

## Ground-State and Thermal Entanglement in Three-Spin Heisenberg- $XXZ$ Chain with Three-Spin Interaction\*

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**Abstract** *The entanglement properties of a three-spin  $XXZ$  Heisenberg chain with three-spin interaction are studied by means of concurrence of pairwise entanglement. We show that ground-state pairwise entanglement, pairwise thermal entanglement, or quantum phase transition is not present in antiferromagnetic spin chain. For the ferromagnetic case, quantum phase transition takes place at  $\Delta = 1$  for anisotropic interaction and at some values of three-spin coupling strength, and pairwise thermal entanglement increases when the value of  $J/T$  increases and with anisotropic interaction and three-spin interaction decrease. In addition, we find that increasing the anisotropic interaction and the three-spin interaction will decrease critical temperature.*

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**Key words:** spin chain, thermal entanglement, quantum phase transition

### 1 Introduction

Quantum entanglement plays a significant role in the field of quantum information and quantum computation.<sup>[1]</sup> The security of quantum communication and the advantage of quantum computation rely crucially on the entanglement in quantum system. Entanglement can be of different types, e.g., bipartite, multipartite, localizable, bound, etc. The entanglement between a pair of interacting spins provides an example of bipartite entanglement. Entanglement in bipartite states has been well studied in many literatures,<sup>[2,3]</sup> as well as in various types of tripartite entangled states.<sup>[4–6]</sup>

Moreover, during recent years low-dimensional quantum spin system, especially, exactly solvable 1D quantum spin system (spin chain), has drawn considerable attention. A great number of spin chain models with various ranges of interaction, spin representations, and anisotropies have been extensively studied in the context of quantum information science.<sup>[7,8]</sup> Bipartite entanglement in spin chains has been also thoroughly investigated.<sup>[9–11]</sup> Recently, thermal entanglement, i.e., the entanglement of thermal states, presented in spin chain with various Heisenberg interaction models at finite temperature and including an external magnetic field has attracted much attention.<sup>[12–17]</sup>

One of the interesting generalizations of a standard Heisenberg model with nearest-neighbor interaction is the inclusion of interactions between next-nearest spin neighbors

and between three or more successive spins. Interaction terms containing bilinear couplings between the next-nearest neighbors and terms involving the product of three spin operators or more, have been shown to be important for the theoretical description of many physical systems.<sup>[18,19]</sup> Recently, the influence of three- and four-spin interaction to quantum phase transition and other quantum behavior at zero and finite low temperature has been discussed.<sup>[20,21]</sup> Thermal entanglement properties of small spin clusters with four-spin interaction has been studied by Bose and Tribedi.<sup>[22]</sup>

The present paper further explores the well-known  $XXZ$  Heisenberg model of a three-spin chain (ring) with three-spin interaction. We present explicit solution of the system Hamiltonian in the standard basis, and investigate the entanglement properties of all eigenstates (Sec. 2). We further analyze pairwise entanglement in the ground state, and discuss quantum phase transition at zero temperature. In particular, we show that no quantum phase transition occurs in antiferromagnetic spin chain, and quantum phase transition takes place at  $\Delta = 1$  for anisotropic interaction and at some values of three-spin coupling strength in ferromagnetic model (Sec. 3). We also discuss thermal entanglement associated with the three-spin chain at finite temperature by focusing on the concurrence of pairwise entanglement (Sec. 4). We prove that the pairwise thermal entanglement increases as the value of  $J/T$  increases. In addition we show the influence of

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anisotropic interaction and three-spin interaction to the pairwise entanglement and to critical temperature beyond which the thermal entanglement disappears.

## 2 Interaction Hamiltonian and Solution

The Hamiltonian of a three-spin  $XXZ$  chain with three-spin interaction can be expressed as

$$\mathcal{H} = J\mathcal{H}_2 + J'\mathcal{H}_3. \quad (1)$$

The first term in the Hamiltonian is the nearest-neighbor Heisenberg- $XXZ$  interaction with the coupling strength  $J$ ,

$$\mathcal{H}_2 = \sum_{i=1}^3 (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z), \quad (2)$$

where  $\sigma_i^\alpha$  ( $\alpha = x, y, z$ ) are the well-known Pauli operators on site  $i$ , and  $\Delta$  represents the anisotropy and  $\Delta \geq 0$ . The second term is three-spin interaction with the interaction constant  $J'$ ,

$$\begin{aligned} \mathcal{H}_3 &= \sum_{i=1}^3 \vec{\sigma}_i \cdot [\vec{\sigma}_{i-1} \times \vec{\sigma}_{i+1}] \\ &= \sigma_2^x (\sigma_1^y \sigma_3^z - \sigma_1^z \sigma_3^y) + \sigma_2^y (\sigma_1^z \sigma_3^x - \sigma_1^x \sigma_3^z) \\ &\quad + \sigma_2^z (\sigma_1^x \sigma_3^y - \sigma_1^y \sigma_3^x). \end{aligned} \quad (3)$$

The spin  $XXZ$  chain with three-spin interaction was constructed by Tsverlik<sup>[23]</sup> and Frahm<sup>[24]</sup> using the quantum inverse scattering method. The model reduces to the  $XX$  model when  $\Delta = 0$  and  $J' = 0$ , and  $XXZ$  model when  $J' = 0$ .<sup>[25]</sup> Here we will consider the system in one spatial dimension with cyclic boundary condition

$$\sigma_4^\alpha = \sigma_1^\alpha, \quad \alpha = x, y, z. \quad (4)$$

In other words, the above Hamiltonian denotes a three-spin Heisenberg ring.

For simplicity, it is convenient to put  $\beta = J'/J$  and assume  $\beta \geq 0$ . Then, the Hamiltonian in Eq. (1) can be rewritten as

$$\mathcal{H} = J(\mathcal{H}_2 + \beta\mathcal{H}_3). \quad (5)$$

Easily, one can find that the total magnetization  $M_z = \sum_i \sigma_i^z$  commutes with the system Hamiltonian, i.e.,  $[M_z, \mathcal{H}] = 0$ . Therefore, in the paper we investigate the common eigenstates of  $\mathcal{H}$  and  $M_z$ , which is a general method used by many authors. Due to the simplicity of the three-spin Hamiltonian, its eigenvalues and eigenvectors can be easily obtained as

$$E_1 = E_2 = 3\Delta J, \quad \psi_1 = |000\rangle, \quad \psi_2 = |111\rangle; \quad (6)$$

$$E_3 = E_4 = (4 - \Delta)J,$$

$$\psi_3 = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle),$$

$$\psi_4 = \frac{1}{\sqrt{3}}(|011\rangle + |101\rangle + |110\rangle); \quad (7)$$

$$E_5 = E_6 = -(2 + 2\sqrt{3}\beta + \Delta)J,$$

$$\psi_5 = \frac{1}{\sqrt{3}} \left( -\frac{1 + \sqrt{3}i}{2} |001\rangle - \frac{1 - \sqrt{3}i}{2} |010\rangle + |100\rangle \right),$$

$$\psi_6 = \frac{1}{\sqrt{3}} \left( -\frac{1 - \sqrt{3}i}{2} |011\rangle - \frac{1 + \sqrt{3}i}{2} |101\rangle + |110\rangle \right); \quad (8)$$

$$E_7 = E_8 = -(2 - 2\sqrt{3}\beta + \Delta)J,$$

$$\psi_7 = \frac{1}{\sqrt{3}} \left( -\frac{1 - \sqrt{3}i}{2} |001\rangle - \frac{1 + \sqrt{3}i}{2} |010\rangle + |100\rangle \right),$$

$$\psi_8 = \frac{1}{\sqrt{3}} \left( -\frac{1 + \sqrt{3}i}{2} |011\rangle - \frac{1 - \sqrt{3}i}{2} |101\rangle + |110\rangle \right). \quad (9)$$

Obviously, one finds every energy level is double-degenerate. It is well known that any linear combinations of two degenerate eigenstates in a two-dimensional subspace are always eigenstates of system Hamiltonian, and an  $N$ -dimensional space may be spanned with many series of bases. Here, we choose to investigate properties of common eigenstates of  $\mathcal{H}$  and  $M_z$ .

Consider first the properties of entanglement associated with these eigenvectors. Here, we investigate the entanglement properties by focusing on the pairwise entanglement associated with any two qubits in terms of *Concurrence*.<sup>[2,5]</sup> To calculate this, we define reduced density matrices  $\rho_{12} = \text{Tr}_3[\rho_{123}]$ ,  $\rho_{23} = \text{Tr}_1[\rho_{123}]$ ,  $\rho_{13} = \text{Tr}_2[\rho_{123}]$ . Due to the symmetric and periodic condition of the system, we have  $\rho_{12} = \rho_{23} = \rho_{13} = \rho$ . Based on the computational basis  $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ , two-qubit reduced matrices can be expressed as

$$\rho = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & \gamma & \mu^* & 0 \\ 0 & \mu & \omega & 0 \\ 0 & 0 & 0 & y \end{pmatrix}. \quad (10)$$

Now concurrence — the quantity of entanglement between two spins related to the density matrix  $\rho$ , was given by

$$\mathcal{C}(\rho) = 2 \max\{|\mu| - \sqrt{xy}, 0\}. \quad (11)$$

As usual for entanglement measures, the values of the concurrence range from zero, for an unentangled state, to one, for a maximally entangled state.

Clearly, the eigenvectors  $\psi_1$  and  $\psi_2$  are both product states, consequently, no entanglement is present. In contrast, all other eigenvectors are entirely entangled. Now we calculate pairwise entanglement of the eigenvector  $\psi_5$ . For  $\psi_5$ , we can obtain  $x_5 = 1/3$ ,  $y_5 = 0$ ,  $\gamma_5 = \omega_5 = 1/3$ ,  $\mu_5 = -(1 + \sqrt{3}i)/6$  in Eq. (10). Therefore, the pairwise entanglement between any two qubits in the eigenvector,  $\psi_5$ , will be  $\mathcal{C}_5 = 2/3$ . In similar method, the pairwise entanglement of other eigenvectors all can be calculated as  $\mathcal{C}_i = 2/3$ , ( $i = 3, 4, 5, 6, 7, 8$ ). The result shows that every eigenvectors have equivalent entanglement except  $\psi_1$  and  $\psi_2$ .

For describing the global property of tripartite entanglement, Coffman *et al.*<sup>[5]</sup> presented an entanglement measure for three-qubit pure states. It is called the residual

entanglement  $\tau$ , which is defined as  $\tau = \mathcal{C}_{i(jk)}^2 - \mathcal{C}_{ij}^2 - \mathcal{C}_{ik}^2$ , ( $i, j, k \in \{1, 2, 3\}$ ). The residual entanglement can be thought of as the amount of entanglement between qubits on sites  $i$  and  $jk$  that cannot be accounted for by the entanglements of qubit  $i$  with  $j$  and  $k$  separately. For all the above entangled eigenvectors, we can obtain the concurrence of bipartite entanglement  $\mathcal{C}_{1(23)} = 2\sqrt{2}/3$ . Then, the residual entanglement  $\tau = 0$ , which shows all of them are contained in  $W$ -class entangled state. It should be noted that corresponding to every two-dimensional subspaces the linear combinations of two degenerate eigenstates may have different pairwise and residual entanglement. For the linear combinations of product states  $\psi_1$  and  $\psi_2$  instance,  $\psi_1 + \eta e^{i\theta} \psi_2$  are entangled. When, especially  $\eta = e^{i\theta} = 1$ , the state will be the maximally entangled-GHZ state.

### 3 Ground-State Entanglement and Quantum Phase Transition at Zero Temperature

In this section, we discuss the entanglement properties of ground state at temperature  $T = 0$ . The values of parameters  $J$ ,  $\Delta$ , and  $\beta$  determine which of eigenvectors is the ground state. Let us first consider the antiferromagnetic case,  $J > 0$ . In this case, the ground state is two-fold degenerate with eigenvalue

$$E_g = E_5 = E_6 = -(2 + 2\sqrt{3}\beta + \Delta)J. \quad (12)$$

Here, the reduced density matrix of  $\psi_6$  related to  $E_6$  can be obtained with  $x_6 = 0$ ,  $y_6 = 1/3$ ,  $\gamma_6 = \omega_6 = 1/3$ , and  $\mu_6 = -(1 - \sqrt{3}i)/6$ . As shown in Sec. 1, the pairwise entanglement associated with the ground states is  $2/3$ . It is well known that if the ground state is degenerate, the  $T = 0$  ensemble is described by a density matrix that is an equal mixture of contributions from all possible ground states. Therefore, the reduced density matrix of the system in the ground state is given by  $\psi_5$  and  $\psi_6$  with  $x = y = 1/6$ ,  $\gamma = \omega = 1/3$ , and  $\mu = -1/6$ . Obviously,  $\mathcal{C}_g^{(\text{AF})} = 0$ . Here, the superscript AF corresponds to the antiferromagnet. In other words, for antiferromagnetic Heisenberg model, the pairwise ground-state entanglement is not presented. It is interestingly found that two degenerated entangled states contribute zero entanglement to the mixed state of system at temperature  $T = 0$ .

When the value of  $J$  is negative, i.e., in the ferromagnetic case, the ground-state energy is more complicated. When  $\Delta < 1$  and  $\beta < \sqrt{3}$ , the ground state is two-fold degenerate with eigenvalue  $E_g = E_3 = E_4 = (4 - \Delta)J$ . Now, the system is in the equal mixture of  $\psi_3$  and  $\psi_4$ . For the mixed ground state,  $x = y = 1/6$ ,  $\gamma = \omega = \mu = 1/3$ , namely,  $\mathcal{C}_g^{(F)} = 1/3$ . When  $\Delta < 1$  and  $\beta > \sqrt{3}$ , the ground state is two-fold degenerate with eigenvalue  $E_g = E_7 = E_8 = -(2 - 2\sqrt{3}\beta + \Delta)J$ . For this case, the

element of reduced density matrix related to the mixed ground state is  $x = y = 1/6$ ,  $\mu = -1/6$ , and  $\gamma = \omega = 1/3$ , so  $\mathcal{C}_g^{(F)} = 0$ .

When  $\Delta > 1$  and  $\beta < (2\Delta + 1)/\sqrt{3}$ , the ground state energy is  $E_g = E_1 = E_2 = 3\Delta J$  with two-fold degenerated product states  $\psi_1$  and  $\psi_2$ . Obviously, the mixed ground state is separable. For  $\beta > (2\Delta + 1)/\sqrt{3}$ ,  $E_g = E_7 = E_8$ , the state of system at  $T = 0$  is an equal mixture of entangled states  $\psi_7$  and  $\psi_8$ . However, the concurrence of pairwise entanglement of the mixed ground state is zero, too.

Moreover, for a quantum ensemble at  $T = 0$ , a singular quantum phenomenon—quantum phase transition (QPT) may be found due to small variations in a given external parameter of interaction Hamiltonian between subsystems. In our case, we clearly find that the ground states  $\psi_5$  and  $\psi_6$  related to eigenvalues  $E_5$  and  $E_6$  for antiferromagnet will not change with the variation of external parameters, which implies that no quantum phase transition occurs. As discussed above, when  $J < 0$ , the ground energy level varies with the change of the parameters  $\Delta$  and  $\beta$ . When  $\Delta < 1$ , increasing three-spin coupling strength over the point  $\beta = \sqrt{3}$  changes the ground-state energy from  $E_3 = E_4$  to  $E_7 = E_8$ . Thus  $\beta = \sqrt{3}$  is a quantum critical point, at which a quantum phase transition occurs. For the case of  $\Delta > 1$ , quantum phase transition occurs at  $(2\Delta + 1)/\sqrt{3}$ . Furthermore, for some values of  $\beta$ , when the anisotropic constant  $\Delta$  varies over the special point  $\Delta = 1$ , the ground state energy was described by  $E_1$ ,  $E_3$  and  $E_7$ , respectively. Therefore, quantum phase transition will occur at critical point  $\Delta = 1$ . It should be noted that when  $\beta > \max\{\sqrt{3}, (2\Delta + 1)/\sqrt{3}\}$  the ground state energy is always  $E_7 = E_8$ , then, no quantum phase transition occurs.

### 4 Thermal Entanglement at Finite Temperature

In the following, we discuss properties of a natural entanglement associated with thermal equilibrium state of the three-spin system at finite temperature. At thermal equilibrium of  $T$ , quantum state of the three-spin system is a mixture of all the eigenvectors noted above with probability related to every level, respectively. The density matrix of the thermal state can be expressed as

$$\rho_{123} = \frac{1}{Z} \sum_{i=1}^8 e^{-E_i/kT} |\psi_i\rangle\langle\psi_i|, \quad (13)$$

where  $Z$  is the partition function with  $Z = \text{Tr}(e^{-\mathcal{H}/kT})$ ,  $k$  denotes the Boltzmann constant and  $k \equiv 1$  is assumed herein. Therefore, the two-qubit reduced matrix is given by  $\rho = (1/Z) \sum_{i=1}^8 e^{-E_i/T} \text{Tr}_3(|\psi_i\rangle\langle\psi_i|)$ . The matrix elements  $x$ ,  $y$ ,  $\mu$ ,  $\gamma$ , and  $\omega$  of the reduced thermal matrix

are

$$\begin{aligned} x = y &= \frac{1}{3Z}(3e^{-E_1/T} + e^{-E_3/T} + e^{-E_5/T} + e^{-E_7/T}), \\ \gamma = \omega &= \frac{2}{3Z}(e^{-E_3/T} + e^{-E_5/T} + e^{-E_7/T}), \\ \mu &= \frac{1}{3Z}(2e^{-E_3/T} - e^{-E_5/T} - e^{-E_7/T}). \end{aligned} \quad (14)$$

The pairwise thermal entanglement between two qubits in the system can be calculated as

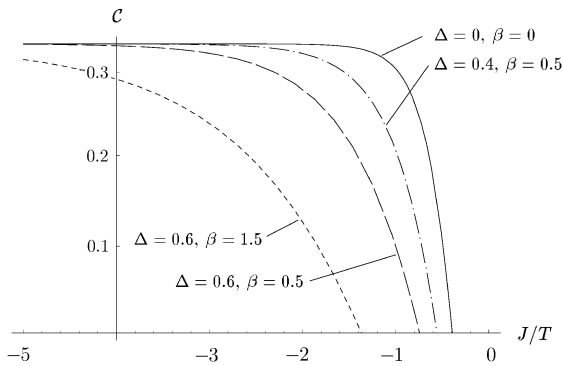
$$C_T = \frac{2}{Z} \max \left\{ 0, -e^{-E_1/T} + \frac{1}{3}e^{-E_3/T} - \frac{2}{3}e^{-E_5/T} - \frac{2}{3}e^{-E_7/T} \right\}. \quad (15)$$

For the antiferromagnetic case of  $J > 0$ , from the above equation, we obtain

$$\begin{aligned} & -e^{-E_1/T} + \frac{1}{3}e^{-E_3/T} - \frac{2}{3}e^{-E_5/T} - \frac{2}{3}e^{-E_7/T} \\ &= e^{\Delta J/T} \left( \frac{1}{3}e^{-4J/T} - e^{-4\Delta J/T} - \frac{2}{3}e^{2J/T} [e^{2\sqrt{3}\beta J/T} + e^{-2\sqrt{3}\beta J/T}] \right) \\ &< e^{\Delta J/T} \left( \frac{1}{3}e^{-4J/T} - e^{-4\Delta J/T} - \frac{2}{3}e^{2J/T} \right) \\ &< e^{\Delta J/T} \left( \frac{1}{3} - e^{-4\Delta J/T} - \frac{2}{3} \right) \\ &< 0. \end{aligned} \quad (16)$$

Here, we have used inequalities  $e^{2J/T} > 1$  and  $e^{2\sqrt{3}\beta J/T} > 1$ . Therefore, for the antiferromagnetic Heisenberg spin chain, we have  $C_T^{(AF)} = 0$  irrespective of  $\Delta$  and  $\beta$ .

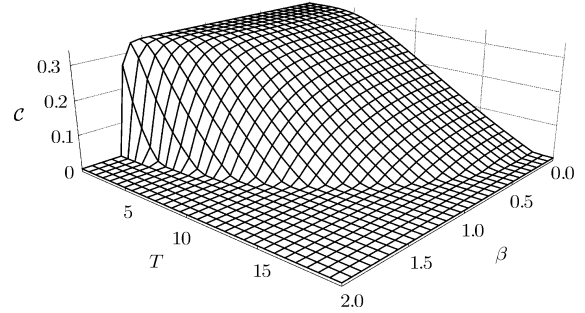
When  $J < 0$ , the inequality (16) will not always hold. The pairwise entanglement may be presented. The variation of pairwise thermal entanglement of the system with parameters  $J, T$  for different  $\Delta$  and  $\beta$  is illustrated in Fig. 1. At fixed  $\Delta$  and  $\beta$ , the concurrence of pairwise thermal entanglement increases as the value of  $|J|/T$  increases.



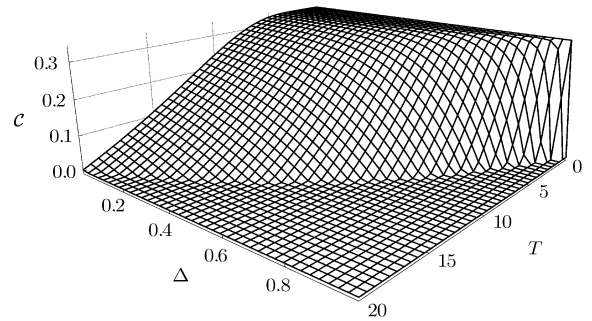
**Fig. 1** The pairwise thermal entanglement with  $J/T$  for different  $\Delta$  and  $\beta$ .

When  $\Delta \geq 1$ , we have  $-e^{-E_1/T} + (1/3)e^{-E_3/T} \leq 0$ . In this case, we can easily check that  $C_T^{(F)} = 0$ . Similarly, one

can establish that when  $\beta \geq \sqrt{3}$ , the above inequality (16) holds. Therefore, no thermal entanglement was present for ferromagnetic Heisenberg model with anisotropic constant  $\Delta \geq 1$  and three-spin interaction  $\beta \geq \sqrt{3}$ . For investigating effects of more general values of the anisotropy and three-spin interaction parameters, we need to resort to numerical calculation. Figure 2 is a plot of the concurrence as a function of  $\beta$  and  $T$ , and as a function of  $\Delta$  and  $T$  in Fig. 3. From them we see that the concurrence of pairwise thermal entanglement decreases as  $\Delta$  and  $\beta$  increase, and maximum entanglement is attained when  $\Delta = \beta = 0$ . It follows that the anisotropy and three-spin coupling permit one to obtain entanglement at higher  $|J|/T$  than it is possible in  $XX$  Heisenberg model. From this point, the interaction along  $x$  and  $y$  orientation in Hamiltonian (2) acts as a most effective *entangler*, the coupling along  $z$ -direction and three-spin interaction, however, acts as a *disentangler*.



**Fig. 2** The pairwise thermal entanglement with  $J = -10$  and  $\Delta = 0.4$ .



**Fig. 3** The pairwise thermal entanglement with  $J = -10$  and  $\beta = 1.0$ .

At special points ( $T = 0, \Delta = 1$ ) and ( $T = 0, \beta = \sqrt{3}$ ), the pairwise thermal entanglement undergoes a sudden change due to the variation of parameters  $\Delta$  and  $\beta$ . As discussed above, these special points are critical points,

at which quantum phase transition occurs. When  $\Delta < 1$  and  $\beta < \sqrt{3}$  at  $T = 0$ , the concurrence of pairwise entanglement associated with the mixture of  $\psi_3$  and  $\psi_4$  is  $1/3$ . When these parameters are adjusted to be larger than quantum critical points, other phases of ground state exhibit no thermal entanglement with  $\mathcal{C}_g = 0$ .

At finite temperature, the mixed state in thermal equilibrium is composed of different eigenvectors corresponding to every energy levels. As noted in Sec. 3, the concurrence of pairwise entanglement associated with mixed degenerate eigenstates corresponding to various energy levels is different. Corresponding to  $E_3 = E_4$ , for example, the pairwise thermal entanglement associated with the mixture of  $\psi_3$  and  $\psi_4$  is  $\mathcal{C} = 1/3$ , and to  $E_5 = E_6$  is  $\mathcal{C} = 0$ . Therefore, the pairwise entanglement in thermal state is contributed by these components with probability related to energy level. However, we find that when  $\Delta \geq 1$  and  $\beta \geq \sqrt{3}$  for ferromagnet, the concurrence of thermal entanglement is zero. In Ref. [26], it has been shown that entanglement of formation of a mixed state  $\rho$  is the minimum average entanglement of an ensemble of pure states that represents  $\rho$ . The concurrence of thermal entanglement is given by the minimum average of concurrence of entanglement associated with all the eigenvectors. Thus, even some of components in mixed state are entangled, the mixed state may be not entangled. In our case, the minimum average concurrence reaches zero for some  $\Delta$  and  $\beta$  at finite temperature. Moreover, the properties of the mixed state of the system in thermal equilibrium at finite temperature will irrelevant to the choice of basis for every degenerate subspaces. Thus, the thermal entanglement of the system will not be affected by the arbitrary choice of eigenstate bases.

One can define a critical temperature  $T_c$  beyond which the pairwise entanglement disappears. The critical temperature  $T_c$  is derived from  $|\mu| - \sqrt{xy} = 0$  in Eq. (11). In our case, the critical temperature is a solution of the equation

$$-e^{-E_1/T} + \frac{1}{3}e^{-E_3/T} - \frac{2}{3}e^{-E_5/T} - \frac{2}{3}e^{-E_7/T} = 0. \quad (17)$$

Obviously, the critical temperature  $T_c$  is a function of parameters  $J$ ,  $\Delta$ , and  $\beta$ . When  $\Delta = \beta = 0$ , corresponding to three-spin  $XX$  Heisenberg model,<sup>[25]</sup> we obtain that the critical temperature is determined by  $4e^{6J/T} + 3e^{4J/T} - 1 = 0$ . Thus,  $T_c = 2.542\ 66|J|$ , above that there is no thermal entanglement. Maximal entanglement is attained when  $J \rightarrow \infty$  or  $T \rightarrow 0$ . From Fig. 1, as  $\Delta$  and  $\beta$  increase the value of  $|J|/T_c$  corresponding to  $\mathcal{C}_T^{(F)} = 0$  increases, then, the critical temperature  $T_c$  decreases. When  $\Delta \rightarrow 1$  or  $\beta \rightarrow \sqrt{3}$ , the critical temperature trends to zero.

## 5 Conclusions

In summary, in this paper we have performed explicitly study on entanglement properties of a three-spin  $XXZ$  Heisenberg chain with three-spin interactions. We have investigated pairwise entanglement and residual entanglement of all eigenstates of system Hamiltonian, and have shown that all entangled eigenstates are contained in  $W$  class. We have further analyzed ground-state pairwise entanglement and quantum phase transition at zero temperature. For ground state of the spin chain, no ground-state entanglement and quantum phase transition are exhibited in antiferromagnetic model, and ground-state entanglement is affected by anisotropic interaction and three-spin interaction in the ferromagnet case. We have shown that, in particular, quantum phase transition takes place at  $\Delta = 1$  for anisotropic interaction and at some values of three-spin coupling strength in ferromagnetic model. We also have discussed thermal entanglement associated with the three-spin chain at finite temperature. For the antiferromagnetic spin chain, there exhibits no pairwise thermal entanglement. The concurrence of pairwise entanglement of two nearest-neighbor ferromagnetic spins increases as the value of  $J/T$  increases and anisotropic interaction and three-spin interaction decrease. In addition, we have found that the critical temperature, beyond which the thermal entanglement disappears, is a function of  $J$ ,  $\Delta$ , and  $\beta$ , and increasing  $\Delta$  and  $\beta$  will decrease this critical temperature.

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