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Ising Model on an Infinite Ladder Lattice

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Abstract *In this paper we propose an Ising model on an infinite ladder lattice, which is made of two infinite Ising spin chains with interactions. It is essentially a quasi-one-dimessional Ising model because the length of the ladder lattice is infinite, while its width is finite. We investigate the phase transition and dynamic behavior of Ising model on this quasi-one-dimessional system. We use the generalized transfer matrix method to investigate the phase transition of the system. It is found that there is no nonzero temperature phase transition in this system. At the same time, we are interested in Glauber dynamics. Based on that, we obtain the time evolution of the local spin magnetization by exactly solving a set of master equations.*

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Key words: Ising model, general transfer matrix, phase transition, Glauber dynamics, critical slow down

1 Introduction

As we know, the one-dimensional Ising model with nearest neighbor interactions in a most simple spin model, the partition function of which has been exactly calculated by spin variables transformation and transfer matrix methods. It has been found that there is no finite temperature phase transition, namely, the critical temperature is zero. Since then various variations of the model were investigated, such as one-dimensional long-range interaction Ising mode,[1−3] non-period Ising chain, alternate interactions linear Ising chain,[4,5] and so on.

On the irreversible dynamical problem, Glauber[6] and Kawasaki[7] have respectively investigated dynamical behaviour of Ising system under un-conservation and conservation condition of order parameter. In this paper, we only focus on Glauber dynamics. Glauber assumed that only single-spin-flip is allowed each time in the process of evolution, he then exactly solved the evolution equation — master equation, and successfully obtained the solution exhibiting the critical slowing down phenomena. In the last several decades the Glauber dynamics was extensively extended to one-dimensional system with different kinds of interactions, such as nonperiodic Ising chains[8] and alternating linear chains.[9] Droz, Kamphorst Leal da Silva, Malaspina (DKM) investigated the critical dynamics of an Ising chain with two different near-neighbor interaction strength.[10] Z.R. Yang[11] considered one-dimensional Ising model with not only nearest-neighbor but also next nearest-neighbor interactions in the absence of external field. The higher-dimensional Ising system was extensively investigated.[12] Very recently, the Glauber dynamics was also extended to Gauss spin system.[13] However, for higher-dimensional spin system, the solution of master equation cannot be exact, this is because the evolution equation becomes non-linear. Usually it is solved by decoupling approximation methods,[14] renormalization-group methods,[15] and computer simulations.[16−19]

In this paper we investigate an Ising model on an infinite ladder lattice. The length of this ladder lattice is infinite while the width of the ladder is finite. Thus it essentially is a quasi-one-dimensional lattice. Our purpose is to study the phase transition and dynamical behaviour of Ising model on such a system.

This paper is organized as follows. In Sec. 2, we describe the model and investigate equilibrium phase transition. In Sec. 3, we study Glauber dynamics, and give the expression of time evolution of local magnetization. Finally, in Sec. 4, a short conclusion is presented.

2 Model and Phase Transition

We now consider the Ising model on an infinite ladder lattice, which is constructed with two parallel lattice chains and $2(2N+1)$ spins located on the lattice sites, as shown in Fig. 1.

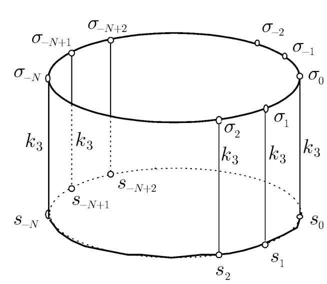

Fig. 1 The infinite ladder Ising system and periodic boundary condition $\sigma_{-N}=\sigma_{N+1}, s_{-N}=s_{N+1}$.

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Ising spin σ_i and s_i $(i = -N, \ldots, N)$ take the values ± 1 only. Suppose the Hamiltonian of the system is

$$-\beta H = k_1 \sum_{i=-N}^{N} \sigma_i \sigma_{i+1} + k_2 \sum_{i=-N}^{N} s_i s_{i+1} + k_3 \sum_{i=-N}^{N} \sigma_i s_i \,, \tag{1}$$

where $\beta = 1/k_B T$, k_B is Boltzmann constant, and T temperature, $k_i = \beta J_i$ $(i = 1, 2, 3)$ are interaction parameters, in which J_i represents the exchange integral between nearest neighbor particles. For ferromagnetic model, $J_i > 0$, k_1 and k_2 respectively indicate interactions between spins on the same chain, and k_3 interactions between spins on the different chain.

To investigate phase transition property of the system, we firstly calculate the partition function of the model Hamiltonian (1). As usual, the partition function is defined as

$$Z_N = \sum_{\{\sigma_i, s_i\}} \exp \sum_{i=-N}^{N} (k_1 \sigma_i \sigma_{i+1} + k_2 s_i s_{i+1} + k_3 \sigma_i s_i) \,. \tag{2}$$

By introducing the following 4×4 transfer matrix:

$$\begin{aligned} f(\sigma, s, \sigma', s') &= \exp\Big[k_1 \sigma\sigma' + k_2 s s' + \frac{1}{2} k_3 (\sigma s + \sigma' s')\Big] \\ &= \begin{pmatrix} f_{(1,1)(1,1)} & f_{(1,1)(1,-1)} & f_{(1,1)(-1,1)} & f_{(1,1)(-1,-1)} \\ f_{(1,-1)(1,1)} & f_{(1,-1)(1,-1)} & f_{(1,-1)(-1,1)} & f_{(1,-1)(-1,-1)} \\ f_{(-1,1)(1,1)} & f_{(-1,1)(1,-1)} & f_{(-1,1)(-1,1)} & f_{(-1,1)(-1,-1)} \\ f_{(-1,-1)(1,1)} & f_{(-1,-1)(1,-1)} & f_{(-1,-1)(-1,1)} & f_{(-1,-1)(-1,-1)} \end{pmatrix} \\ &= \begin{pmatrix} \exp(k_1+k_2+k_3) & \exp(k_1-k_2) & \exp(-k_1+k_2) & \exp(-k_1-k_2+k_3) \\ \exp(k_1-k_2) & \exp(k_1+k_2-k_3) & \exp(-k_1-k_2-k_3) & \exp(-k_1+k_2) \\ \exp(-k_1+k_2) & \exp(-k_1-k_2-k_3) & \exp(k_1+k_2-k_3) & \exp(k_1-k_2) \\ \exp(-k_1-k_2+k_3) & \exp(-k_1+k_2) & \exp(k_1-k_2) & \exp(k_1+k_2+k_3) \end{pmatrix} . \end{aligned} \tag{3}$$

The partition function Z can be rewritten as

$$\begin{aligned} Z_N &= \sum_{\sigma_i, s_i} f(\sigma_{-N} s_{-N} \sigma_{-N+1} s_{-N+1}) \cdots f(\sigma_1 s_1 \sigma_2 s_2) f(\sigma_2 s_2 \sigma_3 s_3) \cdots f(\sigma_N s_N \sigma_{-N} s_{-N}) \\ &= \sum_{\sigma_{-N}, s_{-N}} f^{2N+1}(\sigma_{-N} s_{-N} \sigma_{-N} s_{-N}) = \mathrm{Tr}\, f^{2N+1} = \lambda_1^{2N+1} + \lambda_2^{2N+1} + \lambda_3^{2N+1} + \lambda_4^{2N+1} \,, \end{aligned} \tag{4}$$

where we have considered the periodic boundary condition: $\sigma_{N+1} = \sigma_{-N}$, $s_{N+1} = s_{-N}$, and λ_i $(i = 1, 2, 3, 4)$ are the four eigenvalues of the transfer matrix f, which can be found through solving the following secular equation:

$$\begin{vmatrix} \exp(k_1+k_2+k_3) - \lambda & \exp(k_1-k_2) & \exp(-k_1+k_2) & \exp(-k_1-k_2+k_3) \\ \exp(k_1-k_2) & \exp(k_1+k_2-k_3) - \lambda & \exp(-k_1-k_2-k_3) & \exp(-k_1+k_2) \\ \exp(-k_1+k_2) & \exp(-k_1-k_2-k_3) & \exp(k_1+k_2-k_3) - \lambda & \exp(k_1-k_2) \\ \exp(-k_1-k_2+k_3) & \exp(-k_1+k_2) & \exp(k_1-k_2) & \exp(k_1+k_2+k_3) - \lambda \end{vmatrix} = 0 \,. \tag{5}$$

The explicit expressions of the eigenvalues are given as follows:

$$\begin{aligned} \lambda_1 &= 2\cosh(k_1+k_2)\cosh k_3 + \sqrt{\cosh(2k_1+2k_2)(\cosh 2k_3 - 1) + 2\cosh(2k_1 - 2k_2) + \cosh 2k_3 + 1} \,, \\ \lambda_2 &= 2\cosh(k_1+k_2)\cosh k_3 - \sqrt{\cosh(2k_1+2k_2)(\cosh 2k_3 - 1) + 2\cosh(2k_1 - 2k_2) + \cosh 2k_3 + 1} \,, \\ \lambda_3 &= 2\sinh(k_1+k_2)\cosh k_3 + \sqrt{\cosh(2k_1+2k_2)(\cosh 2k_3 - 1) + 2\cosh(2k_1 - 2k_2) - \cosh 2k_3 - 1} \,, \\ \lambda_4 &= 2\sinh(k_1+k_2)\cosh k_3 - \sqrt{\cosh(2k_1+2k_2)(\cosh 2k_3 - 1) + 2\cosh(2k_1 - 2k_2) - \cosh 2k_3 - 1} \,. \end{aligned} \tag{6}$$

To compare the four eigenvalues, we set

$$\begin{aligned} &A = 2\cosh(k_1+k_2)\cosh k_3 \,, \quad A' = 2\sinh(k_1+k_2)\cosh k_3 \,, \quad B = \cosh(2k_1+2k_2)(\cosh 2k_3 - 1) \,, \\ &C = 2\cosh(2k_1 - 2k_2) \,, \quad D = \cosh 2k_3 + 1 \,. \end{aligned} \tag{7}$$

Thus λ_i $(i = 1, 2, 3, 4)$ can be rewritten as

$$\lambda_1 = A + \sqrt{B + C + D} \,, \quad \lambda_2 = A - \sqrt{B + C + D} \,, \quad \lambda_3 = A' + \sqrt{B + C - D} \,, \quad \lambda_4 = A' - \sqrt{B + C - D} \,. \tag{8}$$

Because of the function $\cosh x \geq 1$, the values of A, B, C, and D are non-negative. The value of A is always larger than A' because of $\cosh x \geq \sinh x$. Comparing the above four expressions of eigenvalues, it is obvious that the eigenvalue λ_1 is the largest one among the four eigenvalues.

In the thermodynamics limit $(N \to \infty)$, the free energy per site can be expressed as follows:

$$F = \lim_{N\to\infty} \Big[\frac{-k_B T}{2(2N+1)} \ln Z_N\Big] = \lim_{N\to\infty} \Big[\frac{-k_B T}{2(2N+1)} \ln(\lambda_1^{2N+1} + \lambda_2^{2N+1} + \lambda_3^{2N+1} + \lambda_4^{2N+1})\Big]$$

$$= \lim_{N\to\infty} \frac{-k_BT}{2(2N+1)} \ln \lambda_1^{2N+1}\Big[1+\Big(\frac{\lambda_2}{\lambda_1}\Big)^{2N+1}+\Big(\frac{\lambda_3}{\lambda_1}\Big)^{2N+1}+\Big(\frac{\lambda_4}{\lambda_1}\Big)^{2N+1}\Big] = \Big(-\frac{k_BT}{2}\Big)\ln\lambda_1\,. \tag{9}$$

Obviously, function F is only related to the largest eigenvalue λ_1.

From Eq. (7), it is very evident that the values of A, B, C, and D are continuous and gradually increased with the increase of k_1, k_2, and k_3. Therefore according to the formula (6), the value λ_1 is also continuously increased with the increase of k_1, k_2, and k_3, and approaches infinite at $k_1 \to \infty$, $k_2 \to \infty$, $k_3 \to \infty$. From Eq. (9), it is clear that the free energy is divergent just only at zero temperature ($T_C = 0$). Therefore we can obtain the conclusion that there is no nonzero temperature phase transition in our model.

If we choose $k_1 = k$, $k_2 = 0$, $k_3 = 0$, from Eq. (6), the four eigenvalues reduce to

$$\lambda_1 = 4\cosh k\,, \quad \lambda_2 = 0\,, \quad \lambda_3 = 4\sinh k\,, \quad \lambda_4 = 0\,, \tag{10}$$

which reproduces precisely the result of one-dimensional Ising chain.[20]

3 Irreversible Dynamics

In this section, we study the time evolution of the system from the nonequilibrium state to the equilibrium state. The evolution originates from interaction between spins and interaction between the system and heat resource. Since the latter is stochastic, the process of evolution is considered as a stochastic one, in which the spin can randomly flip between values ± 1.

3.1 *Master Equation*

Glauber[6] assumed that in the transition process only single spin can change spin value each time. Therefore the transition probability from a spin configuration $\{\sigma_{-N}(t),\ldots\sigma_j(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_j(t),\ldots,s_N(t)\}$ to another one $\{\sigma_{-N}(t),\ldots,-\sigma_j(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_j(t),\ldots,s_N(t)\}$ can be written as $W_j(\sigma_j(t))$, which means that only spin $\sigma_j(t)$ flips, while the others remain unchanged. Now let us suppose the probability distribution function of the system being in configuration $\{\sigma_{-N}(t),\ldots,\sigma_j(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_j(t),\ldots,s_N(t)\}$ at time t is $P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))$, then the time derivative of $P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))$ can be expressed as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))\\
=&-\Big[\sum_j W_j(\sigma_j(t))+W_j(s_j(t))\Big]P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))\\
&+\sum_j W_j(-\sigma_j(t))P(\sigma_{-N}(t),\ldots,-\sigma_j(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))\\
&+\sum_j W_j(-s_j(t))P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,-s_j(t),\ldots,s_N(t))\,,
\end{aligned} \tag{11}$$

where the first and second terms on the right-hand side denote the decrease of probability distribution function $P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))$ per unit time due to transition of spin from $\sigma_j(t)$ to $-\sigma_j(t)$ and from $s_j(t)$ to $-s_j(t)$, respectively, and the third and fourth terms denote the increase of $P(\sigma_1(t),\ldots,\sigma_N(t),s_1(t),\ldots,s_N(t))$ per unit time due to transition of spin from $-\sigma_j(t)$ to $\sigma_j(t)$ and from $-s_j(t)$ to $s_j(t)$, respectively. Equation (11) is usually called master equation.

It is most important that to solve the equation we must firstly find the transition probability $W_j(\sigma_j(t))$ (or $W_j(s_j(t))$). In principle, it can be found by solving quantum many-body problem. However it is so difficult that it cannot be done. Usually, we employ some general principle and argument to determinate the transition probability $W_j(\sigma_j(t))$. They are, for example, $W_j(\sigma_j)$ should be positive and satisfies the detailed balance condition. Based on the above, we can write the following:

$$\frac{W_j(\sigma_j)}{W_j(-\sigma_j)} = \frac{P_e(\sigma_{-N},\ldots,-\sigma_j,\ldots,\sigma_N,s_{-N},\ldots,s_N)}{P_e(\sigma_{-N},\ldots,\sigma_j,\ldots,\sigma_N,s_{-N},\ldots,s_N)} = \frac{\exp[-k_1\sigma_j(\sigma_{j-1}+\sigma_{j+1})-k_3\sigma_js_j]}{\exp[k_1\sigma_j(\sigma_{j-1}+\sigma_{j+1})+k_3\sigma_js_j]}\,, \tag{12}$$

$$\frac{W_j(s_j)}{W_j(-s_j)} = \frac{P_e(\sigma_{-N},\ldots,\sigma_N,s_{-N},\ldots,-s_j,\ldots,s_N)}{P_e(\sigma_{-N},\ldots,\sigma_N,s_{-N},\ldots,s_j,\ldots,s_N)} = \frac{\exp[-k_2s_j(s_{j-1}+s_{j+1})-k_3\sigma_js_j]}{\exp[k_2s_j(s_{j-1}+s_{j+1})+k_3\sigma_js_j]}\,, \tag{13}$$

where $P_e(\sigma_{-N},\ldots,\sigma_j,\ldots,\sigma_N,s_{-N},\ldots,s_N)$ denotes the equilibrium canonical distribution function. However, equations (12) and (13) still do not uniquely determine the transition probability $W_j(\sigma_j)$.

The exponential functions, which occur in the ratios (12) and (13), may be written in the forms:

$$\exp(\pm k_1\sigma_j(\sigma_{j-1}+\sigma_{j+1}) = \cosh[k_1(\sigma_{j-1}+\sigma_{j+1})]\Big\{1\pm\frac{1}{2}\sigma_j(\sigma_{j-1}+\sigma_{j+1})\tanh 2k_1\Big\}\,,$$

$$\exp(\pm k_2s_j(s_{j-1}+s_{j+1}) = \cosh[k_2(s_{j-1}+s_{j+1})]\Big\{1\pm\frac{1}{2}s_j(s_{j-1}+s_{j+1})\tanh 2k_2\Big\}\,,$$

$$\exp(\pm k_3\sigma_j s_j) = (\cosh k_3)(1 \pm \sigma_j s_j \tanh k_3)\,. \tag{14}$$

Similar to what Glauber did we choose $W_j(\sigma_j)$ and $W_j(s_j)$ as

$$W_j(\sigma_j) = \frac{1}{2}\alpha\Big[1 - \frac{1}{2}\delta_1\sigma_j(\sigma_{j-1} + \sigma_{j+1}) - \gamma_3\sigma_j s_j + \frac{1}{2}\delta_1\gamma_3 s_j(\sigma_{j-1} + \sigma_{j+1})\Big]\,, \tag{15}$$

$$W_j(s_j) = \frac{1}{2}\alpha\Big[1 - \frac{1}{2}\delta_2 s_j(s_{j-1} + s_{j+1}) - \gamma_3\sigma_j s_j + \frac{1}{2}\delta_2\gamma_3\sigma_j(s_{j-1} + s_{j+1})\Big] \tag{16}$$

with $\delta_1 = \tanh(2k_1)$, $\delta_2 = \tanh(2k_2)$, and $\gamma_3 = \tanh(k_3)$, where we have used the equality (14), which is valid special for Ising spin and α is an arbitrary constant indicating relaxation time of non-interacting spin system.

We are interested in the evolution of local magnetization (local order parameter). It is defined as

$$\langle\sigma_k(t)\rangle = \sum_{\{\sigma,s\}} \sigma_k(t)P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))\,, \tag{17}$$

$$\langle s_k(t)\rangle = \sum_{\{\sigma,s\}} s_k(t)P(\sigma_{-N}(t),\ldots,\sigma_N(t),s_{-N}(t),\ldots,s_N(t))\,. \tag{18}$$

According to the above definition and the master equation (11), we can derive the time-evolution equations of $\langle\sigma_k(t)\rangle$ and $\langle s_k(t)\rangle$ (see Appendix A),

$$\frac{\mathrm{d}}{\mathrm{d}(\alpha t)}\langle\sigma_k(t)\rangle = -\langle\sigma_k(t)\rangle + \frac{1}{2}\delta_1(\sigma_{k-1}(t) + \sigma_{k+1}(t)) + \gamma_3\langle s_k(t)\rangle - \frac{1}{2}\delta_1\gamma_3(\langle\sigma_k(t)s_k(t)\sigma_{k-1(t)}\rangle + \langle\sigma_k(t)s_k(t)\sigma_{k+1}(t)\rangle)\,, \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}(\alpha t)}\langle s_k(t)\rangle = -\langle s_k(t)\rangle + \frac{1}{2}\delta_2(s_{k-1}(t) + s_{k+1}(t)) + \gamma_3\langle\sigma_k(t)\rangle - \frac{1}{2}\delta_2\gamma_3(\langle\sigma_k(t)s_k(t)s_{k-1}(t)\rangle + \langle\sigma_k(t)s_k(t)s_{k+1}(t)\rangle)\,. \tag{20}$$

They are a set of two coupling equations.

Noting that there are three-spin correlation functions in Eqs. (19) and (20). For getting the solution, we have to employ some approximate technique. Here we use decoupling technique as[11]

$$\begin{aligned}&\langle\sigma_k s_k\sigma_{k-1}\rangle \approx \langle\sigma_k s_k\rangle\langle\sigma_{k-1}\rangle = r_{k,k}\langle\sigma_{k-1}\rangle\,, \quad \langle\sigma_k s_k\sigma_{k+1}\rangle \approx \langle\sigma_k s_k\rangle\langle\sigma_{k+1}\rangle = r_{k,k}\langle\sigma_{k+1}\rangle\,,\\ &\langle\sigma_k s_k s_{k-1}\rangle \approx \langle\sigma_k s_k\rangle\langle s_{k-1}\rangle = r_{k,k}\langle s_{k-1}\rangle\,, \quad \langle\sigma_k s_k s_{k+1}\rangle \approx \langle\sigma_k s_k\rangle\langle s_{k+1}\rangle = r_{k,k}\langle s_{k+1}\rangle\,.\end{aligned} \tag{21}$$

In which $r_{k,k} \equiv \langle\sigma_k s_k\rangle$ is two-spin, located at two different spin chains, correlation function. For simplicity, we approximately use $r^e_{k,k} \equiv \theta_0$, an equilibrium correlation function, instead of $r_{k,k}$, then equations (19) and (20) are simplified as

$$\frac{\mathrm{d}}{\mathrm{d}(\alpha t)}\langle\sigma_k(t)\rangle = -\langle\sigma_k(t)\rangle + \frac{1}{2}\delta'_1(\sigma_{k-1}(t) + \sigma_{k+1}(t)) + \gamma_3\langle s_k(t)\rangle\,, \tag{22}$$

$$\frac{\mathrm{d}}{\mathrm{d}(\alpha t)}\langle s_k(t)\rangle = -\langle s_k(t)\rangle + \frac{1}{2}\delta'_2(s_{k-1}(t) + s_{k+1}(t)) + \gamma_3\langle\sigma_k(t)\rangle\,, \tag{23}$$

where

$$\delta'_1 = \delta_1(1 - \theta_0\gamma_3)\,, \quad \delta'_2 = \delta_2(1 - \theta_0\gamma_3)\,. \tag{24}$$

3.2 *Solutions of Master Equations*

To solve the coupling master equations (22) and (23), let us construct two generating functions,

$$F(\lambda,t) = \sum_{k=-\infty}^{\infty}\lambda^k\langle\sigma_k(t)\rangle\,, \qquad G(\lambda,t) = \sum_{k=-\infty}^{\infty}\lambda^k\langle s_k(t)\rangle\,. \tag{25}$$

According to Eqs. (22) and (23), the two generating functions $F(\lambda,t)$ and $G(\lambda,t)$ satisfy the differential equations,

$$\begin{aligned}\frac{\partial}{\partial(\alpha t)}F(\lambda,t) &= -F(\lambda,t) + \frac{1}{2}\delta'_1(\lambda^{-1} + \lambda)F(\lambda,t) + \gamma_3 G(\lambda,t)\,,\\ \frac{\partial}{\partial(\alpha t)}G(\lambda,t) &= -G(\lambda,t) + \frac{1}{2}\delta'_2(\lambda^{-1} + \lambda)G(\lambda,t) + \gamma_3 F(\lambda,t)\,.\end{aligned} \tag{26}$$

Let us define

$$\gamma_1 = -1 + \frac{1}{2}\delta'_1(\lambda^{-1} + \lambda)\,, \quad \gamma_2 = -1 + \frac{1}{2}\delta'_2(\lambda^{-1} + \lambda)\,. \tag{27}$$

The differential equations (26) can be simplified as

$$\frac{\partial}{\partial(\alpha t)}F(\lambda,t) = \gamma_1 F(\lambda,t) + \gamma_3 G(\lambda,t)\,, \quad \frac{\partial}{\partial(\alpha t)}G(\lambda,t) = \gamma_2 G(\lambda,t) + \gamma_3 F(\lambda,t)\,, \tag{28}$$

which are typical linear and homogeneous differential equations. Evidently, we can obtain their solutions as follows:

$$F(\lambda,t) = F(\lambda,0)\frac{\beta_2 - \gamma_1}{\beta_2 - \beta_1}\exp(\beta_1\alpha t) + G(\lambda,0)\frac{\gamma_3}{\beta_1 - \beta_2}\exp(\beta_1\alpha t)$$

$$+ F(\lambda,0)\frac{\beta_1-\gamma_1}{\beta_1-\beta_2}\exp(\beta_2\alpha t) + G(\lambda,0)\frac{\gamma_3}{\beta_2-\beta_1}\exp(\beta_2\alpha t)\,,$$

$$G(\lambda,t) = F(\lambda,0)\frac{(\beta_1-\gamma_1)(\beta_2-\gamma_1)}{(\beta_2-\beta_1)\gamma_3}\exp(\beta_1\alpha t) + G(\lambda,0)\frac{\beta_1-\gamma_1}{\beta_1-\beta_2}\exp(\beta_1\alpha t)$$
$$+ F(\lambda,0)\frac{(\beta_1-\gamma_1)(\beta_2-\gamma_1)}{(\beta_1-\beta_2)\gamma_3}\exp(\beta_2\alpha t) + G(\lambda,0)\frac{\beta_2-\gamma_1}{\beta_2-\beta_1}\exp(\beta_2\alpha t)\,, \tag{29}$$

where $F(\lambda,0)$ and $G(\lambda,0)$ are respectively the initial values of $F(\lambda,t)$ and $G(\lambda,t)$, and β_1 and β_2 are respectively as follows:

$$\beta_1 = \frac{\gamma_1+\gamma_2+2\gamma_3\sqrt{1+[(\gamma_1-\gamma_2)^2/4\gamma_3^2]}}{2}\,,\quad \beta_2 = \frac{\gamma_1+\gamma_2-2\gamma_3\sqrt{1+[(\gamma_1-\gamma_2)^2/4\gamma_3^2]}}{2}\,. \tag{30}$$

If we consider the case that k_1 is close to k_2 , and thus $(\gamma_1-\gamma_2)/2\gamma_3 \ll 1$, then the expression (30) can be expressed approximately as

$$\beta_1 = \frac{\gamma_1+\gamma_2}{2} + \gamma_3 + \frac{(\gamma_1-\gamma_2)^2}{8\gamma_3}\,,\quad \beta_2 = \frac{\gamma_1+\gamma_2}{2} - \gamma_3 - \frac{(\gamma_1-\gamma_2)^2}{8\gamma_3}\,. \tag{31}$$

We note that each term on the right-hand side of expression (29) can be expressed in virtue of the Bessel functions of the imaginary argument. The detailed process is given in Appendix B.

Then the four terms of $F(\lambda,t)$ in the expression (29) can be written respectively as follows:

$$F(\lambda,0)\frac{\beta_2-\gamma_1}{\beta_2-\beta_1}\exp(\beta_1\alpha t) = -\frac{1}{2}F(\lambda,0)\exp[(-1+\gamma_3+\eta)\alpha t]$$
$$\times\sum_{k,m}\lambda^k\,\mathrm{I}_m(\eta\alpha t)\Big\{-\mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\gamma_-}{2\gamma_3}[\,\mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)]\Big\}\,, \tag{32}$$

$$G(\lambda,0)\frac{\gamma_3}{\beta_1-\beta_2}\exp(\beta_1\alpha t) = \frac{1}{2}G(\lambda,0)\exp[(-1+\gamma_3+\eta)\alpha t]$$
$$\times\sum_{k,m}\lambda^k\,\mathrm{I}_m(\eta\alpha t)\Big\{(1-\eta)\,\mathrm{I}_{k-2m}(\gamma_+\alpha t) - \frac{\eta}{2}[\mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)]\Big\}\,, \tag{33}$$

$$F(\lambda,0)\frac{\beta_1-\gamma_1}{\beta_1-\beta_2}\exp(\beta_2\alpha t) = \frac{1}{2}F(\lambda,0)\exp[(-1-\gamma_3-\eta)\alpha t]$$
$$\times\sum_{k,m}\lambda^k\,\mathrm{I}_m(-\eta\alpha t)\Big\{\mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\gamma_-}{2\gamma_3}[\,\mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)]\Big\}\,, \tag{34}$$

$$G(\lambda,0)\frac{\gamma_3}{\beta_2-\beta_1}\exp(\beta_2\alpha t) = \frac{1}{2}G(\lambda,0)\exp[(-1-\gamma_3-\eta)\alpha t]$$
$$\times\sum_{k,m}\lambda^k\,\mathrm{I}_m(-\eta\alpha t)\Big\{(-1+\eta)\,\mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\eta}{2}[\mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)]\Big\}\,, \tag{35}$$

where $\mathrm{I}_m(x)$ is the modified Bessel function, at the same time γ_+, γ_- and η are respectively

$$\gamma_+ = \frac{1}{2}[\tanh(2k_2)+\tanh(2k_1)](1-\theta_0\tanh k_3)\,,\quad \gamma_- = \frac{1}{2}[\tanh(2k_2)-\tanh(2k_1)](1-\theta_0\tanh k_3)\,,\quad \eta = \frac{\gamma_-^2}{4\gamma_3}\,. \tag{36}$$

In the same way, the four terms of $G(\lambda,t)$ in the expression (29) can be written respectively as follows:

$$F(\lambda,0)\frac{(\beta_1-\gamma_1)(\beta_2-\gamma_1)}{(\beta_2-\beta_1)\gamma_3}\exp(\beta_1\alpha t)$$
$$= -\frac{1}{4}F(\lambda,0)\exp[(-1+\gamma_3+\eta)\alpha t]$$
$$\times\sum_{k,m}\lambda^k\,\mathrm{I}_m(\eta\alpha t)\Big\{2\Big(\frac{\eta}{\gamma_3}-1\Big)\mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\eta}{\gamma_3}[\,\mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)]\Big\}\,, \tag{37}$$

$$G(\lambda,0)\frac{\beta_1-\gamma_1}{\beta_1-\beta_2}\exp(\beta_1\alpha t)$$
$$= \frac{1}{2}G(\lambda,0)\exp[(-1+\gamma_3+\eta)\alpha t]$$
$$\times\sum_{k,m}\lambda^k\,\mathrm{I}_m(\eta\alpha t)\Big\{\mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\gamma_-}{2\gamma_3}[\,\mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)]\Big\}\,, \tag{38}$$

$$F(\lambda,0)\frac{(\beta_1-\gamma_1)(\beta_2-\gamma_1)}{(\beta_1-\beta_2)\gamma_3}\exp(\beta_2\alpha t)$$

$$
\begin{aligned}
&= \frac{1}{4} F(\lambda,0) \exp[(-1-\gamma_3-\eta)\alpha t] \\
&\quad \times \sum_{k,m} \lambda^k \, \mathrm{I}_m(-\eta\alpha t) \Big\{ 2\Big(\frac{\eta}{\gamma_3}-1\Big) \mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\eta}{\gamma_3} [\, \mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)] \Big\} ,
\end{aligned} \tag{39}
$$

$$
\begin{aligned}
&G(\lambda,0) \frac{\beta_2-\gamma_1}{\beta_2-\beta_1} \exp(\beta_2\alpha t) \\
&= \frac{1}{2} G(\lambda,0) \exp[(-1-\gamma_3-\eta)\alpha t] \\
&\quad \times \sum_{k,m} \lambda^k \, \mathrm{I}_m(-\eta\alpha t) \Big\{ I_{k-2m}(\gamma_+\alpha t) - \frac{\gamma_-}{2\gamma_3} [\, \mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)] \Big\} .
\end{aligned} \tag{40}
$$

We consider the case in which all of the spin expectations $\langle \sigma_k(t) \rangle$ and $\langle s_k(t) \rangle$ vanish initially except for one, which we may choose to be the one at the origin,

$$
\langle \sigma_k(0) \rangle = \delta_{k,0} \, , \quad \langle s_k(0) \rangle = \delta_{k,0} \, . \tag{41}
$$

From the formula (25), the initial value of the generating function is just one ($F(\lambda,0) = G(\lambda,0) = 1$). For convenience, let us assume that the spin expectation is

$$
\langle \sigma_k(t) \rangle = A(1) + A(2) + A(3) + A(4) \, , \quad \langle s_k(t) \rangle = B(1) + B(2) + B(3) + B(4) \, . \tag{42}
$$

By comparing with the formula (25), we conclude that

$$
\begin{aligned}
A(1) = &-\frac{1}{2} \exp[(-1+\gamma_3+\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(\eta\alpha t) \Big\{ -\mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\gamma_-}{2\gamma_3} [\, \mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)] \Big\} , \\
A(2) = &\frac{1}{2} \exp[(-1+\gamma_3+\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(\eta\alpha t) \Big\{ \Big(1-\frac{\eta}{\gamma_3}\Big) \mathrm{I}_{k-2m}(\gamma_+\alpha t) - \frac{\eta}{2\gamma_3} [\, \mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)] \Big\} , \\
A(3) = &\frac{1}{2} \exp[(-1-\gamma_3-\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(-\eta\alpha t) \Big\{ \mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\gamma_-}{2\gamma_3} [\, \mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)] \Big\} , \\
A(4) = &-\frac{1}{2} \exp[(-1-\gamma_3-\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(-\eta\alpha t) \Big\{ \Big(1-\frac{\eta}{\gamma_3}\Big) \mathrm{I}_{k-2m}(\gamma_+\alpha t) - \frac{\eta}{2\gamma_3} [\, \mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)] \Big\} ;
\end{aligned} \tag{43}
$$

and

$$
\begin{aligned}
B(1) = &-\frac{1}{4} \exp[(-1+\gamma_3+\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(\eta\alpha t) \Big\{ 2\Big(\frac{\eta}{\gamma_3}-1\Big) \mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\eta}{\gamma_3} [\, \mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)] \Big\} , \\
B(2) = &\frac{1}{2} \exp[(-1+\gamma_3+\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(\eta\alpha t) \Big\{ \mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\gamma_-}{2\gamma_3} [\, \mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)] \Big\} , \\
B(3) = &\frac{1}{4} \exp[(-1-\gamma_3-\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(-\eta\alpha t) \Big\{ 2\Big(\frac{\eta}{\gamma_3}-1\Big) \mathrm{I}_{k-2m}(\gamma_+\alpha t) + \frac{\eta}{\gamma_3} [\, \mathrm{I}_{k-2m+2}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-2}(\gamma_+\alpha t)] \Big\} , \\
B(4) = &\frac{1}{2} \exp[(-1-\gamma_3-\eta)\alpha t] \\
&\times \sum_m \mathrm{I}_m(-\eta\alpha t) \Big\{ \mathrm{I}_{k-2m}(\gamma_+\alpha t) - \frac{\gamma_-}{2\gamma_3} [\, \mathrm{I}_{k-2m+1}(\gamma_+\alpha t) + \mathrm{I}_{k-2m-1}(\gamma_+\alpha t)] \Big\} .
\end{aligned} \tag{44}
$$

The expressions (42) $\sim$ (44) denote time evolutions of the spin expectations $\langle \sigma_k(t) \rangle$ and $\langle s_k(t) \rangle$.

3.3 *Discussion of the Result*

Now we discuss our result. At first we investigate the asymptotic behavior of $\langle\sigma_k(t)\rangle$ and $\langle s_k(t)\rangle$ as $t\to\infty$ by the asymptotic expressions of $\mathrm{I}_m(x)$. Then we discuss two special cases: (i) $k_1=k_2=k$ and $k_3\neq 0$, which means the two same Ising chains are dependent; (ii) $k_1=k_2=k$ and $k_3=0$, which means the two same Ising chains are independent.

Since the asymptotic expressions of $\mathrm{I}_m(x)$ as $x\to\infty$ is

$$\mathrm{I}_m(x)=\frac{1}{2\sqrt{x}}\exp(x)\,,\quad \mathrm{I}_m(-x)=(-1)^m\frac{1}{2\sqrt{x}}\exp(x)\,. \tag{45}$$

The asymptotic expressions of $\langle\sigma_k(t)\rangle$ and $\langle s_k(t)\rangle$ as $t\to\infty$ are

$$\langle\sigma_k(t)\rangle\propto\exp((-1+\gamma_3+2\eta+\gamma_+)\alpha t)=\exp\Big(-\frac{t}{\tau}\Big)\,,\quad \langle s_k(t)\rangle\propto\exp((-1+\gamma_3+2\eta+\gamma_+)\alpha t)=\exp\Big(-\frac{t}{\tau}\Big)\,, \tag{46}$$

where

$$\tau=\frac{1}{\alpha(1-\gamma_3-2\eta-\gamma_+)}\,, \tag{47}$$

which is the relaxation time of evolution of the system.

From the above expression and Eq. (36) we can find that the relaxation time depends on the temperature of the system. For our model, in Sec. 2, we have known that the critical temperature is $T_c=0$. At the critical point ($T_c=0$), $k_i\to\infty$ ($i=1,2,3$), then $\gamma_3=\tanh(k_3)=1$, $\gamma_+=(1-\theta_0)$ and $\eta=0$, therefore the relaxation time is

$$\tau=\frac{1}{\alpha(\theta_0-1)}\,. \tag{48}$$

We can prove that the equilibrium value $\theta_0=1$ (See Appendix C). From the expression (48), we get the relaxation time is infinite at the critical point. It implies that the system will very slowly approach the equilibrium state, thus the system will exhibit the critical slowing down phenomenon.

Now we discuss a special case in which the interactions are $k_1=k_2=k$ and $k_3\neq 0$. The parameters in the expression (36) are as follows:

$$\gamma_+=\tanh(2k)(1-\tanh k_3)\,,\quad \gamma_-=0\,,\quad \eta=0\,. \tag{49}$$

We apply the relation of the modified Bessel function

$$\mathrm{I}_m(0)=\delta_{m,0}\,. \tag{50}$$

The expression (43) reduces to

$$\begin{aligned}A(1)&=\frac{1}{2}\exp[(-1+\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)]\,,\quad A(2)=\frac{1}{2}\exp[(-1+\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)\,,\\ A(3)&=\frac{1}{2}\exp[(-1-\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)\,,\quad A(4)=-\frac{1}{2}\exp[(-1-\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)\,,\end{aligned} \tag{51}$$

and expression (44) reduces to

$$\begin{aligned}B(1)&=\frac{1}{2}\exp[(-1+\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)\,,\quad B(2)=\frac{1}{2}\exp[(-1+\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)\,,\\ B(3)&=-\frac{1}{2}\exp[(-1-\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)\,,\quad B(4)=\frac{1}{2}\exp[(-1-\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)\,.\end{aligned} \tag{52}$$

Therefore the expression of the spin expectation $\langle\sigma_k(t)\rangle$ and $\langle s_k(t)\rangle$ is completely the same and very simple.

$$\langle\sigma_k(t)\rangle=\exp[(-1+\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)]\,,\quad \langle s_k(t)\rangle=\exp[(-1+\gamma_3)\alpha t]\,\mathrm{I}_k(\gamma_+\alpha t)]\,, \tag{53}$$

As $t\to\infty$, using the asymptotic expressions of $\mathrm{I}_k(\gamma_+\alpha t)$, we obtain the asymptotic expressions of $\langle\sigma_k(t)\rangle$ and $\langle s_k(t)\rangle$ as follows:

$$\langle\sigma_k(t)\rangle=\langle s_k(t)\rangle\propto\exp[(-1+\gamma_3+\gamma_+)\alpha t]\,. \tag{54}$$

So we can get the relaxation time

$$\tau=\frac{1}{\alpha(1-\gamma_3-\gamma_+)}=\frac{1}{\alpha(1-\tanh 2k)(1-\tanh k_3)}\,. \tag{55}$$

At the critical point ($T_c=0$), $k\to\infty$, and $k_3\to\infty$, the relaxation time is also infinite. The system exhibits the critical slowing down phenomenon again.

When $k_3=0$, then $\gamma_3=0$, it implies that there is no interaction between the two infinite chains of the ladder. The system becomes two independent chains. The expression (54) reproduces precisely Glauber's one-dimensional chain result.[6]

4 Conclusions

In this paper, we studied the equilibrium phase transition and dynamical properties of Ising model on an infinite ladder lattice. In the first part, we investigate phase transition property of the system by using the transfer matrix method. It is found that there is no nonzero temperature phase transition as expected. In the special case of $k_1 = k$, $k_2 = 0$, $k_3 = 0$, our model reproduces precisely the result of one-dimensional Ising chain. In the second part, we study the time evolution of the system from the nonequilibrium state to the equilibrium state. Based on Glauber assumption, we solve a set of decoupling master equations and obtain approximation solution of the master equations under a special condition ($(\gamma_1 - \gamma_2)/2\gamma_3 \ll 1$). We found the system will exhibit the critical slowing down phenomenon.

Finally, let us point out that the result about dynamics in this paper is obtained under the condition $(\gamma_1 - \gamma_2)/2\gamma_3 \ll 1$, which implies that the conclusion is valid, only when k_1 is close to k_2 or $k_3 = J_3/k_\beta T$ is large enough, i.e., J_3 large enough or temperature low enough. Therefore for a ladder spin system with strong coupling between both spin chains or a low-temperature ladder spin system, our result and conclusion are valid, especially, at zero temperature the conclusion is true and credible.

Appendix A: Derivation of Eqs. (19) and (20)

Let us multiply both sides of the master equation (11) by $\sigma_k(t)$ and sum over all values of the σ and s variables, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{\{\sigma,s\}}\sigma_k(t)P(\sigma_1(t),\ldots,\sigma_N(t),s_1(t),\ldots,s_N(t)) = -2\sum_{\{\sigma,s\}}\sigma_k(t)W_k(\sigma_k(t))P(\sigma_1(t),\ldots,\sigma_N(t),s_1(t),\ldots,s_N(t))\,. \tag{A1}$$

Based on the definition (17) of $\langle\sigma_k(t)\rangle$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_k(t)\rangle = -2\langle\sigma_k(t)W_k(\sigma_k(t))\rangle\,. \tag{A2}$$

Substituting the form (15) for the transition probabilities into Eq. (A2) and considering $\sigma_k^2 = 1$, we obtain a differential equation for the expectation values $\langle\sigma_k(t)\rangle$,

$$\frac{\mathrm{d}}{\mathrm{d}(\alpha t)}\langle\sigma_k(t)\rangle = -\langle\sigma_k(t)\rangle + \frac{1}{2}\delta_1(\sigma_{k-1}(t)+\sigma_{k+1}(t)) + \gamma_3\langle s_k(t)\rangle - \frac{1}{2}\delta_1\gamma_3(\langle\sigma_k(t)s_k(t)\sigma_{k-1}(t)\rangle + \langle\sigma_k(t)s_k(t)\sigma_{k+1}(t)\rangle)\,. \tag{A3}$$

Similarly we can get a differential equation for the expectation values $\langle s_k(t)\rangle$. So we derived a set of coupling equations (19) and (20) from the master equation (11).

Appendix B: Solutions for Generating Functions

From the expressions (31) and (27), we have

$$\frac{(\beta_1-\gamma_1)(\beta_2-\gamma_1)}{\gamma_3(\beta_1-\beta_2)} = \frac{1}{4}\Big[\frac{\gamma_-^2}{4\gamma_3^2}(\lambda^{-2}+\lambda^2+2) - 2\Big]\,, \tag{A4}$$

$$\frac{\beta_1-\gamma_1}{\beta_1-\beta_2} = \frac{1}{2}\Big[1+\frac{\gamma_-}{2\gamma_3}(\lambda^{-1}+\lambda)\Big]\,, \tag{A5}$$

$$\frac{(\beta_2-\gamma_1)}{(\beta_2-\beta_1)} = \frac{1}{2}\Big[1-\frac{\gamma_-}{2\gamma_3}(\lambda^{-1}+\lambda)\Big]\,, \tag{A6}$$

where $\gamma_- = (1/2)[\tanh(2k_2) - \tanh(2k_1)][1-\theta_0\tanh(k_3)]$.

We also have

$$\exp(\beta_1\alpha t) = \exp\Big[\Big(-1+\gamma_3+\frac{\gamma_-^2}{4\gamma_3}\Big)\alpha t\Big]\exp\Big[\frac{\gamma_+}{2}(\lambda^{-1}+\lambda)\alpha t\Big]\exp\Big[\frac{\gamma_-^2}{8\gamma_3}(\lambda^{-2}+\lambda^2)\alpha t\Big]\,, \tag{A7}$$

$$\exp(\beta_2\alpha t) = \exp\Big[\Big(-1-\gamma_3-\frac{\gamma_-^2}{4\gamma_3}\Big)\alpha t\Big]\exp\Big[\frac{\gamma_+}{2}(\lambda^{-1}+\lambda)\alpha t\Big]\exp\Big[-\frac{\gamma_-^2}{8\gamma_3}(\lambda^{-2}+\lambda^2)\alpha t\Big]\,, \tag{A8}$$

where $\gamma_+ = (1/2)[\tanh(2k_2) + \tanh(2k_1)][1-\theta_0\tanh(k_3)]$.

Noting the generating function of the Bessel function of imaginary argument,[6]

$$\exp\Big[\frac{x}{2}(\lambda^{-1}+\lambda)\Big] = \sum_{k=-\infty}^{k=\infty}\lambda^k\,\mathrm{I}_k(x)\,, \tag{A9}$$

and

$$\exp\Big[\frac{x}{2}(\lambda^{-2}+\lambda^2)\Big] = \sum_{k=-\infty}^{k=\infty}\lambda^{2k}\,\mathrm{I}_k(x)\,, \tag{A10}$$

we obtain

$$\exp(\beta_1\alpha t) = \exp\Big[\Big(-1+\gamma_3+\frac{\gamma_-^2}{4\gamma_3}\Big)\alpha t\Big]\sum_{n,m}\lambda^{n+2m}\,\mathrm{I}_n(\gamma_+\alpha t)\,\mathrm{I}_m\Big(\frac{\gamma_-^2}{4\gamma_3}\alpha t\Big)\,, \tag{A11}$$

and

$$\exp(\beta_2\alpha t) = \exp\Big[\Big(-1-\gamma_3-\frac{\gamma_-^2}{4\gamma_3}\Big)\alpha t\Big]\sum_{n,m}\lambda^{n+2m}\,\mathrm{I}_n(\gamma_+\alpha t)\,\mathrm{I}_m\Big(-\frac{\gamma_-^2}{4\gamma_3}\alpha t\Big)\,. \tag{A12}$$

From the formulas (A6) and (A11), we obtain the first term of $F(\lambda,t)$,

$$\begin{aligned} & F(\lambda,0)\frac{\beta_2-\gamma_1}{\beta_2-\beta_1}\exp(\beta_1\alpha t) \\ = & -\frac{1}{2}F(\lambda,0)\exp\Big[\Big(-1+\gamma_3+\frac{\gamma_-^2}{4\gamma_3}\Big)\alpha t\Big]\Big\{\Big[1-\frac{\gamma_-}{2\gamma_3}(\lambda^{-1}+\lambda)\Big]\sum_{n,m}\lambda^{n+2m}\,\mathrm{I}_n(\gamma_+\alpha t)\,\mathrm{I}_m\Big(\frac{\gamma_-^2}{4\gamma_3}\alpha t\Big)\Big\} \\ = & -\frac{1}{2}F(\lambda,0)\exp\Big[\Big(-1+\gamma_3+\frac{\gamma_-^2}{4\gamma_3}\Big)\alpha t\Big]\sum_{n,m}\Big\{\lambda^{n+2m}\,\mathrm{I}_n(\gamma_+\alpha t)\,\mathrm{I}_m\Big(\frac{\gamma_-^2}{4\gamma_3}\alpha t\Big) \\ & -\frac{\gamma_-}{2\gamma_3}\Big[\lambda^{n+2m-1}\,\mathrm{I}_n(\gamma_+\alpha t)\,\mathrm{I}_m\Big(\frac{\gamma_-^2}{4\gamma_3}\alpha t\Big)+\lambda^{n+2m+1}\,\mathrm{I}_n(\gamma_+\alpha t)\,\mathrm{I}_m\Big(\frac{\gamma_-^2}{4\gamma_3}\alpha t\Big)\Big]\Big\}\,. \end{aligned} \tag{A13}$$

Consider the following property of the Bessel function of imaginary argument:

$$\begin{aligned} & \sum_{n,m}\lambda^{n+2m}\,\mathrm{I}_n(x)\,\mathrm{I}_m(y)=\sum_{k,m}\lambda^k\,\mathrm{I}_{k-2m}(x)\,\mathrm{I}_m(y)\,, \quad \sum_{n,m}\lambda^{n+2m+1}\,\mathrm{I}_n(x)\,\mathrm{I}_m(y)=\sum_{k,m}\lambda^k\,\mathrm{I}_{k-2m-1}(x)\,\mathrm{I}_m(y)\,, \\ & \sum_{n,m}\lambda^{n+2m-1}\,\mathrm{I}_n(x)\,\mathrm{I}_m(y)=\sum_{k,m}\lambda^k\,\mathrm{I}_{k-2m+1}(x)\,\mathrm{I}_m(y)\,, \end{aligned} \tag{A14}$$

simplifying the formula (A13), we can get the first term (32) of $F(\lambda,t)$. In the same way, the second term (33), the third term (34) and the fourth term (35) can be obtained.

Appendix C: Proof of $\theta_0 = 1$

We define an equilibrium correlation function

$$r^e_{k,k}\equiv\theta_0\,, \tag{A15}$$

and the partition function is

$$Z_N=\sum_{\{\sigma_i,s_i\}}\exp\sum_{i=-N}^{N}(k_1\sigma_i\sigma_{i+1}+k_2s_is_{i+1}+k_3\sigma_is_i)\,. \tag{A16}$$

We have

$$\begin{aligned} \frac{\partial Z_N}{\partial k_3} &= \sum_{\{\sigma_i,s_i\}}\Big(\sum_{i=-N}^{N}\sigma_is_i\Big)\exp\sum_{i=-N}^{N}(k_1\sigma_i\sigma_{i+1}+k_2s_is_{i+1}+k_3\sigma_is_i) \\ &= \sum_{\{\sigma_i,s_i\}}(2N+1)\sigma_is_i\exp\sum_{i=-N}^{N}(k_1\sigma_i\sigma_{i+1}+k_2s_is_{i+1}+k_3\sigma_is_i)\,. \end{aligned} \tag{A17}$$

From the definition of $\langle\sigma_is_i\rangle$,

$$\langle\sigma_is_i\rangle=\frac{1}{Z_N}\sum_{\{\sigma_i,s_i\}}\sigma_is_i\exp\sum_{i=-N}^{N}(k_1\sigma_i\sigma_{i+1}+k_2s_is_{i+1}+k_3\sigma_is_i)\,, \tag{A18}$$

we can obtain

$$\frac{1}{Z_N}\frac{\partial Z_N}{\partial k_3}=(2N+1)\langle\sigma_is_i\rangle\,. \tag{A19}$$

From the transfer matrix, we have the expression of the partition function

$$Z_N=\lambda_1^{2N+1}+\lambda_2^{2N+1}+\lambda_3^{2N+1}+\lambda_4^{2N+1}\,, \tag{A20}$$

where λ_i $(i=1,2,3,4)$ are the four eigenvalues and λ_1 is the maximum value among these four eigenvalues. Therefore when N is very large $(N\to\infty)$, we have

$$Z_N=\lambda_1^{2N+1}\,. \tag{A21}$$

From the formula (A19) we have

$$\langle\sigma_is_i\rangle=\frac{1}{\lambda_1}\frac{\partial\lambda_1}{\partial k_3}\,, \tag{A22}$$

where

$$\lambda_1=2\cosh(k_1{+}k_2)\cosh k_3+\sqrt{\cosh(2k_1{+}2k_2)(\cosh 2k_3-1)+2\cosh(2k_1-2k_2)+\cosh 2k_3+1}\,,$$

since $\cosh 2x=2\cosh^2x-1$. λ_1 can be rewritten as

$$\lambda_1=2\Big[\cosh(k_1{+}k_2)\cosh k_3+\sqrt{\cosh^2(k_1{+}k_2)\cosh^2k_3-\cosh^2(k_1{+}k_2)+\cosh^2(k_1{-}k_2)}\Big]\,,$$

from the above formula, we have

$$\frac{\partial\lambda_1}{\partial k_3} = 2\left[\cosh(k_1{+}k_2)\sinh k_3 + \frac{\cosh^2(k_1{+}k_2)\cosh k_3\sinh k_3}{\sqrt{\cosh^2(k_1{+}k_2)\cosh^2 k_3 - \cosh^2(k_1{+}k_2) + \cosh^2(k_1{-}k_2)}}\right]$$
$$= 2\cosh(k_1{+}k_2)\sinh k_3\left[1 + \frac{\cosh k_3}{\sqrt{\cosh^2 k_3 - 1 + \cosh^2(k_1{-}k_2)/\cosh^2(k_1{+}k_2)}}\right].$$

Because

$$\frac{\cosh(k_1{-}k_2)}{\cosh(k_1{+}k_2)} = \frac{1-\tanh k_1\tanh k_2}{1+\tanh k_1\tanh k_2},$$

and $\tanh k_i = 1$ when $k_i\to\infty$ $(i=1,2)$, we have

$$\frac{\cosh(k_1{-}k_2)}{\cosh(k_1{+}k_2)} = 0,$$

then

$$\frac{\partial\lambda_1}{\partial k_3} = 2\cosh(k_1{+}k_2)\sinh k_3(1+\coth k_3),$$

and

$$\lambda_1 = 2\cosh(k_1{+}k_2)\cosh k_3(1+\tanh k_3).$$

According to the formula (A22), we have

$$\theta_0 = \langle\sigma_i s_i\rangle = \tanh k_3\frac{(1+\coth k_3)}{(1+\tanh k_3)} = 1.$$

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References

[1] George A. Baker, Jr., Phys. Rev. **130** (1963) 1406.
[2] J.S. Hoye, Phys. Rev. B **6** (1972) 4261.
[3] Per Bak, Phys. Rev. Lett. **49** (1982) 249.
[4] M. Barati and A. Ramazani, Phys. Rev. B **64** (2001) 024407.
[5] Maria Gloria Pini, Phys. Rev. B **48** (1993) 3240.
[6] R.J. Glauber, J. Math. Phys. **4** (1963) 294.
[7] K. Kawasaki, Phys. Rev. **145** (1966) 224.
[8] Pei-Qing Tong, Phys. Rev. E **56** (1997) 1371.
[9] L.L. Goncalves, Phys. Rev. E **63** (2001) 026114.
[10] M. Droz, J. Kamphorst Leal da Silva, and A. Malaspinas, Phys. Lett. A **115** (1986) 448.
[11] Z.R. Yang, Phys. Rev. B **46** (1992) 11578.
[12] B. Sutherland, Phys. Rev. Lett. **31** (1973) 1504.
[13] J.Y. Zhu and Z.R. Yang, Phys. Rev. E **61** (2000) 210.
[14] K.-t. Lrung, Phys. Rev. E **63** (2001) 016102.
[15] M.C. Yalabik and J.D. Gunton, Phys. Rev. B **25** (1982) 534.
[16] R. Bausch, V. Dohm, H.K. Janssen, and R.K.P. Zia, Phys. Rev. Lett. **47** (1981) 1837.
[17] S. Wansleben and P. Landau, J. Appl. Phys. **61** (1987) 3968.
[18] S. Tang and D.P. Landau, Phys. Rev. B **36** (1987) 567.
[19] E. Pytte, Phys. Rev. B **34** (1986) 2060.
[20] Z.R. Yang, Progress in Phys. **1(2)** (1981) 317.