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Euler Characteristic and Topological Phase Transition of NUT-Kerr-Newman Black Hole*

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Abstract *From the Gauss–Bonnet–Chern theorem, the Euler characteristic of NUT-Kerr-Newman black hole is calculated to be some discrete numbers from 0 to 2. We find that the Bekenstein–Hawking entropy is the largest entropy in topology by taking into account of the relationship between the entropy and the Euler characteristic. The NUT-Kerr-Newman black hole evolves from the torus-like topological structure to the spherical structure with the changes of mass, angular momentum, electric and NUT charges. In this process, the Euler characteristic and the entropy are changed discontinuously, which give the topological aspect of the first-order phase transition of NUT-Kerr-Newman black hole. The corresponding latent heat of the topological phase transition is also obtained. The estimated latent heat of the black hole evolving from the star just lies in the range of the energy of gamma ray bursts.*

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The Euler characteristic is one of the topological invariants for black holes. Its study is of great importance and of great current interest for its relation with the entropy of black hole.[1,2] In decades, the Euler characteristics of various black holes have been calculated as 2 or 0. In Refs. [2] ∼ [5], the authors performed the direct calculations for the Euler characteristics of four-dimensional spherically symmetric black holes. The Euler characteristics for Kerr and Kerr–Newman black holes were obtained to be 2 as well as other four-dimensional rotating black holes,[6] such as those that appear in heterotic string theory.[7−10] However, since the topology of NUT charged black hole is changed dramatically by some special kinds of singularities, such as the Misner strings, the concrete result for the Euler characteristic seems missing from the literature. In this paper, we investigate the Euler characteristic of the NUT-Kerr–Newman black hole from the Gauss–Bonnet–Chern theorem and show that the Euler characteristic is some discrete numbers from 0 to 2. We also discuss the aspect of topological phase transition of NUT-Kerr-Newman black hole, which can be generalized to other black holes straightforwardly.

First of all, we use the Gauss–Bonnet–Chern theorem to study the Euler characteristic. For a closed N(even)-dimensional Riemannian manifold M^N, the Euler characteristic $\chi(M^N)$ can be expressed as the volume integral of the Gauss–Bonnet–Chern differential N-form Λ

$$\chi(M^N) = \int_{M^N} \Lambda \,, \tag{1}$$

$$\Lambda = \frac{(-1)^{N/2}}{2^N \pi^{N/2} (N/2)!} \epsilon_{a_1 a_2 \cdots a_{N-1} a_N} R^{a_1 a_2} \wedge \cdots \wedge R^{a_{N-1} a_N} \,, \tag{2}$$

where R^{ab} is the curvature 2-form of M^N. Here the integral area of the density Λ is taken to be the area outside the horizon, because the observer outside the horizon cannot obtain the information for the division of the area inside the horizon. Therefore the manifold of a black hole is treated as a compact manifold, we do not consider again the boundary corrections of the Euler characteristic on the horizons.[6] In the opinion of decomposition of gauge potential, the N-form Λ can be formulated as[11]

$$\Lambda = \frac{1}{(N-1)! A(S^{N-1})} \epsilon^{\mu_1 \cdots \mu_N} \epsilon_{a_1 \cdots a_N} \times \partial_{\mu_1} n^{a_1} \cdots \partial_{\mu_N} n^{a_N} \, \mathrm{d}^N x \,, \tag{3}$$

where $A(S^{N-1})$ is the area of S^{N-1} and $n^a(x)$ is a section of the sphere bundle $S(M^N)$ of M^N, i.e. a unit tangent vector field over M^N satisfying

$$n^a(x) n^a(x) = 1 \qquad (a = 1, 2, \ldots, N) \,. \tag{4}$$

For a black hole manifold, since there exists the Killing vector field $\phi^a(x)$, which is tangent to the spacetime, $n^a(x)$ can be taken as the normalization of the Killing vector field

$$n^a(x) = \frac{\phi^a(x)}{\|\phi(x)\|} \,, \quad \|\phi(x)\|^2 = \phi^a(x) \phi^a(x) \,. \tag{5}$$

Applying the ϕ-mapping method,[11] we obtain the δ-function form of Λ

$$\Lambda = \delta^N(\phi) J\Big(\frac{\phi}{x}\Big) \mathrm{d}^N x \,, \tag{6}$$

where $J(\phi/x)$ is the Jacobian determinant of $\phi^a(x)$. It is obvious that $\Lambda \neq 0$ only when $\phi(x) = 0$. So the consideration of the zeros of $\phi^a(x)$ in detail is necessary. Suppose the Killing vector field possesses l zeros and let the i-th zero be

$$x^\mu = z_i^\mu \quad (\mu = 1, 2, \ldots, N, \quad i = 1, 2, \ldots, l) \,. \tag{7}$$

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The δ-function $\delta^N(\phi)$ can be expanded by these zeros as[11]

$$\delta^N(\phi)=\sum_{i=1}^{l}\frac{\beta_i}{|J(\phi/x)_{z_i}|}\delta\left(x^1-z_i^1\right)\cdots\delta\left(x^N-z_i^N\right)\,, \quad (8)$$

where the positive β_i is called the Hopf index of ϕ-mapping at z_i and it means that, when the spacetime point x covers the neighborhood of z_i one time, the function $\phi(x)$ covers the corresponding region β_i times, which is a topological number of the first Chern class and relates to the localized winding number of ϕ-mapping. Substituting Eq. (8) into Eq. (6) gives

$$\Lambda=\sum_{i=1}^{l}\beta_i\eta_i\delta\left(x^1-z_i^1\right)\cdots\delta\left(x^N-z_i^N\right)\mathrm{d}^Nx\,. \quad (9)$$

Here η_i is called the Brouwer degree of ϕ-mapping at z_i[11] and $\eta_i=\text{sign }J(\phi/x)|_{z_i}=\pm1$ according to the clockwise or anti-clockwise rotation of $\phi(x)$ when x covers z_i clockwise. So, from Eq. (1), the Euler characteristic of black hole manifold is determined by the sum of these Hopf indices and Brouwer degrees of the Killing vector field at its zeros, i.e.

$$\chi=\sum_{i=1}^{l}\beta_i\eta_i\,, \quad (10)$$

which is just the content of the Hopf index theorem.

Next, we calculate the Euler characteristic of the NUT-Kerr-Newman black hole from Eq. (10). The metric of NUT-Kerr-Newman black hole in Boyer–Lindquist coordinate is given by

$$\begin{aligned}\mathrm{d}s^2=&\frac{\Delta}{\rho^2}\left[\mathrm{d}t-\left(a-\frac{(n+a\cos\theta)^2}{a}\right)\mathrm{d}\varphi\right]^2\\&+\frac{\rho^2}{\Delta}\mathrm{d}r^2+\rho^2\mathrm{d}\theta^2\\&+\frac{\sin^2\theta}{\rho^2}\left[(r^2-a^2)\mathrm{d}\varphi+a\,\mathrm{d}t\right]^2\,,\end{aligned} \quad (11)$$

where

$$\begin{aligned}\Delta&=r^2+a^2+Q^2-n^2-2Mr\,,\\\rho^2&=r^2-(n+a\cos\theta)^2\,,\end{aligned} \quad (12)$$

in which M, a, Q, and n are the mass, the angular momentum per unit mass, the electric and the NUT charges, respectively. The NUT-Kerr–Newman black hole has the event horizon r_+ and the Cauchy horizon r_- at

$$r_\pm=M\pm\sqrt{M^2+n^2-a^2-Q^2}\,. \quad (13)$$

The metric in Eq. (11) corresponds to the following vielbein 1-forms:

$$\begin{aligned}e^0&=\frac{\sqrt{\Delta}}{\rho}\left[\mathrm{d}t-\left(a-\frac{(n+a\cos\theta)^2}{a}\right)\mathrm{d}\varphi\right]\,,\\e^1&=\frac{\rho}{\sqrt{\Delta}}\mathrm{d}r,\qquad e^2=\rho\mathrm{d}\theta\,,\\e^3&=\frac{\sin\theta}{\rho}[(r^2-a^2)\mathrm{d}\varphi+a\,\mathrm{d}t]\,.\end{aligned} \quad (14)$$

In terms of the static and the axisymmetric properties of the NUT-Kerr–Newman black hole

$$\partial_0\phi^\mu=\partial_3\phi^\mu=0,\quad \mu=0,1,2,3\,, \quad (15)$$

and the null vector property of the Killing vector field at the event horizon

$$\begin{aligned}\phi^a\phi^a|_{r=r_+}&=e^a_\mu e^a_\nu\phi^\mu\phi^\nu|_{r=r_+}\\&=g_{\mu\nu}\phi^\mu\phi^\nu|_{r=r_+}=0\,,\end{aligned} \quad (16)$$

one can solve the Killing equation and obtain the Killing vector field $\phi^a(x)$,

$$\begin{aligned}\phi^0&=\frac{\sqrt{\triangle}}{a}[(n+a\cos\theta)^2-r_+^2]\,,\\\phi^1&=0,\qquad \phi^2=0\,,\\\phi^3&=(r^2-r_+^2)\sin\theta\,.\end{aligned} \quad (17)$$

Setting $\phi^0=0$ and $\phi^3=0$, we finally have 13 cases of zeros under the different conditions of the parameters n, a, and r_+ (for details see Table 1). The distributions of the Killing vector field around these zeros are shown in Fig. 1, in which the Hopf indices and the Brouwer degrees are also indicated.

Then, from Eq. (10) we get the Euler characteristics of the 13 cases in Table 1. This displays clearly that the Euler characteristics are some discrete numbers 0, 1/2, 3/2, and 2. When the parameters n, a, and r_+ are changed, the NUT-Kerr–Newman black hole will evolve from the torus-like topological structure to the spherical structure.

In the following, we discuss the topological aspect of phase transition of the NUT-Kerr–Newman black hole. In Ref. [2], Liberati and Pollifrone presented a new formulation of the Bekenstein–Hawking law of entropy by calculating almost all known gravitational instantons

$$S=\frac{A}{8}\chi\,, \quad (18)$$

where S and A denote the entropy and the area of event horizon of black hole, respectively. From this new formulation and the results in Table 1 we obtain

$$S=0,\quad \frac{A}{16},\quad \frac{3A}{16},\quad \frac{A}{4}\,. \quad (19)$$

This shows that the entropy of NUT-Kerr–Newman black hole varies from 0 to $A/4$ discontinuously, and corresponding to the spherical topological structure with the Euler characteristic $\chi=2$, the Bekenstein–Hawking entropy is the largest entropy in topology. When the parameters M, Q, a, and n of the NUT-Kerr–Newman black hole modulate, the zeros of Killing vector field will split or merge, and then the Euler characteristic and the entropy will also modulate discontinuously. It manifests that the NUT-Kerr–Newman black hole takes place the first-order phase transition which we name the topological phase transition in the present paper. Now, we calculate the latent heat of the topological first-order phase transition. From Eq. (19) we get

$$\Delta S=\frac{A}{16},\quad \frac{A}{8},\quad \frac{3A}{16},\quad \frac{A}{4}\,. \quad (20)$$

By the area A of event horizon and the temperature T of Hawking radiation

$$A=4\pi(r_+^2+n^2+a^2)\,, \quad (21)$$

$$T=\frac{\kappa}{2\pi}=\frac{1}{2\pi}\cdot\frac{r_+-M}{r_+^2+n^2+a^2}\,, \quad (22)$$

the latent heat L of the topological first-order phase transition is

$$L = \Delta S \cdot T = \frac{1}{8}\sqrt{M^2+n^2-Q^2-a^2}, \quad \frac{1}{4}\sqrt{M^2+n^2-Q^2-a^2}\,, \quad \frac{3}{8}\sqrt{M^2+n^2-Q^2-a^2}, \quad \frac{1}{2}\sqrt{M^2+n^2-Q^2-a^2}\,. \tag{23}$$

Table 1 The Euler characteristics of the 13 cases.

Cases	Conditions	Zeros	Figures	χ
1	$n-a>r_+$	$(r=r_+,\theta=0,\pi),\ r=r_+$	a, e	**0**
2	$n-a>-r_+,\ n-a=r_+$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+,\ \theta=\pi$	d, f	**3/2**
3	$n-a>-r_+,$ $n-a<r_+<n+a$	$(r=r_+,\theta=0,\pi),\ r=r_+,$ $(\theta=\arccos\frac{-n+r_+}{a},\ r=r_+)$	b, g, k	**2**
4	$n-a>-r_+,\ n+a=r_+$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+,\ \theta=0$	c, f	**3/2**
5	$n-a>-r_+,\ n+a<r_+$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+$	a, e	**0**
6	$n-a=-r_+,$ $n-a<r_+<n+a$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+,\ \theta=\pi,$ $(\theta=\arccos\frac{-n+r_+}{a},\ r=r_+)$	d, h, k	**1/2**
7	$n-a=-r_+,$ $n+a=-r_+$	$(r=r_+,\theta=0,\pi),\ r=r_+,$ $\theta=0,\ \theta=\pi$	i, j	**0**
8	$n-a=-r_+,\ n-a<r_+$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+,\ \theta=\pi$	d, f	**3/2**
9	$n-a<-r_+<n+a,$ $n-a<r_+<n+a$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+,$ $(\theta=\arccos\frac{-n\pm r_+}{a},\ r=r_+)$	a, e, k, l	**0**
10	$n-a<-r_+<n+a, (r=r_+,\ \theta=0,\ \pi),$ $n+a=r_+$	$r=r_+,\theta=0,$ $(\theta=\arccos\frac{-n-r_+}{a},\ r=r_+)$	c, h, k	**1/2**
11	$n-a<-r_+<n+a,$ $n+a<r_+$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+,$ $(\theta=\arccos\frac{-n-r_+}{a},\ r=r_+)$	b, g, k	**2**
12	$n+a=-r_+,\ n+a<r_+$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+,\ \theta=0$	c, f	**3/2**
13	$n+a<-r_+$	$(r=r_+,\ \theta=0,\ \pi),\ r=r_+$	a, e	**0**

It is obvious that the latent heat is completely determined by the physical parameters of the NUT-Kerr–Newman black hole and it is interesting that the existence condition of L is just that of event horizon in Eq. (13).

In the following, we will estimate the order of the magnitude of the latent heat of the black hole. For simplicity, we only consider the case when M dominates L, which means the latent heat

$$L \doteq \alpha M, \qquad \alpha = \frac{1}{8}, \quad \frac{1}{4}, \quad \frac{3}{8}, \quad \frac{1}{2}\,. \tag{24}$$

For the normal range of the mass of the black hole 10^{12} kg $\rightarrow$ 10^{39} kg, the corresponding latent heat are $\alpha(6\times10^{35})$ ergs $\rightarrow$ $\alpha(6\times10^{62})$ ergs. For example, the mass of the black hole in the central of the galaxy is 6×10^{36} kg, so the latent heat are $\alpha(5.4\times10^{60})$ ergs. Especially, for the typical black hole evolved from the star whose mass is 10^{31} kg, the latent heat are 7.5×10^{52} ergs, 1.5×10^{53} ergs, 2.25×10^{53} ergs and 3×10^{53} ergs, which just lie in the span of the energy of gamma ray bursts 10^{51} ergs $\rightarrow$ 10^{54} ergs.

So the Euler characteristic of the NUT-Kerr–Newman black hole is some discrete numbers 0, 1/2, 3/2 and 2. By virtue of the relationship between the entropy and the Euler characteristic, this result gives the corresponding entropies 0, $A/16$, $3A/16$, and $A/4$, which show the Bekenstein–Hawking entropy is the largest entropy in topology. When the parameters modulate, the NUT-Kerr-Newman black hole will evolve from the torus-like topological structure to the spherical structure. In this process, the Euler characteristic and the entropy will modulate discontinuously, which leads to the topological first-order phase transition with the latent heat in Eq. (23). The order of the magnitude of the latent heat for the black hole is estimated. Especially the latent heat of the black hole evolved from the star lies in the span of the energy of gamma ray bursts. At last, we would like to point out that the methods presented in this paper, mainly the Eq. (10) and the aspect of topological phase transition, have some universality and can be applied to other black holes straightforwardly.

Appendix: Hopf Indices and Brouwer Degree of Killing Vector Field

As examples, we demonstrate the achievement of some of the Hopf indices, Brouwer degrees and the Euler characteristics in this appendix. In Case 11, the zeros of Killing vector field are $(r=r_+)$, $(r=r_+,\theta=0,\pi)$ and $(r=r_+,\theta=\arccos\frac{-n-r_+}{a})$ which are called the bolt, nut and Misner string in Ref. [12]. The distributions of Killing vector field on the neighborhoods of these zeros are shown in Figs. 1(b), 1(g), and 1(k), where ϵ denotes the radius of the neighborhoods. For the nuts $(r_+, 0)$ and (r_+, π) in Fig. 1(g), in the limit of $\epsilon \longrightarrow 0$, the Killing vector field (ϕ^0, ϕ^3) rotates from 0 to 0 and π to π when the spacetime point (r, θ) circles π angles. So we have the Hopf indices

$$\beta_1 = 0, \qquad \beta_2 = 0\,, \tag{A1}$$

at the nuts $(r_+, 0)$ and (r_+, π). For the Misner string $(r_+, \arccos\frac{-n-r_+}{a})$ in Fig. 1(k), when (r, θ) circles an angle of π clockwise, $\phi^a(x)$ rotates from 0 to π clockwise, which gives the Hopf index and the Brouwer degree

$$\beta_3 = 1, \qquad \eta_3 = +1\,. \tag{A2}$$

Similarly, for the bolt $(r = r_+)$ in Fig. 1(b), when (r, θ) circles from 0 to 2π clockwise, $\phi^a(x)$ rotates from 0 to 2π clockwise too, then

$$\beta_4 = 1, \qquad \eta_4 = +1\,. \tag{A3}$$

So in terms of Eq. (10), we obtain the Euler characteristic of the NUT-Kerr-Newman black hole in Case 11

$$\chi = \sum_{i=1,2,3,4} \beta_i \eta_i = 2\,. \tag{A4}$$

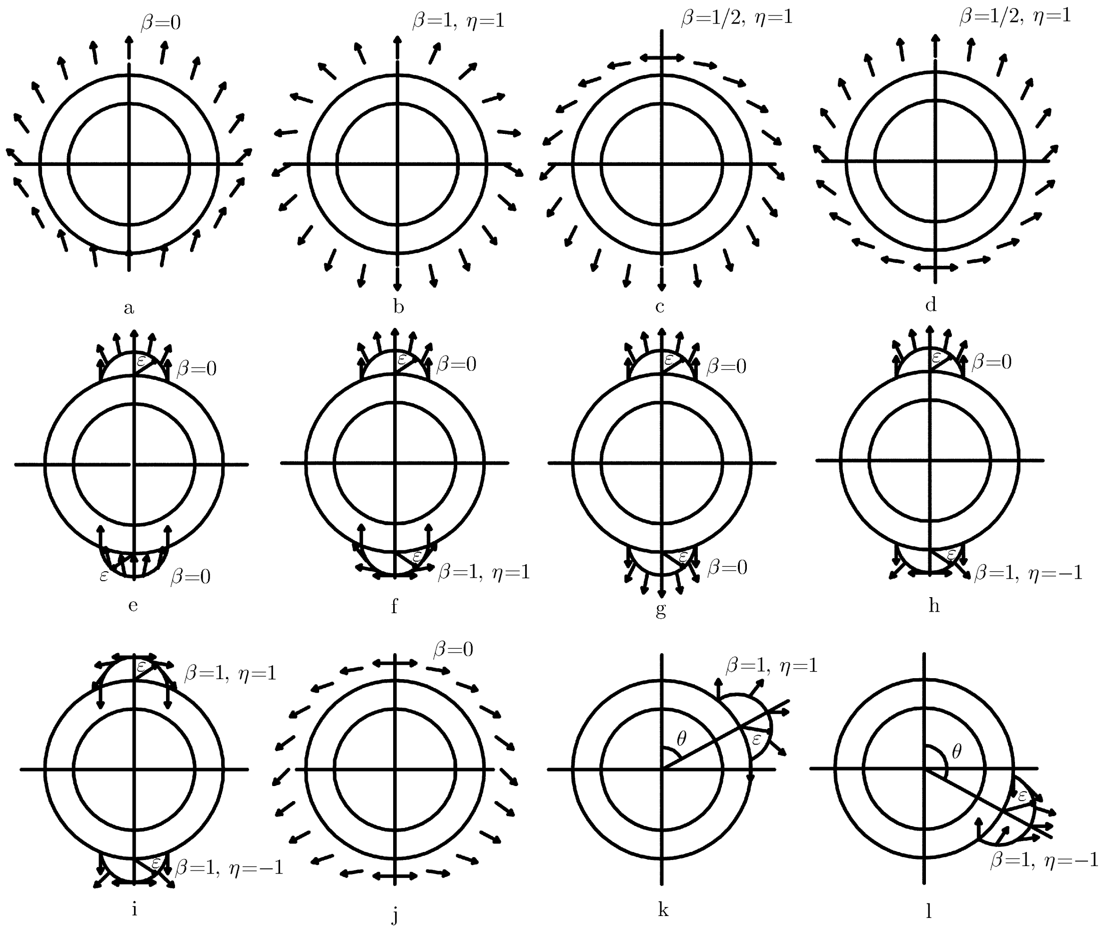

Fig. 1 The distributions of the Killing vector field around the zeros.

References

[1] S.W. Hawking, G.T. Horowtz, and S.F. Ross, Phys. Rev. D **51** (1995) 4302.

[2] S. Liberati and G. Pollifrone, Phys. Rev. D **56** (1997) 6458.

[3] G.W. Gibbons and R.E. Kallosh, Phys. Rev. D **51** (1995) 2839.

[4] S. Liberati, Nuovo Cimento Soc, Ital. Fis. B **112** (1997) 405; G. Pollifrone, Nucl. Phys. B (Proc. Suppl.) **57** (1997) 197.

[5] K. Suzuki, Phys. Rev. D **56** (1997) 7846.

[6] Z.Z. Ma, Phys. Rev. D **67** (2003) 024027.

[7] A. Sen, Phys. Rev. Lett. **69** (1992) 1006.

[8] A. Sen, Nucl. Phys. B **440** (1995) 421.

[9] M. Cvetic and D. Youm, Phys. Rev. D **54** (1996) 2612.

[10] G.T. Horowitz and A. Sen, Phys. Rev. D **53** (1996) 808.

[11] Y.S. Duan, S. Li, and G.H. Yang, Nucl. Phys. B **514** (1998) 705.

[12] G.W. Gibbons and S.W. Hawking, Commun. Math. Phys. **66** (1979) 291.