon, the characteristic wavelength of any wave packet is always arbitrarily small there. In the semi-classical limit, the radial wave-number approaches infinity and the point particle, or Wentzel–Kramers–Brillouin approximation is justified,[18,19]

$$\Gamma \approx \exp(-2\mathrm{Im}\, I) \,, \tag{7}$$

where the imaginary part of the action can be expressed as

$$\mathrm{Im}\, I = \int_{r_i}^{r_f} p_r \, \mathrm{d}r = \int_{r_i}^{r_f}\int_0^{p_r} \mathrm{d}p_r \, \mathrm{d}r \,, \tag{8}$$

in which p_r is the canonical momentum conjugate to r, while r_i and r_f are initial and final location of the event horizon before and after the particle emission. To proceed with an explicit calculation, it is useful to adopt Hamilton equation

$$\dot{r} = \frac{\mathrm{d}H}{\mathrm{d}p_r} = \frac{\mathrm{d}(E-\omega)}{\mathrm{d}p_r} = -(1-\eta^2)\frac{\mathrm{d}\omega}{\mathrm{d}p_r} \,. \tag{9}$$

Substituting Eq. (9) into Eq. (8) yields

$$\mathrm{Im}\, I = -\mathrm{Im}\int_{r_i}^{r_f}\int_0^{(1-\eta^2)\omega} \frac{(1-\eta^2)}{\dot{r}}\mathrm{d}\omega' \mathrm{d}r \,. \tag{10}$$

When the massive particles self-gravitation is taken into account, we have

$$\dot{r} = -\frac{1}{2r}\sqrt{\frac{\lambda r}{3}}\frac{(r-r'_-)(r-r'_h)(r-r'_+)}{\sqrt{r^3 + 6(1-\eta^2)(m-\omega')/\lambda}} \,, \tag{11}$$

where

$$r'_+ = 2\sqrt{\frac{1}{\lambda}}\cos\frac{\varphi'}{3}, \quad r'_h = -2\sqrt{\frac{1}{\lambda}}\cos\frac{\varphi'+\pi}{3} \,,$$
$$r'_- = -2\sqrt{\frac{1}{\lambda}}\cos\frac{\varphi'-\pi}{3} \tag{12}$$

with φ' satisfying $\cos\varphi' = -3(1-\eta^2)(m-\omega')\sqrt{\lambda}$. At the event horizon, r'_h is a pole and $r_i < r'_h < r_f$. According to Cauchy theorem we have

$$\mathrm{Im}\, I = -\frac{6(1-\eta^2)\pi}{\lambda} \times \mathrm{Im}\int_0^{(1-\eta^2)\omega} \frac{r'_h}{(r'_h - r'_-)(r'_h - r'_+)}\mathrm{d}\omega' \,. \tag{13}$$

From Eq. (12), we have

$$\mathrm{d}\omega' = \frac{\sin\varphi'}{3\sqrt{\lambda}}\mathrm{d}\varphi' \,. \tag{14}$$

Substituting Eq. (14) into Eq. (13) and doing the integral yield

$$\mathrm{Im}\, I = -\frac{\pi(1-\eta^2)(r_f^2 - r_i^2)}{2} = -\frac{1}{2}\Delta S_{\mathrm{BH}} \,, \tag{15}$$

where ΔS_{BH} is the change of the entropies of the black hole before and after the particle emission. So the emission rate is

$$\Gamma \approx \exp(-2\mathrm{Im}\, I) = \exp(\Delta S_{\mathrm{BH}}) \,. \tag{16}$$

Obviously, the emission rate is related to the change of Bekenstein–Hawking entropies and the radiation spectrum deviates from the pure thermal one.

In fact, we can also get the result by Hamilton–Jacobi method. Near the event horizon, ordering $\Delta = 1 - 2m/r - \lambda r^2/3$, the line element (3) takes on the forms as

$$\mathrm{d}s^2 = -\Delta_{,r}(r_h)(r-r_h)\mathrm{d}t^2 + \frac{1}{\Delta_{,r}(r_h)(r-r_h)}\mathrm{d}r^2 + r_h^2(1-\eta^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2) \,, \tag{17}$$

where

$$\Delta_{,r}(r_h) = \frac{\partial\Delta}{\partial r}\Big|_{r=r_h} = \frac{2m}{r_h^2} - \frac{2}{3}\lambda r_h \,.$$

The action of the radiation particle satisfies the relativistic Hamilton–Jacobi equation, namely

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + u^2 = 0 \,, \tag{18}$$

where u and I are the mass and action of the radiation particle, and $g^{\mu\nu}$ is the inverse metric tensors derived from the line element (17). Substituting them into the Hamilton-Jacobi equation yields

$$-\frac{1}{\Delta_{,r}(r_h)(r-r_h)}(\partial_t I)^2 + \Delta_{,r}(r_h)(r-r_h)(\partial_r I)^2 + g^{22}(\partial_\theta I)^2 + g^{33}(\partial_\varphi I)^2 + u^2 = 0 \,, \tag{19}$$

in which

$$g^{22} = \frac{1}{r_h^2(1-\eta^2)}, \quad g^{22} = \frac{1}{r_h^2(1-\eta^2)\sin^2\theta} \,.$$

Obviously, it is difficult to solve the action I for it is the function of t, r, θ, and φ. But considering that there is a Killing vector in the black hole space-time, we can carry out the following variable separation

$$I = -\omega t + R(r) + P_\theta\theta + P_\varphi\varphi \,, \tag{20}$$

where ω is the energy of the particle, P_θ and P_φ are the generalized momentums corresponding to θ and φ respectively. Substituting Eq. (20) into Eq. (19) and solving $R(r)$ we can get

$$R(r) = \frac{1}{\Delta_{,r}(r_h)} \int \frac{\mathrm{d}r}{r - r_h} \sqrt{\omega^2 - \Delta_{,r}(r_h)(r - r_h)\big[g^{22}(\partial_\theta I)^2 + g^{33}(\partial_\varphi I)^2 + u^2\big]}\,. \tag{21}$$

Obviously, the imaginary part can be produced at the event horizon. If we do integral directly, we cannot derive the correct result. So we introduce the proper spatial distance so as to make sure the correctness of the result.[20] The distance of any two points at a fixed time is defined by

$$\mathrm{d}\sigma^2 = \frac{1}{\Delta_{,r}(r_h)(r - r_h)}\mathrm{d}r^2 + r_h^2(1 - \eta^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\,. \tag{22}$$

Noting that the particle is treated as S-wave to tunnels across the event horizon and should not have motion in θ-direction, the proper spatial distance is

$$\sigma = 2\sqrt{\frac{r - r_h}{\Delta_{,r}(r_h)}}\,. \tag{23}$$

And then we have

$$R(\sigma) = \frac{2}{\Delta_{,r}(r_h)} \int \frac{\mathrm{d}\sigma}{\sigma} \sqrt{\omega^2 - (\Delta_{,r}^2(r_h)\sigma^2/4)\big[g^{22}(\partial_\theta I)^2 + g^{33}(\partial_\varphi I)^2 + u^2\big]}\,. \tag{24}$$

Now the pole is produced from $\sigma = 0$, which corresponds to the event horizon. Doing the integral and substituting the integral result into Eq. (20), we can get the imaginary part as

$$\operatorname{Im} I = \frac{2\pi\omega}{\Delta_{,r}(r_h)} = \frac{3\pi r_h^2\omega}{3m - \lambda r_h^3}\,. \tag{25}$$

Using WKB approximation, we can get the tunneling rate of the radiation particle. However, we find that the derived radiation spectrum is only the leading term, and the reason is that the self-gravitation interaction and the unfixed background space-time were not taken into account. For getting the actual radiation spectrum, we incorporate these and move on discussing the Hawking radiation of the black hole. As mentioned in Sec. 2, when a particle with energy ω tunnels out, the mass parameter of the black hole should change. Considering the self-gravitation interaction, therefore, the imaginary part of the actual action is

$$\operatorname{Im} I = \int_0^{(1-\eta^2)\omega} \frac{3\pi r_h'^2\,\mathrm{d}\omega'}{3(1 - \eta^2)m - \lambda r_h'^3}\,. \tag{26}$$

Calculating the integral we can get

$$\operatorname{Im} I = -\frac{(1 - \eta^2)\pi}{2}(r_f^2 - r_i^2) = -\frac{1}{2}\Delta S_{\mathrm{BH}}\,, \tag{27}$$

which clearly is fully consistent with Eq. (15). Therefore this can also produce the emission rate.

In this paper, the Hawking radiation as a tunneling from Schwarzschild-anti-de Sitter black hole with topological defect process has been viewed by two methods. Both of the results shows that the emission rate is related to the change of Bekenstein–Hawking entropies and the emission spectrum deviates from the pure thermal one, which are fully accordant with Parikh and Wilczek's and give a correction to the Hawking radiation of the black hole.

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