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Topological Aspects of Solitons in Ferromagnets*

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Abstract *Two kinds of topological soliton (skyrmion and magnetic vortex ring) in ferromagnets are studied. They have the common topological origin, a tensor $H_{\alpha\beta} = \vec{n}\cdot(\partial_\alpha\vec{n}\times\partial_\beta\vec{n})$, which describes the non-trivial distribution of local orientation of magnetization $\vec{n}$ at large distances in space. The topological stability of skyrmion is protected by the winding number. Knot-like topological defect as magnetic vortex rings is also studied. On the assumption that magnetic vortex rings are geometric lines, we present their δ-function distribution in ferromagnetic materials. Furthermore, it is briefly shown that Hopf invariant is a proper topological invariant to describe the topology of magnetic vortex rings.*

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1 Introduction

For a single-domain ferromagnetic nanoparticle at sufficiently low temperatures, all the spins are locked together by the strong exchange interaction, and therefore only the orientation of the total magnetization vector can change but not its absolute value. In many situations ferromagnetic materials may be viewed as continuous media, in which the state of the system is represented by a vector field indicating the local orientation of the magnetization. The dynamics of the ferromagnet then follows from the time evolution of this vector field, which obeys a nonlinear differential equation known as the Landau–Lifshitz equation. Representing the local orientation of the magnetization by the three-component vector field $\vec{n}(\vec{r},t)$, the Landau–Lifshitz equation[1−3] takes the form

$$\rho J\frac{\partial n_i}{\partial t} = -\epsilon_{ijk}n_j\frac{\delta E}{\delta n_k} \tag{1}$$

in the absence of dissipation. ρ is the density of magnetic moments, each of angular momentum J, and $E(n_i,\partial_\alpha n_j)$ is an energy functional of the magnetization and its spatial derivatives.

The total energy E and the modulus n_i^2 are conserved by Eq. (1). There are several other conserved quantities that are important in what follows: the winding number W, the linear momentum P_α, and the total number of spin reversals N, defined by[4]

$$W \equiv \int \mathrm{d}^3\vec{r}\,\partial_\alpha\Omega^\alpha\,,\quad P_\alpha \equiv 2\pi\rho J\epsilon_{\alpha\beta\gamma}\int\mathrm{d}^3\vec{r}\,r_\beta\Omega^\gamma(\vec{r},t)\,,$$
$$N \equiv \frac{\rho J}{\hbar}\int\mathrm{d}^3\vec{r}(1-n_z)\,, \tag{2}$$

where

$$\Omega^\alpha \equiv \frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}\epsilon_{ijk}n^i\partial_\beta n^j\partial_\gamma n^k \tag{3}$$

is called the "magnetic vorticity", which is the topological density. The linear momentum of three-dimensional Landau–Lifshitz ferromagnets P_α is proportional to the "magnetic vorticity" Ω^α, which is a topological term.

We will focus on the configuration for which the magnetization tends to $\vec{n}_0 = (0,0,1)$ at spatial infinity. In this case, we can introduce a tensor as

$$H_{\alpha\beta} \equiv \vec{n}\cdot(\partial_\alpha\vec{n}\times\partial_\beta\vec{n}) = \epsilon_{ijk}n^i\partial_\alpha n^j\partial_\beta n^k\,. \tag{4}$$

So the magnetic vorticity Ω^α is

$$\Omega^\alpha = \frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}H_{\beta\gamma}\,. \tag{5}$$

The tensor $H_{\beta\gamma}$ can be reexpressed in a U(1) gauge field tensor form[5]

$$H_{\beta\gamma} = \epsilon_{ijk}n^i\partial_\beta n^j\partial_\gamma n^k = \partial_\beta A_\gamma - \partial_\gamma A_\beta\,, \tag{6}$$

where A_β is a U(1) gauge potential.

Here $H_{\alpha\beta}$ describes the non-trivial distribution of the local orientation of the magnetization $\vec{n}$ at large distances in space.[6] It plays a key role in the construction of a sequence of topological soliton in an isotropic ferromagnet.

The purpose of this paper is to get a theoretical insight into the topological aspects of defects in ferromagnetic materials. Using the so-called ϕ-mapping topological current theory,[5] we study the two kinds of topological soliton (skyrmion and magnetic vortex ring) in ferromagnetic materials. It is revealed that they have the common topological origin, a tensor $H_{\alpha\beta}$. The topological stability of skyrmions is protected by the winding number W_l while the topological stability of magnetic vortex rings is preserved by the Hopf invariant.

This paper is organized as follows. In Sec. 2 we will prove that there is a three-dimensional topological charge density, which can be derived from $H_{\alpha\beta}$, and leads to the skyrmions. In Sec. 3, it is shown that the magnetic vorticity is a two-dimensional topological current by using the vector parameter $\vec{\phi}$, and the vortex ring structure is just

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inhering in this topological current. We also show that the Hopf invariant is a proper topological invariant to describe this topology of magnetic vortex rings.

2 Skyrmions in Ferromagnets

Although many works have been devoted to exploring the properties of skyrmion soliton in isotropic ferromagnets,[7,8] its topological origin is still of great worth to study in detail. These solitons have been of recent interest in quantum Hall systems.[9,11] In this section, it is shown that there is a three-dimensional topological current, which can be derived from $H_{\alpha\beta}$, when it is expressed in a three-component order parameter $\vec{\varphi}$, and the point-like defects (i.e. skyrmion soliton) are just inhering in this three-dimensional topological current. The winding number plays a key role in the construction of a sequence of static skyrmion solitons of an isotropic ferromagnet.[12]

Introduce a three-component vector order parameter $\vec{\varphi} = (\varphi^1, \varphi^2, \varphi^3)$,

$$\|\varphi\|^2 = \varphi^a\varphi^a = 1\,, \quad (a = 1, 2, 3)\,. \tag{7}$$

Then $\vec{n}$ can be expressed in terms of φ^a,

$$n^a = \frac{\varphi^a}{\|\varphi\|}\,. \tag{8}$$

Obviously, the zero point of the $\vec{\varphi}$ field is just the three-dimensional singular point of the $\vec{n}$ field.

The generalized winding number is defined by the Gauss map n: $\partial\Theta \to S^2$,[13]

$$W = \frac{1}{8\pi}\int_{\partial\Theta} n^*(\epsilon_{ijk}n^i \mathrm{d}n^j \wedge \mathrm{d}n^k)\,, \tag{9}$$

i.e.

$$\begin{aligned} W &= \frac{1}{8\pi}\int_{\partial\Theta} \epsilon_{ijk}n^i\partial_\beta n^j\partial_\gamma n^k \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \\ &= \frac{1}{8\pi}\int_{\partial\Theta} H_{\beta\gamma}\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \quad (\beta, \gamma = 1, 2, 3)\,, \end{aligned} \tag{10}$$

where n^* is the pull back of map n, Θ is a spatial volume, and $\partial\Theta$ is its boundary. The generalized winding number is a topological invariant and is also called the degree of Gauss map. It is well known that W is corresponding to the second homotopy group $\pi_2(S^2) = Z$ (the set of integers).

On the other hand, W also is the total topological charge of the point defects (i.e., the singular point of $\vec{n}$). Using the Stokes' theorem, we get

$$W = \frac{1}{8\pi}\int_\Theta \epsilon_{ijk}\epsilon^{\alpha\beta\gamma}\partial_\alpha n^i\partial_\beta n^j\partial_\gamma n^k \mathrm{d}^3x = \int_\Theta \rho \mathrm{d}^3x\,. \tag{11}$$

So the density of point defects is

$$\begin{aligned} \rho &= \frac{1}{8\pi}\epsilon_{ijk}\epsilon^{\alpha\beta\gamma}\partial_\alpha n^i\partial_\beta n^j\partial_\gamma n^k \\ &= \frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}\partial_\alpha H_{\beta\gamma} = \partial_\alpha\Omega^\alpha\,. \end{aligned} \tag{12}$$

By making use of

$$\partial_\alpha n^a = \partial_\alpha\Big(\frac{\varphi^a}{\|\phi\|}\Big) = (\partial_\alpha\varphi^a)\frac{1}{\|\varphi\|} + \varphi^a\partial_\alpha\Big(\frac{1}{\|\varphi\|}\Big)\,, \tag{13}$$

and the Green function equation in ϕ-space

$$\partial_p\partial_p \ln\|\varphi\| = 2\pi\delta^2(\vec{\varphi}\,)\,, \quad \Big(\partial_p = \frac{\partial}{\partial\varphi^p}\Big)\,. \tag{14}$$

It can be proved that[14]

$$\rho = \delta^3(\vec{\varphi}\,)D\Big(\frac{\varphi}{x}\Big)\,, \tag{15}$$

where $D(\varphi/x)$ is the Jacobian

$$\epsilon^{ijk}D\Big(\frac{\varphi}{x}\Big) = \epsilon^{\alpha\beta\gamma}\partial_\alpha n^i\partial_\beta n^j\partial_\gamma n^k\,. \tag{16}$$

This is similar to Eq. (35). It is necessary to study the zero points of $\vec{\varphi}$ to determine the non-zero solutions of ρ. According to the implicit function theory,[15] under the regular condition $D(\varphi/x) \neq 0$, the general solution of

$$\varphi^a(x^1, x^2, x^3) = 0\,, \quad (a = 1, 2, 3)\,, \tag{17}$$

can be expressed as

$$\begin{aligned} &x^1 = x_l^1(\xi)\,, \quad x^2 = x_l^2(\xi)\,, \\ &x^3 = x_l^3(\xi)\,, \quad (l = 1, 2, \ldots, N)\,, \end{aligned} \tag{18}$$

which represent N isolated singular points P_l. These singular points are just the skyrmions in ferromagnets.

In δ-function theory,[16] we can prove that

$$\delta^3(\vec{\varphi}\,) = \sum_{l=1}^N \frac{\beta_l\eta_l}{D(\varphi/x)_{\vec{x}_l}}\delta^3(\vec{x} - \vec{x}_l)\,, \tag{19}$$

where the positive integer β_l is the Hopf index, and η_l is the Brouwer degree. Then we obtain

$$\rho = \sum_{l=1}^N \beta_l\eta_l\delta^3(\vec{x} - \vec{x}_l)\,. \tag{20}$$

The expression (20) for ρ just leads to that for the topological charges of skyrmions. Using Eqs. (12) and (20) we get the total winding number on $\partial\Theta$ around the N skyrmions:

$$W = \sum_{l=1}^N W_l = \sum_{l=1}^N \beta_l\eta_l\,, \tag{21}$$

where W_l is the winding number of $\vec{\varphi}$ around the l-th skyrmion P_l, which means that when $\vec{x}$ covers the boundary of the neighborhood of P_l in three-dimensional space once, the unit vector $\vec{n}$ will cover S^2 for W_l times.

3 Vortex Rings in Ferromagnets

We know that another nontrivial homotopy group is $\pi_3(S^2)$. The following analysis indicates that this nontrivial property admits the existence of line-like defects (i.e. magnetic vortex lines) and when those vortex lines are closed (i.e. they are in knot-like configuration) it leads to an important topological invariant Hopf invariant to describe the topology of these knots.

We have defined the topological tensor,

$$H_{\beta\gamma} \equiv \epsilon_{ijk}n^i\partial_\beta n^j\partial_\gamma n^k = \partial_\beta A_\gamma - \partial_\gamma A_\beta\,, \tag{22}$$

and A_β is a U(1) gauge potential.

In the following we will define a new two-dimensional vector order parameter $\vec{\phi}$ from the unit vector $\vec{n}$. The vector field $\vec{n}$ is the section of sphere bundle S^2. Define two-dimensional unit vectors $\vec{e}_1$ and $\vec{e}_2$ in this (S^2) (i.e., $\vec{e}_1 \perp \vec{n}$, $\vec{e}_2 \perp \vec{n}$), which are normal to each other, then ($\vec{e}_1$, $\vec{e}_2$, $\vec{n}$) forms an orthogonal frame:

$$\vec{e}_1 \cdot \vec{e}_2 = \vec{e}_1 \cdot \vec{n} = \vec{e}_2 \cdot \vec{n} = 0\,, \tag{23}$$

$$\vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{n} \cdot \vec{n} = 1\,. \tag{24}$$

Consider another two-component vector order parameter $\vec{\phi} = (\phi^1, \phi^2)$ in space, which resides in the plane formed by the unit vectors $\vec{e}_1$ and $\vec{e}_2$ and satisfies

$$e_1^a = \frac{\phi^a}{\|\phi\|}\,, \quad e_2^a = \epsilon_{ab}\frac{\phi^b}{\|\phi\|}\,, \tag{25}$$

it can be proved that equation (25) naturally satisfies the restrictions (23) and (24). Obviously the zero points of ϕ field are just the two-dimensional singular points of $\vec{e}_1$ and $\vec{e}_2$. Using the $\vec{\phi}$ field, the U(1) gauge potential can be expressed as

$$A_\beta = \epsilon_{ab}\frac{\phi^a}{\|\phi\|}\partial_\beta \frac{\phi^b}{\|\phi\|}\,, \tag{26}$$

and the topological tensor $H_{\beta\gamma}$ is

$$H_{\beta\gamma} = 2\epsilon_{ab}\partial_\beta \frac{\phi^a}{\|\phi\|}\partial_\gamma \frac{\phi^b}{\|\phi\|}\,. \tag{27}$$

So the magnetic vorticity Ω^α is

$$\Omega^\alpha = \frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}H_{\beta\gamma} \tag{28}$$

$$= \frac{1}{8\pi}\epsilon^{\alpha\beta\gamma}2\epsilon_{ab}\partial_\beta \frac{\phi^a}{\|\phi\|}\partial_\gamma \frac{\phi^b}{\|\phi\|} \tag{29}$$

$$= \frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\epsilon_{ab}\partial_\beta \frac{\phi^a}{\|\phi\|}\partial_\gamma \frac{\phi^b}{\|\phi\|}\,. \tag{30}$$

According to the ϕ-mapping theory,[17] the two-dimensional topological tensor current is defined as

$$\tilde{K}^{\delta\alpha} = \frac{1}{4\pi}\epsilon^{\delta\alpha\beta\gamma}\epsilon_{ab}\partial_\beta \frac{\phi^a}{\|\phi\|}\partial_\gamma \frac{\phi^b}{\|\phi\|}\,, \quad (\delta, \alpha, \beta, \gamma = 0, 1, 2, 3)\,, \tag{31}$$

and the spatial components of $\tilde{K}^{\delta\alpha}$ are

$$\Omega^\alpha = \tilde{K}^{0\alpha} = \frac{1}{4\pi}\epsilon^{\alpha\beta\gamma}\epsilon_{ab}\partial_\beta \frac{\phi^a}{\|\phi\|}\partial_\gamma \frac{\phi^b}{\|\phi\|}\,, \quad (\alpha, \beta, \gamma = 1, 2, 3)\,. \tag{32}$$

It can be proved that

$$\tilde{K}^{\delta\alpha} = \delta^2(\vec{\phi})D^{\delta\alpha}\Big(\frac{\phi}{x}\Big)\,, \tag{33}$$

where

$$D^{\delta\alpha}\Big(\frac{\phi}{x}\Big) = \frac{1}{2}\epsilon^{\delta\alpha\beta\gamma}\epsilon_{ab}\partial_\beta \frac{\phi^a}{\|\phi\|}\partial_\gamma \frac{\phi^b}{\|\phi\|}\,; \tag{34}$$

and

$$\Omega^\alpha = \delta^2(\vec{\phi})D^\alpha\Big(\frac{\phi}{x}\Big)\,, \tag{35}$$

where

$$D^\alpha\Big(\frac{\phi}{x}\Big) = \frac{1}{2}\epsilon^{\alpha\beta\gamma}\epsilon_{ab}\partial_\beta \frac{\phi^a}{\|\phi\|}\partial_\gamma \frac{\phi^b}{\|\phi\|}\,. \tag{36}$$

Equation (35) requires that the zero-points of $\vec{\phi}$ should be studied in order to determine the non-zero solutions of Ω^α. The implicit function theory shows[15] that under the regular condition $D^\alpha(\phi/x) \neq 0$, the general solution of

$$\phi^a(x^1, x^2, x^3) = 0\,, \quad (a = 1, 2)\,, \tag{37}$$

can be expressed as

$$x^1 = x_k^1(\xi)\,, \quad x^2 = x_k^2(\xi)\,, \quad x^3 = x_k^3(\xi)\,, \quad (k = 1, 2, \ldots, M)\,, \tag{38}$$

which represents M isolated singular strings L_k with string parameter ξ. These topological string structures are just the vortex line structures in ferromagnets.

According to δ-function theory,[16] we can prove

$$\delta^2(\vec{\phi}) = \sum_{k=1}^{M}\beta_k \int_{L_k}\frac{\delta^3(\vec{x} - \vec{x}_k(\xi))\,\mathrm{d}\xi}{|D(\phi/u)|_{\Sigma_k}}\,, \tag{39}$$

where

$$D(\phi/u) = \frac{1}{2}\epsilon^{jk}\epsilon_{mn}\frac{\partial\phi^m}{\partial u^j}\frac{\partial\phi^n}{\partial u^k}\,, \tag{40}$$

and Σ_k is the k-th planar element transversal to L_k with local coordinate (u^1, u^2). The positive integer β_k is the Hopf index of ϕ-mapping, which means that when the point $\vec{x}$ covers the neighborhood of the zero point $\vec{x}_k(\xi)$ once, the vector field $\vec{\phi}$ covers the corresponding region in ϕ-space for β_k times. With the definition of vector Jacobians (36), we can obtain general velocity of the k-th vortex,

$$v_k^\alpha = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\xi}\Big|_{\vec{x}_k} = \frac{D^\alpha(\phi/x)}{D(\phi/u)_{\Sigma_k}}\Big|_{\vec{x}_k}\,. \tag{41}$$

Then from Eqs. (39) and (41) we find the inner topological structure of Ω^α,

$$\Omega^\alpha = \sum_{k=1}^{M}\beta_k\eta_k \int_{L_k}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\xi}\delta^3(\vec{x} - \vec{x}_k(\xi))\,\mathrm{d}\xi = \sum_{k=1}^{M}W_k \int_{L_k}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\xi}\delta^3(\vec{x} - \vec{x}_k(\xi))\,\mathrm{d}\xi\,, \tag{42}$$

where W_k is the winding number of $\vec{\phi}$ around L_k, with $\eta_k = \operatorname{sgn} D(\phi/x)_{\vec{x}_k} = \pm 1$ being the Brouwer degree of ϕ-mapping.

From Eq. (42) one can obtain the topological charge of vortex line L_k,

$$q_k = \frac{1}{2\pi}\int_{\Sigma_k}\frac{1}{2}H_{\beta\gamma}\,\mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma = \int_{\Sigma_k}\Omega^\alpha\,\mathrm{d}\sigma_\alpha = W_k\,. \tag{43}$$

Furthermore the total topological charge on surface Σ is

$$q = \sum_{k=1}^{M}W_k\,. \tag{44}$$

It is obvious that there exist M isolated vortices of which the k-th vortex possesses charge W_k. And the vortex corresponds to $\eta_k = 1$, while the anti-vortex corresponds to $\eta_k = -1$.

From Eq. (42) we can see that when these M vortex lines are M closed curves, it can lead to the vortex rings,

$$\Omega^\alpha = \sum_{k=1}^{M} W_k \oint_{L_k} \frac{\mathrm{d}x^\alpha}{\mathrm{d}\xi} \delta^3(\vec{x} - \vec{x}_k(\xi))\,\mathrm{d}\xi\,. \tag{45}$$

We have mentioned above, that the topological stability of magnetic vortex rings is preserved by the Hopf invariant. In the following, we will establish the relationship between the Hopf invariant (which describes the vortex rings in ferromagnets) and the linking number of knots family.

The topological quantum number of knotted solitons in terms of $H_{\alpha\beta}$ is given by[18]

$$Q = \frac{1}{32\pi^2} \int_{R_3} \epsilon^{\alpha\beta\gamma} A_\alpha H_{\beta\gamma}\,\mathrm{d}^3 x\,, \tag{46}$$

where A_α is a U(1) gauge potential. The definition of $\vec{n}$ defines a mapping $S^3 \to S^2$, which falls into the third homotopy classes $\pi_3(S^2) = Z$. So the definition (46) leads to a Hopf invariant.

It can be proved that the expression for Hopf invariant is[19]

$$Q = \sum_{k=1}^{M} W_k^2 \mathrm{SL}(\gamma_k) + \sum_{k,l=1,k\neq l}^{M} W_k W_l\,\mathrm{Lk}(\gamma_k, \gamma_l)\,, \tag{47}$$

where $\mathrm{Lk}(\gamma_k, \gamma_l)$ is the Gauss linking number and $\mathrm{SL}(\gamma_k)$ is the self-linking number of γ_k. W_k is the winding number. So the expression of Hopf invariant reveals the relationship between the Hopf invariant Q and the self-linking and the linking numbers of the knots family. Because the linking numbers and self-linking numbers are topological invariant in topology, Hopf invariant is a proper quantity to describe the topology of these magnetic vortex rings.

At last, we must point out that in this paper the vortex rings are treated as geometric lines, i.e., the width of a vortex ring is zero; but in experiments, this width does not vanish. Then, in experiments, since the magnetic vortex rings and the skyrmions originate from the non-trivial $\vec{n}$ field distributions at large distances, the width of the core of a magnetic vortex line should be large. We have used the regular condition $D^i(\phi/x) \neq 0$ in this paper. But in mathematics, this condition is not always tenable. When this condition fails, branch process will occur. Since Hopf invariant is a topological invariant, in branch processes during the evolution of knot (splitting, merging, and intersection), the Hopf invariant is preserved.

4 Conclusion

In this paper, by making use of the ϕ-mapping theory, we show that there exist two kinds of topological defects, i.e., the skyrmions and the vortex rings in ferromagnets. Noting that there are different $\vec{n}$ field configurations in space, we respectively define the vector order parameters $\vec{\varphi}$ in Sec. 2 and $\vec{\phi}$ in Sec. 3. Using these two new order parameters, we can get a three-dimensional topological charge density ρ, which can be derived from $H_{\alpha\beta}$ and leads to the skyrmions, also we can see that the "magnetic vorticity" Ω^α is a two-dimensional topological current, which leads to the vortex rings. These two kinds of topological defects are characterized in terms of Hopf indices and Brouwer degrees of ϕ-mapping. And at last we prove that for the vortex rings the Hopf invariant is just total sum of all the self-linking and linking number of knots family.

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