Commun. Theor. Phys. (Beijing, China) **50** (2008) pp. 517–520
Vol. 50, No. 2, August 15, 2008

Quantum Group Symmetry in Hofstadter Problem

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(Received September 17, 2007)

Abstract *We show that there is a quantum $Sl_q(2)$ group symmetry in Hofstadter problem on square lattice. The cyclic representation of the quantum group is discussed and its application for computing the degeneracy density of the model is shown.*

PACS numbers: 72.10.-b, 02.20.Uw
Key words: Hofstadter problem, quantum $Sl_q(2)$ group, cyclic representation

Hofstadter problem of two-dimensional Bloch electron in a uniform magnetic field is an old problem and it still attracts a lot of attention today.[1−14] The first character of this problem is the interplay of two intrinsic periods, the period of the lattice and that of the phase induced by the magnetic field. This character brings many unexpected structures into physics, such as the fractal and self-similar structure, which can be observed in the energy spectrum and the wave function.[1] The problem also has a connection to the phenomenon of quasiperiodic system[2] and high-temperature superconductivity.[3] Furthermore, the problem is focused in the study of quantum Hall effect. The topological character of the Hall conductance in this lattice model was revealed both for periodic boundary conditions and for systems with edges,[4] and recently, the interest in the Hofstadter problem on triangular and honeycomb lattice is renewed by the experiment observation of the quantum Hall effect in graphene.[5−7]

In the study of Hofstadter problem on square lattice, Wiegmann and Zabrodin found a relation between the magnetic translation and the quantum group $Sl_q(2)$.[8] Using the relation, they rewrote the Schrödinger equation as a functional equation and derived the Bethe ansatz equations. This discovery initiated the application of quantum group technique to the problem,[8−10] but it should be noted that $Sl_q(2)$ group mentioned in the above works is not the symmetry of the system and in this letter, we will concentrate on the quantum group symmetry of the system.

In the following, we firstly show that there is a quantum group symmetry in Hofstadter problem. The basic observation is that the problem has a magnetic translation group symmetry, and from this magnetic translation group we can construct a quantum $Sl_q(2)$ group, which plays the role as the symmetry of the Hamiltonian. Then we establish the cyclic representation of the quantum group based on the eigenstates of the Hamiltonian. At last, we discuss the degeneracy density of the system based on the knowledge of cyclic representation theory of the quantum group.

The Hamiltonian of Hofstadter problem on a square lattice can be written as

$$H = \hat{T}_x + \hat{T}_{-x} + \hat{T}_y + \hat{T}_{-y}\,, \tag{1}$$

where

$$\hat{T}_\mu = \sum_n \mathrm{e}^{\mathrm{i}\theta_\mu(n)} |n+\mu\rangle\langle n| \tag{2}$$

is the magnetic translation operator. $|n\rangle$ is the Wannier function for an electron at site $n = (n_x, n_y)$ and $\mu = \pm x, \pm y$ is the lattice vector that $x = (1, 0)$, $y = (0, 1)$. The phase factors satisfy the condition

$$\prod_{\text{plaquette}} \mathrm{e}^{\mathrm{i}\theta_\mu(n)} = \mathrm{e}^{-2\mathrm{i}\pi\phi}\,, \tag{3}$$

where $\phi = P/Q = Be/hc$ is the magnetic flux per plaquette in the unit of magnetic flux quanta hc/e, and P and Q are mutually prime integers. Similar to the usual electron in the magnetic field, the Hamiltonian is gauge-invariant when the number of the lattice's sites tends to be infinite, so naturally the physics is independent of the gauge we choose and for convenience, we take Landau gauge of $\theta_x(n) = 2\pi\phi n_y$, $\theta_y(n) = 0$ in the following discussions.

Now let us define another kind of magnetic translation operators

$$T_\nu = \sum_n \mathrm{e}^{\mathrm{i}\tau_\nu(n)} |n+\nu\rangle\langle n|\,, \tag{4}$$

where

$$\tau_x(n) = \theta_x(n) - 2\pi\phi n_y\,,$$
$$\tau_y(n) = \theta_y(n) + 2\pi\phi n_x\,. \tag{5}$$

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The phase factors of these operators satisfy the condition contrary to Eq. (3)

$$\prod_{\text{plaquette}} \mathrm{e}^{\mathrm{i}\tau_\nu(n)} = \mathrm{e}^{2\mathrm{i}\pi\phi}\,, \tag{6}$$

so T_ν is different from $\hat{T}_\mu$ with an opposite magnetic field. These two series of operators have their continuous origins that $\hat{T}_\mu$ corresponds to $\mathrm{e}^{-\mathrm{i}\mu\cdot[(P+e)/cA]/\hbar}$ and T_ν corresponds to $\mathrm{e}^{-\mathrm{i}\nu\cdot[(P+e)/cA']/\hbar}$, where $A' = A + r \times B$ and $\bigtriangledown \times A = B$, $\bigtriangledown \times A' = -B$. Straightforward computation shows that $[\hat{T}_\mu, T_\nu] = 0$, corresponds to $[P+(e/c)A, P+(e/c)A'] = 0$.[15] Thus the system has the symmetry under the new defined magnetic translations T_ν, that $[H,T_\nu] = \sum_\mu [\hat{T}_\mu, T_\nu] = 0$ and we will obtain a quantum group based on them. The magnetic translations satisfy the well-known Heisenberg–Weyl relation

$$T_\mu T_\nu = \mathrm{e}^{\mathrm{i}2\pi\phi(\mu\times\nu)\cdot z} T_\nu T_\mu\,. \tag{7}$$

From this relation, we can construct the $Sl_q(2)$ algebra as follows:

$$J_+ = \frac{q^\lambda T_x - q^{-\lambda} T_{-x}}{q-q^{-1}} T_{-y}\,,$$
$$J_- = \frac{q^\lambda T_{-x} - q^{-\lambda} T_x}{q-q^{-1}} T_y\,,$$
$$K^\pm = T_{\pm x}\,, \tag{8}$$

where $q = \mathrm{e}^{2\mathrm{i}\pi P/Q}$, and λ is an arbitrary real number. It is easy to check that these operators obey the commutation relations of the quantum group $Sl_q(2)$

$$[J_+, J_-] = \frac{K^{+2} - K^{-2}}{q-q^{-1}}\,,$$
$$K^+ J_\pm K^- = q^{\pm 1} J_\pm\,. \tag{9}$$

Comparing Eqs. (8) and the $Sl_q(2)$ group mentioned in Ref. [8] ~ [10]:

$$J_+ = \frac{\lambda \mathrm{i}}{q-q^{-1}}(\hat{T}_{-x} + \hat{T}_{-y})\,,$$
$$J_- = \frac{\mathrm{i}}{q-q^{-1}}(\hat{T}_x + \hat{T}_y)\,,$$
$$K^{+2} = \lambda q \hat{T}_{-y}\hat{T}_x\,,$$
$$K^{-2} = \lambda q^{-1}\hat{T}_{-x}\hat{T}_y\,, \tag{10}$$

where $q = \mathrm{e}^{\mathrm{i}\pi P/Q}$, and $\lambda = 1(-1)$ for P is even (odd), the most important difference is that the definition of $K^\pm$ is replaced by $K^{\pm 2}$ in the latter case. It has to be emphasized that the latter definition is taken in the discussion about rewriting the Hamiltonian with the quantum group operators,[8−10] and the detailed form of $K^\pm$ has nothing to do with this object. However, when we consider the symmetry of the system and the representation of quantum group, there is an additional constraint that $K^\pm$ should also be the combination of T_ν, otherwise it is not commutative with the Hamiltonian. From the definition of $K^{\pm 2}$, we cannot find such form of $K^\pm$ to satisfy the constraint, and so the definition of Eqs. (10) should not be chosen in our discussion. An interesting character of quantum group is that it has a Q-dimensional irreducible cyclic representation shown as follows:[16]

$$K^\pm|s\rangle = q^{\pm(\lambda-\mu-s)}|s\rangle\,,$$
$$J_+|s\rangle = [2\lambda - \mu - s + 1]_q |s-1\rangle\,,$$
$$J_-|s\rangle = [s+\mu+1]_q |s+1\rangle\,,$$
$$|s+Q\rangle = |s\rangle\,, \tag{11}$$

where s is the integer number ranging from 0 to $Q-1$, $[x]_q = (q^x - q^{-x})/(q-q^{-1})$ and $q = \mathrm{e}^{2\mathrm{i}\pi P/Q}$. As will be shown in the following, this cyclic representation can also be established in this Hofstadter problem.

To establish the representation, the key is to find out the relation between the basis $|s\rangle$ and the eigenstate of the Hamiltonian. Starting from the Landau gauge $\theta_x(n) = 2\pi\phi n_y$,$\theta_y(n) = 0$, and corresponding $\tau_x(n) = 0$, $\tau_y(n) = 2\pi\phi n_x$, the magnetic translation operator along x direction T_x is equal to the normal translation operator in the absence of magnetic field, i.e $T_x = \sum_n |n+x\rangle\langle n|$. Because $[T_x, H] = 0$, we can define the common eigenstate of Hamiltonian and T_x as

$$|E,k\rangle = \sum_n C_{n_y}(k)\,\mathrm{e}^{\mathrm{i}kn_x}|n\rangle\,, \tag{12}$$

where E and $\mathrm{e}^{-\mathrm{i}k}$ are the eigenvalues of the operators H and T_x. Then applying $T_{\pm y}$ on $|E,k\rangle$ we have $T_{\pm y}|E,k\rangle = \sum_n C_{n_y\mp 1}(k)\,\mathrm{e}^{\mathrm{i}(k\mp 2\pi\phi)n_x}|n\rangle$, which is the eigenstate of T_x with eigenvalue $\mathrm{e}^{-\mathrm{i}(k\mp 2\pi\phi)}$. Because $HT_{\pm y}|E,k\rangle = ET_{\pm y}|E,k\rangle$ results from $[T_{\pm y}, H] = 0$, we conclude that

$$T_{\pm y}|E,k\rangle = |E, k\mp 2\pi\phi\rangle\,. \tag{13}$$

From $\phi = P/Q$, we also know that $T_{\pm y}^Q|E,k\rangle = |E,k\rangle$. Now we define the basis of representation

$$|s=0\rangle = |E,k_0\rangle\,,$$
$$|s\rangle = (T_y)^s|E,k_0\rangle = |E, k_0 - 2\pi\phi s\rangle\,, \tag{14}$$

then $|s+Q\rangle = |s\rangle$ and it is easy to check that we have Eqs. (11) with $\lambda - \mu = k_0/(2\pi\phi)$ when the operators of Eqs. (8) are applied to the basis $|s\rangle$. Thus the irreducible cyclic representation of the quantum group has been established in the subspace of eigenstates of the Hamiltonian with the eigenvalue E. It should be noted that there is more than one choice of $|s\rangle$ for a general lattice, because when the number of k is bigger than Q, you can replace k_0 with k_0', and get a new basis series if $(k_0' - k_0)/2\pi\phi$ mod $2\pi \neq 0$. If a $Q \times Q$ lattice with periodic boundary condition is considered, the number of levels of the Hamiltonian is Q,[1] and then every subspace of one level can include only one Q-dimensional cyclic representation of quantum group, because the total number of independent states equals the total number of sites.

Now we apply the cyclic representation theory to studying the degeneracy density of the Hofstadter problem. Theoretically, a real solid sample is always viewed as a lattice with infinite sites. To consider the degeneracy density, we firstly suppose that the lattice is finite with each side as N_x and N_y satisfying the periodic boundary condition, then let N_x and N_y tend to infinity at the end of the discussion. The periodic boundary condition $|n'\rangle \equiv |n+N_x x+N_y y\rangle = |n\rangle$ asks the elements in the operator $\hat{T}_x$ has $\mathrm{e}^{2\mathrm{i}\pi\phi n_y}|n+x\rangle\langle n| = \mathrm{e}^{2\mathrm{i}\pi\phi(n_y+N_y)}|n'+x\rangle\langle n'|$, i.e $\mathrm{e}^{2\mathrm{i}\pi\phi N_y}=1$, so N_y should be chosen to be integer times of Q. Similarly, N_x also should be integer times of Q considering the operator T_y. Thus we set $N_x = N_1 Q$ and $N_y = N_2 Q$, where N_1 and N_2 are integer numbers. We can define a new Wannier function through the following Fourier transformation

$$|n;t\rangle = \frac{1}{\sqrt{N}}\sum_R \mathrm{e}^{\mathrm{i}t\cdot(n+R)}|n+R\rangle\,, \tag{15}$$

where $R=(R_x,R_y)Q$ and $N=N_1\times N_2$ is the normalized factor. $t=(t_x,t_y)$ and $t_x = 2t_1\pi/(N_1Q)$, $t_y = 2t_2\pi/(N_2Q)$, where $t_1(t_2)$ is the integer ranging from 1 to $N_1(N_2)$. It is obvious that $|n;t\rangle$ satisfies the periodic boundary condition of $|n+Qx;t\rangle = |n;t\rangle = |n+Qy;t\rangle$, in other words, the subspace consisting of $|n;t\rangle$ with a fixed t can be viewed as a $Q\times Q$ lattice with periodic boundary condition. With this new basis, the Hamiltonian is divided into $N_1\times N_2$ parts: $H=\sum_t H(t)$ and

$$H(t) = \sum_\mu \mathrm{e}^{-\mathrm{i}t\cdot\mu}\hat{T}_\mu(t)\,, \tag{16}$$

where

$$\hat{T}_\mu(t) = {\sum_n}' \mathrm{e}^{\mathrm{i}\theta_\mu(n)}|n+\mu;t\rangle\langle n;t|\,, \tag{17}$$

and $\sum_n'$ ranges from (0,0) to $(Q-1,\,Q-1)$. All the Hamiltonians $H(t)$ and H have Q levels,[1] so they have to share the same Q eigenvalues. Similar to the Hamiltonian, the magnetic translation operator T_μ can be written as

$$T_\mu = \sum_t T_\mu(t)\,\mathrm{e}^{-\mathrm{i}t\cdot\mu}\,, \tag{18}$$

where

$$T_\mu(t) = {\sum_n}' \mathrm{e}^{\mathrm{i}\tau_\mu(n)}|n+\mu;t\rangle\langle n;t|\,. \tag{19}$$

Then there are $N_1\times N_2$ magnetic translation group symmetry $T_\mu(t)$ in the physical system and accordingly $N_1\times N_2$ quantum group symmetry $K^\pm(t)$, $J_\pm(t)$ by replacing T_μ with $T_\mu(t)$ in Eqs. (8). Similar to Eqs. (14), the basis of the cyclic representation of $K^\pm(t)$, $J_\pm(t)$ is

$$|s;t\rangle \equiv (T_y(t))^s|E,k_0;t\rangle\,, \tag{20}$$

where

$$|E,k;t\rangle = {\sum_n}' C_{n_y}(k;t)\,\mathrm{e}^{\mathrm{i}kn_x}|n;t\rangle \tag{21}$$

and the representation is unique because the subspace consisting of $|n;t\rangle$ can be viewed as a $Q\times Q$ lattice. Now we have $N_1\times N_2$ independent quantum group symmetries, and every quantum group has a Q-dimensional irreducible cyclic representation, so every energy level of the system is $N_1\times N_2\times Q$ folds degeneracy. The simple algebra computation shows that the degeneracy density of the system is $1/Q$, and can be rewritten as $(1/P)eB/hc$ because $P/Q = eB/hc$. This value of degeneracy density is independent of N_1 and N_2, so nothing is changed when N_1 and N_2 tend to infinity and the final result is just $(1/P)eB/hc$.

In summary, we have shown that there is a quantum $Sl_q(2)$ group symmetry in the Hofstadter problem. The cyclic representation of the quantum group is established in the subspace of the eigenstates of the Hamiltonian. We have chosen Landau gauge in our discussion, but when the lattice's sites tend to be infinitely large, the system is independent of the gauge, and a different gauge will change nothing but the detailed form of Eq. (12), i.e., eigenstates of $T_{\pm x}$. Based on the theory of cyclic representation, we find that the degeneracy density of the Hofstadter problem is very easy to obtain. The value $(1/P)eB/hc$ is different from the case of the usual Landau level for an additional factor $1/P$ and shows the essential difference between the magnetic field induced dynamics in the lattice system and the continuous system. In the weak magnetic field limit that $P=1$ and Q tends to infinity, this difference is diminished, which is consistent with the usual method treating the lattice system as a continuous system by replacing the lattice momentum $\hbar k_i$ with $-\mathrm{i}\hbar\partial_i + (e/c)A_i$.

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