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Solution of ODE $u'' + p(u)(u')^2 + q(u) = 0$ and Applications to Classifications of All Single Travelling Wave Solutions to Some Nonlinear Mathematical Physics Equations

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Abstract *Under the travelling wave transformation, some nonlinear partial differential equations such as Camassa-Holm equation, High-order KdV equation,etc., are reduced to an integrable ODE expressed by $u'' + p(u)(u')^2 + q(u) = 0$ whose general solution can be given. Furthermore, combining complete discrimination system for polynomial, the classifications of all single travelling wave solutions to these equations are obtained. The equation $u''+p(u)(u')^2+q(u) = 0$ includes the equation $(u')^2 = f(u)$ as a special case, so the proposed method can be also applied to a large number of nonlinear equations. These complete results cannot be obtained by any indirect method.*

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1 Introduction

For seeking for the single travelling wave solutions to nonlinear partial differential equations, a lot of expansion methods have been proposed (for example, see Refs. [1] ∼ [3]). These methods are only indirect methods based on some assumptions about the forms of solutions of equations considered. Applying those indirect methods, we cannot give all single travelling wave solutions to the equations considered. On the other hand, some superficially different solutions obtained by indirect methods are sometimes the same solutions essentially.[24] Therefore, it is worth while to give the classifications of all single travelling wave solutions to some nonlinear mathematical physics equations. We know that direct integral method is a rather simple and powerful methods,[25−28] and sometimes can be used to give the classifications of all single travelling wave solutions to some differential equations[28] by combining with complete discrimination system for polynomial.[25−29] In particular, if a nonlinear equation can be reduced to an integral form as follows:

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}u}{\sqrt{f(u)}}\,, \tag{1}$$

with $f(u) = p_n(u)$ a polynomial of the n-th order, then we can directly give the classification of all solutions to the right integral in Eq. (1) using complete discrimination system for the n-th order polynomial.

There are many nonlinear differential equations that cannot be directly reduced to Eq. (1), or the corresponding function $f(u)$ in Eq. (1) has more complex form, so we need other tricks to solve these equations. In the present paper, we use direct integral method to study some nonlinear mathematical physics equation which can be reduced to the following integrable ODE,

$$u''(\xi) + p(u)(u'(\xi))^2 + q(u) = 0\,. \tag{2}$$

These equations include: Camassa–Holm equation[30]

$$u_t + 2\varepsilon u_x + 3uu_x - u_{xxt} - 2u_x u_{xx} - uu_{xxx} = 0\,; \tag{3}$$

Getmanou equation[31]

$$u_{xt} + \frac{u_x u_t}{1-u^2} - u(1-u^2) = 0\,; \tag{4}$$

Harry–Dym equation[32]

$$u_t + u^3 u_{xxx} = 0\,; \tag{5}$$

Monge–Ampere equation[33]

$$u_{tt} + u_{xx} - u_{tx}^2 = -k, \quad k = \pm 1, 0\,; \tag{6}$$

high-order KdV equation[35]

$$u_t + u_x + \alpha uu_x + \beta u_{xxx} + \alpha^2\rho_1 u^2 u_x + \alpha\beta(\rho_2 uu_{xxx} + \rho_3 u_x u_{xx}) = 0\,; \tag{7}$$

nonlinear variant equation[35]

$$(u_t + uu_x)_{xx} - \frac{1}{2}(u_x^2)_x = 0\,; \tag{8}$$

Drinfel'd–Sokolov–Wilson equation[36]

$$u_t + vv_x = 0\,, \tag{9}$$

$$v_t - \varepsilon_1 uv_x + \varepsilon_2 vu_x + \varepsilon_3 u_{xxx} = 0\,; \tag{10}$$

coupled Harry–Dym equation[37]

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x + v_x + \alpha u_x\,, \tag{11}$$

$$v_t = -u_x v - \frac{1}{2}uv_x + \alpha v_x\,; \tag{12}$$

Fuchssteiner–Focas–Camassa–Holm equation[38,39]

$$u_t = 2f_x u + fu_x\,, \tag{13}$$

$$u = \frac{1}{2}f_{xx} - 2f\,; \tag{14}$$

coupled KdV equation[40]

$$u_t + \varepsilon_1 v_x + \varepsilon_2 u^2 u_x + \varepsilon_3 u_{xxx} = 0\,, \tag{15}$$

$$v_t + \eta_1(uv)_x + \eta_2 vv_x = 0\,; \tag{16}$$

and coupled mKdV equation[41]

$$u_t + \varepsilon_1 vv_x + \varepsilon_2 uu_x + \varepsilon_3 u_{xxx} = 0\,, \tag{17}$$

$$v_t + \eta_1(uv)_x + \eta_2 vv_x = 0\,. \tag{18}$$

Using variable transformation and the method of variation of constants, we obtain the general solution of Eq. (2). Furthermore using the complete discrimination system for polynomial, we can give the classifications of all single travelling wave solutions to the above listed equations. These results cannot be obtained by using any indirect method.

Remark 1 In fact, equation (2) has a more general form $p_1(u)u'' + p_2(u)(u')^2 + q_1(u) = 0$. Therefore, if $p_1(u) = 0$ or $p_2(u) = 0$, then equation (2) can be reduced to Eq. (1), that is, equation (1) is a special case of Eq. (2). Thus our method can be also applied to a large number of equations such as KdV equation, Hirota equation, breaking soliton equation, coupled mKdV equation, Chen–Lee–Liu equation, Kundu equation, cubic Schrödinger equation, sine-Gordon equation, double sine-Gordon equation, triple sine-Gordon equation, sinh-Gordon equation, double sinh-Gordon equation, triple sinh-Gordon equation, combined KdV-mKdV equation, nonlinear Klein–Gordon equation, and so on.

2 General Solution to ODE (2)

We give the general solution of Eq. (2) and a more brief proof in the following.

Theorem[42] The general solution of ODE (2) is given by

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}u}{\sqrt{\exp(-2\int p(u)\mathrm{d}u)[c - 2\int q(u)\exp(2\int p(u)\mathrm{d}u)\mathrm{d}u]}}\,, \tag{19}$$

where c and ξ_0 are two arbitrary constants.

Proof Let $W = (u')^2$, then we have $2u'u'' = W' = u'\mathrm{d}W/\mathrm{d}u$. Therefore, $u'' = \mathrm{d}W/2\mathrm{d}u$. Inserting these terms into Eq. (2) yields

$$\frac{\mathrm{d}W}{\mathrm{d}u} + 2p(u)W + 2q(u) = 0\,. \tag{20}$$

Using the method of variation of constants, we have the general solution of Eq. (2) as follows:

$$W(u) = \exp\left(-2\int p(u)\mathrm{d}u\right)\left[c - 2\int q(u)\exp\left(2\int p(u)\mathrm{d}u\right)\mathrm{d}u\right]. \tag{21}$$

Since $W = (\mathrm{d}u/\mathrm{d}\xi)^2$, so we have the general solution (19) of Eq. (2). The proof is completed.

3 Applications

We take Camassa–Holm equation and Drinfel'd–Sokolov–Wilson equations to illustrate our method for short. Other equations can be dealt with similarly.

Example 1 Classification of all single travelling wave solutions to Camassa–Holm equation (3).

Camassa–Holm equation derived from physics as a water wave equation has been studied extensively.[13,43−48] We now also study its single travelling wave solutions and give a complete classification of this kind of solutions. Taking the travelling wave transformation as $u = u(\xi)$, $\xi = kx + \omega t$, it is reduced to the following ODE

$$u'' + \frac{k}{2(ku+\omega)}(u')^2 - \frac{3ku^2 + 2(\omega + 2\varepsilon k)u + c_0}{2k^2(ku+\omega)} = 0\,, \tag{22}$$

where c_0 is an integral constant. According to the formula (19), the corresponding general solution of Eq. (22) is given by

$$\pm\frac{1}{k}(\xi - \xi_0) = \int \sqrt{\frac{u + \omega/k}{(u+\frac{\omega}{k})^3 + d_2(u+\omega/k)^2 + d_1(u+\omega/k) + d_0}}\,\mathrm{d}u\,, \tag{23}$$

where

$$d_2 = 2\varepsilon - \frac{2\omega}{k}, \qquad d_1 = \frac{\omega^2}{k^2} - \frac{4\varepsilon\omega}{k} + \frac{c_0}{k}\,, \tag{24}$$

and d_0 is an arbitrary constant. We give the classification of all solutions to the integral (23) as follows:

Case 1 $d_0 = 0$. Denote $\triangle = d_2^2 - 4d_1$. There are two cases to be discussed.

Case 1.1 If $\triangle = 0$, then the corresponding solutions are

$$u = \pm\exp\Big(\pm\frac{\xi - \xi_0}{k}\Big) - \varepsilon\,. \tag{25}$$

Case 1.2 If $\triangle > 0$ or $\triangle < 0$, then the corresponding solutions are

$$u = \pm\frac{1}{2}\exp\Big(\frac{\pm(\xi-\xi_0)}{k}\Big) \mp \frac{1}{2}\Big(\Big(\varepsilon - \frac{\omega}{k}\Big)^2 - d_1\Big)\exp\Big(\frac{\mp(\xi-\xi_0)}{k}\Big) - \varepsilon\,. \tag{26}$$

Case 2 $d_0 \neq 0$. Denote

$$\triangle = -27\Big(\frac{2d_2^3}{27} + d_0 - \frac{d_1 d_2}{3}\Big)^2 - 4\Big(d_1 - \frac{d_2^2}{3}\Big)^3\,, \tag{27}$$

$$D_1 = d_1 - \frac{d_2^2}{3}\,, \tag{28}$$

where $\triangle$ and D_1 make up a complete discrimination system for $F(u) = (u+\omega/k)^3 + d_2(u+\omega/k)^2 + d_1(u+\omega/k) + d_0$. There are the following four cases to be discussed.

Case 2.1 If $\triangle = 0$, $D_1 < 0$, then we have $F(u) = (u-\alpha)^2(u-\beta), \alpha \neq \beta$. We make the change of the variable as follows:

$$u = \frac{\beta v^2 - \omega/k}{v^2 + 1}\,, \tag{29}$$

and hence its inverse transformation is

$$\frac{u + \omega/k}{u - \beta} = v^2\,. \tag{30}$$

Therefore, we have

$$\pm\frac{1}{k}(\xi - \xi_0) = \ln\Bigl|\frac{v+1}{v-1}\Bigr| + \sqrt{\frac{\omega + \alpha k}{k(\alpha - \beta)}}\ln\Bigl|\frac{v - \sqrt{(\omega+\alpha k)/k(\alpha-\beta)}}{v + \sqrt{(\omega+\alpha k)/k(\alpha-\beta)}}\Bigr|, \qquad \Bigl(\frac{\omega+\alpha k}{k(\beta-\alpha)} < 0\Bigr)\,, \tag{31}$$

and

$$\pm\frac{1}{k}(\xi - \xi_0) = \ln\Bigl|\frac{v+1}{v-1}\Bigr| - 2\sqrt{\frac{\omega + \alpha k}{k(\beta - \alpha)}}\arctan\Bigl(v\sqrt{\frac{k(\beta-\alpha)}{\omega+\alpha k}}\Bigr)\,, \qquad \Bigl(\frac{\omega+\alpha k}{k(\beta-\alpha)} > 0\Bigr)\,. \tag{32}$$

Case 2.2 If $\triangle = 0$, $D_1 = 0$, then we have $F(u) = (u-\alpha)^3$. The solution is given by

$$\pm\frac{1}{2k}(\xi - \xi_0) = \pm\sqrt{\frac{ku+\omega}{k(u-\alpha)}} + \frac{1}{2}\ln\Bigl|\frac{\pm\sqrt{(ku+\omega)/k(u-\alpha)} - 1}{\pm\sqrt{(ku+\omega)/k(u-\alpha)} + 1}\Bigr|\,. \tag{33}$$

Case 2.3 If $\triangle > 0$, $D_1 < 0$, then $F(w) = (w-\alpha)(w-\beta)(w-\gamma)$. Suppose $\alpha > \beta > \gamma$. By making the change of variable

$$u = \frac{\alpha v^2 + \omega/k}{v^2 - 1}\,, \tag{34}$$

the corresponding integral becomes

$$\pm\frac{\sqrt{(\alpha-\beta)(\alpha-\gamma)}}{2(k\alpha+\omega)}(\xi - \xi_0) = \int\Bigl\{1 + \frac{1}{2}\Bigl(\frac{1}{v-1} - \frac{1}{v+1}\Bigr)\Bigr\}\frac{1}{\sqrt{(v^2+A)(v^2+B)}}\mathrm{d}v\,, \tag{35}$$

where $A = (\omega + k\beta)/k(\alpha-\beta)$, $B = (\omega + k\gamma)/k(\alpha - \gamma)$. It is easy to see that the integral (35) can be expressed by the first kind of elliptic integrals and the third kind of elliptic integrals.

Case 2.4 If $\triangle < 0$, then we have $F(u) = (u-\alpha)(u^2 + pu + q)$, $p^2 - 4q < 0$. Making the change of variable $v = (ku+\omega)/k(u-\alpha)$, then the corresponding integral becomes

$$\pm\frac{\xi - \xi_0}{k\alpha + \omega} = \int\Bigl(1 + \frac{1}{v-1}\Bigr)\frac{1}{\sqrt{v(Av^2 + Bv + C)}}\mathrm{d}v\,, \tag{36}$$

where $A = p\alpha + q + \alpha^2$, $B = (2\alpha + p)(\omega/k) - p\alpha - 2q$, $C = \omega^2/k^2 - p\omega/k + q$, moreover $B^2 - 4AC < 0$. It is easy to see that the corresponding integral (36) can be expressed by the first kind of elliptic integrals and the third kind of elliptic integrals.

Remark 2 Using an algebraic expansion method, Fan[13] obtained three solutions to Camassa–Holm equation. Among those solutions, u_1 is just the solution given in the above Case 1.1, but we must point out that the other two solutions u_2 and u_3 do not satisfy the corresponding reduced ODE, and hence they are not solutions. In fact, if we substitute directly the travelling wave transformation into Camassa–Holm equation, then the corresponding ODE is given by $\omega u' + 2\varepsilon k u' + 3kuu' - k^2\omega u''' - 2k^3 u'u'' - k^3 uu''' = 0$. According to the above Case 1, it is a kind of solutions $u = a\exp(\xi/k) + b\exp(-\xi/k) - \varepsilon$, where a and b are two arbitrary constants. In addition, we must point out that we can directly write out the above solutions as a special case of GCH equation using the results in Ref. [42].

Example 2 Classification of all single travelling wave solutions to Drinfel'd–Sokolov–Wilson equations (9) and (10).

Using the travelling wave transformations $u = u(\xi)$, $v = v(\xi)$, $\xi = kx + \omega t$, we have the reduced ODE as follows:

$$u = c_0 - \frac{k}{2\omega}v^2\,, \tag{37}$$

$$v'' + \frac{1}{v}(v')^2 + \frac{d_2 v^3 + d_1 v + d_0}{v} = 0\,, \tag{38}$$

where $d_2 = (2\varepsilon_2 - \varepsilon_1)/6k\varepsilon_3$, $d_2 = -\omega(\omega - c_0k\varepsilon_1)/k^3\varepsilon_3$ and d_0 is an arbitrary constant. We only need to solve Eq. (38). From the theorem in Sec. 2, we have the general solutions of Eq. (38) as follows:

$$\pm(\xi - \xi_0) = \int \frac{v\,\mathrm{d}v}{\sqrt{-2d_2/5\, v^5 - (2d_1/3)v^3 - d_0v^2 + c_1}}\,. \tag{39}$$

Case 1 $c_1 = 0$. The corresponding integral becomes

$$\pm(\xi - \xi_0) = \int \frac{\mathrm{d}v}{\sqrt{-2d_2v^3/5 - 2d_1v/3 - d_0}}\,. \tag{40}$$

Denote $F(w) = w^3 - 2d_1/3(-2d_2/5)^{-1/3}w - d_0$, $w = (-2d_2/5)^{1/3}v$, and $\triangle = -27d_0^2 - 4(-2d_1/3)^3(-2d_2/5)^{-1}$, $D_1 = -(2d_1/3)(-2d_2/5)^{-1/3}$. Here $\triangle$ and D_1 make up a complete discrimination system for $F(w)$.

Case 1.1 If $\triangle = 0$, $D_1 < 0$, then we have $F(w) = (w-\alpha)^2(w-\beta), \alpha \neq \beta$. If $w > \beta$, the solutions are given by

$$v = \Big(-\frac{2d_2}{5}\Big)^{-1/3}\Big[(\alpha-\beta)\tanh^2\Big(\frac{\sqrt{\alpha-\beta}}{2}\Big(-\frac{2d_2}{5}\Big)^{1/3}(\xi-\xi_0)\Big) + \beta\Big], \qquad \alpha > \beta\,; \tag{41}$$

$$v = \Big(-\frac{2d_2}{5}\Big)^{-1/3}\Big[(\alpha-\beta)\coth^2\Big(\frac{\sqrt{\alpha-\beta}}{2}\Big(-\frac{2d_2}{5}\Big)^{1/3}(\xi-\xi_0)\Big) + \beta\Big], \qquad \alpha > \beta\,; \tag{42}$$

$$v = \Big(-\frac{2d_2}{5}\Big)^{-1/3}\Big[(\beta-\alpha)\sec^2\Big(\frac{\sqrt{\beta-\alpha}}{2}\Big(-\frac{2d_2}{5}\Big)^{1/3}(\xi-\xi_0)) + \alpha\Big], \quad \alpha < \beta\,. \tag{43}$$

Case 1.2 If $\triangle = 0$, $D_1 = 0$, then we have $F(w) = w^3$. Thus the solution is given by

$$v = 4\Big(-\frac{2d_2}{5}\Big)^{-2/3}(\xi-\xi_0)^{-2}\,. \tag{44}$$

Case 1.3 If $\triangle > 0$, $D_1 < 0$, then we have $F(w) = (w-\alpha)(w-\beta)(w-\gamma)$. Suppose $\alpha < \beta < \gamma$. When $\alpha < w < \beta$, we have

$$v = \Big(-\frac{2d_2}{5}\Big)^{-1/3}\Big[\alpha + (\beta-\alpha)\mathrm{sn}^2\Big(\frac{\sqrt{\gamma-\alpha}}{2}\Big(-\frac{2d_2}{5}\Big)^{1/3}(\xi-\xi_0)\Big), m\Big)\Big]\,. \tag{45}$$

When $w > \gamma$, we have

$$v = \Big(-\frac{2d_2}{5})^{-1/3}\Big[\frac{\gamma - \beta\mathrm{sn}^2(\sqrt{\gamma-\alpha}/2(-2d_2/5)^{1/3}(\xi-\xi_0)), m)}{\mathrm{cn}^2(\sqrt{\gamma-\alpha}/2(-2d_2/5)^{1/3}(\xi-\xi_0)), m)}\Big]\,, \tag{46}$$

where $m^2 = (\beta-\alpha)/(\gamma-\alpha)$.

Case 1.4 If $\triangle < 0$, then we have $F(w) = (w-\alpha)(w^2+pw+q)$, $p^2 - 4q < 0$. Thus, we have

$$v = (-\frac{2d_2}{5}\Big)^{-1/3}\Big[\alpha - \sqrt{\alpha^2+p\alpha+q} + \frac{2\sqrt{\alpha^2+p\alpha+q}}{1+\mathrm{cn}((\alpha^2+p\alpha+q)^{1/4}(-2d_2/5)^{1/3}(\xi-\xi_0), m)}\Big]\,, \tag{47}$$

where $m^2 = (1/2)(1 - (\alpha+p/2)/\sqrt{\alpha^2+p\alpha+q})$.

Case 2 $c_1 \neq 0$. Denote $F(w) = w^5 + pw^3 + qw^2 + c_1$, $w = (-2d_2/5)^{1/5}v$, where $p = -2d_1/3(-2d_2/5)^{-3/5}$, $q = -d_0(-2d_2/5)^{-2/5}$. We write the complete discrimination system for $F(w)$ as follows:[29]

$$\begin{aligned} &D_2 = -p, \qquad D_3 = -12p^3 - 45q^2, \qquad D_4 = -4p^3q^2 - 40qc_1p^2 - 27q^4 + 125pc_1^2\,,\\ &D_5 = -3750pqc_1^3 + 16p^3q^3c_1 + 825p^2q^2c_1^2 + 108p^5c_1^2 + 108c_1q^5 + 3125c_1^4\,,\\ &E_2 = 16q^2p^4 - 1100qc_1p^3 + 625c_1^2p^2 - 3375c_1q^3, \qquad F_2 = 3q^2\,. \end{aligned} \tag{48}$$

According to the discrimination system (48), we can classify all solutions of the integral (39). Notice that $F_2 \geq 0$, so there are the following five cases to be discussed:

Case 2.1 If $D_5 = 0$, $D_4 = 0$, $D_3 > 0$, $E_2 \neq 0$, then we have

$$F(w) = (w-\alpha)^2(w-\beta)^2(w-\gamma)\,, \tag{49}$$

where α, β, γ are real numbers, and $\alpha \neq \beta \neq \gamma, \alpha > \beta$. We have

$$\begin{aligned} \pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\alpha-\beta)(\xi-\xi_0) = {}& \frac{\alpha}{\sqrt{\alpha-\gamma}}\ln\Big|\frac{\sqrt{(-2d_2/5)^{1/5}v-\gamma} - \sqrt{\alpha-\gamma}}{\sqrt{(-2d_2/5)^{1/5}v-\gamma} + \sqrt{\alpha-\gamma}}\Big| \\ & - \frac{\beta}{\sqrt{\beta-\gamma}}\ln\Big|\frac{\sqrt{(-2d_2/5)^{1/5}v-\gamma} - \sqrt{\alpha-\gamma}}{\sqrt{(-2d_2/5)^{1/5}v-\gamma} + \sqrt{\alpha-\gamma}}\Big|, \qquad (\gamma < \beta)\,; \end{aligned} \tag{50}$$

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\alpha-\beta)(\xi-\xi_0) = \frac{2\alpha}{\sqrt{\gamma-\alpha}}\arctan\frac{\sqrt{(-2d_2/5)^{1/5}v-\gamma}}{\sqrt{\gamma-\alpha}} - \frac{2\beta}{\sqrt{\gamma-\beta}}\arctan\frac{\sqrt{(-2d_2/5)^{1/5}v-\gamma}}{\sqrt{\gamma-\beta}}\,, \quad (\gamma > \alpha)\,; \tag{51}$$

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\alpha-\beta)(\xi-\xi_0)=\frac{\alpha}{\sqrt{\alpha-\gamma}}\ln\Big|\frac{\sqrt{(-2d_2/5)^{1/5}v-\gamma}-\sqrt{\alpha-\gamma}}{\sqrt{(-2d_2/5)^{1/5}v-\gamma}+\sqrt{\alpha-\gamma}}\Big| -\frac{2\beta}{\sqrt{\gamma-\beta}}\arctan\frac{\sqrt{(-2d_2/5)^{1/5}v-\gamma}}{\sqrt{\gamma-\beta}},\qquad(\alpha>\gamma>\beta)\,;\tag{52}$$

Case 2.2 If $D_5=0$, $D_4=0$, $D_3=0$, $D_2\neq0$, $F_2\neq0$, then we have

$$F(w)=(w-\alpha)^3(w-\beta)^2\,,\tag{53}$$

where α and β are real numbers, $\alpha\neq\beta$. Thus we have

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\alpha-\beta)(\xi-\xi_0)=-\frac{2\alpha}{\sqrt{(-2d_2/5)^{1/5}v-\alpha}}-\frac{\beta}{\sqrt{\beta-\alpha}}\ln\Big|\frac{\sqrt{(-2d_2/5)^{1/5}v-\alpha}-\sqrt{\beta-\alpha}}{\sqrt{(-2d_2/5)^{1/5}v-\alpha}+\sqrt{\beta-\alpha}}\Big|,\quad(\alpha<\beta)\,;\tag{54}$$

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\alpha-\beta)(\xi-\xi_0)=-\frac{2\alpha}{\sqrt{(-2d_2/5)^{1/5}v-\alpha}}-\frac{2\beta}{\sqrt{\alpha-\beta}}\arctan\frac{\sqrt{(-2d_2/5)^{1/5}v-\alpha}}{\sqrt{\alpha-\beta}},\quad(\alpha>\beta)\,.\tag{55}$$

Case 2.3 If $D_5=0$, $D_4=0$, $D_3<0$, $E_2\neq0$, then we have

$$f(w)=(w-\alpha)(w^2+r_1w+s_1)^2\,,\tag{56}$$

where α is a real number, and $r_1^2-4s_1<0$. Taking a change of variable $w=V^2+\alpha$, the corresponding integral becomes

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int(2V^2+2\alpha)\Big\{\frac{2a(V-A)+b}{(V-A)^2+B^2}-\frac{2a(V+A)-b}{(V+A)^2+B^2}\Big\}\mathrm{d}V\,,\tag{57}$$

where

$$A=\frac12\sqrt{\sqrt{\alpha^2+r_1\alpha+s_1}-\frac{r_1+2\alpha}{2}},\qquad B=\frac12\sqrt{\sqrt{\alpha^2+r_1\alpha+s_1}+\frac{r_1+2\alpha}{2}}\,,$$

and

$$a=-\frac{1}{2A(2A^2+B^2)},\qquad b=\frac{1}{2(2A^2+B^2)}\,.$$

Solving this integral yields

$$\begin{aligned}\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)&=\pm(2A^2+2\alpha-2B^2+8Aa)\sqrt{(-\frac{2d_2}{5})^{1/5}v-\alpha}+a(3A^2+\alpha-B^2)\\&\times\ln\frac{(\pm\sqrt{(-2d_2/5)^{1/5}v-\alpha}-A)^2+B^2}{(\pm\sqrt{(-2d_2/5)^{1/5}v-\alpha}+A)^2+B^2}+[b(3A^2+\alpha-B^2)-4AaB^2]\\&\times\frac{b}{B}\Big\{\arctan\frac{\pm\sqrt{(-2d_2/5)^{1/5}v-\alpha}-A}{B}+\arctan\frac{\pm\sqrt{(-2d_2/5)^{1/5}v-\alpha}+A}{B}\Big\}.\end{aligned}\tag{58}$$

Case 2.4 If $D_5=0$, $D_4>0$; or $D_5=0$, $D_4=0$, $D_3<0$, $E_2=0$; or $D_5=0$, $D_4<0$; or $D_5=0$, $D_4=0$, $D_3>0$, $E_2=0$, then $F(w)=0$ has only one real root with multiplicities 2 or 3. In detail, we list the corresponding integrals as follows:

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int\frac{w\,\mathrm{d}w}{(w-\alpha)\sqrt{(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)}}\,;\tag{59}$$

or

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int\frac{w\,\mathrm{d}w}{(w-\alpha)\sqrt{w-\alpha)((w-l_1)^2+s_1^2)}}\,;\tag{60}$$

or

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int\frac{w\,\mathrm{d}w}{(w-\alpha)\sqrt{(w-\beta)((w-l_1)^2+s_1^2)}}\,;\tag{61}$$

or

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int\frac{w\,\mathrm{d}w}{(w-\alpha)\sqrt{(w-\alpha)(w-\beta)(w-\gamma)}}\,.\tag{62}$$

It is easy to see that the corresponding integrals can be expressed by the first kind of elliptic integrals and the third kind of elliptic integrals.

Case 2.5 If $D_5>0$, $D_4>0$, $D_3>0$, $D_2>0$; or $D_5<0$; or $D_5>0\bigwedge(D_4\le0\bigvee D_3\le0\bigvee D_2\le0)$, where $\bigvee$ means 'or', $\bigwedge$ means 'and', then $F(w)=0$ has not multiple roots. Therefore we must express the corresponding integrals by the hyper-elliptic functions or hyper-elliptic integrals. In detail, we list the corresponding integrals as follows:

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int\frac{w\,\mathrm{d}w}{\sqrt{(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)(w-\alpha_4)(w-\alpha_5)}}\,;\tag{63}$$

or

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int\frac{w\,\mathrm{d}w}{\sqrt{(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)((w-l_1)^2+s_1^2)}}\,;\tag{64}$$

or

$$\pm\Big(-\frac{2d_2}{5}\Big)^{2/5}(\xi-\xi_0)=\int\frac{w\,\mathrm{d}w}{\sqrt{(w-\alpha)((w-l_1)^2+s_1^2))((w-l_2)^2+s_2^2)}}\,.\tag{65}$$

4 Conclusions

In summary, we reduce Eq. (2) to the first order ODE and give its general solutions. Furthermore, reducing the equations considered to Eq. (2) and applying complete discrimination system for polynomial, we obtain the classifications of all single travelling wave solutions to these equations. These complete results cannot be given by any indirect method. Of course, the tricks and methods used here can be expected to apply to more equations.

Finally, we must point out that our key idea in fact is very simple, that is, firstly to reduce the equation considered to an integrable equation $u'(\xi)=G(u,\theta_1,\ldots,\theta_m)$, where $\theta_1,\ldots,\theta_m$ are parameters, then we only need to solve the integral $\int[1/G(u)]\,\mathrm{d}u$. According to the different parameters, we will give the different solutions to the integral. This is so-called direct integral method. Although direct integral method is a routine method, to decide the parameter's scope and to reduce the equation considered to an integral ODE are still rather difficult. Thus the most important steps are to do these two things. In order to reduce the order of ODE, symmetry group is a powerful tool. At the same time, it is just because combined with complete discrimination system that direct integral method becomes a powerful and efficient method.

Remark 3 In fact, what we give in the paper is only all atom solutions which are obtained by solving reduced integral (in particular, see Ref. [49]).

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