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Soliton Solutions and Bilinear Bäcklund Transformation for Generalized Nonlinear Schrödinger Equation with Radial Symmetry*

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Abstract *Investigated in this paper is the generalized nonlinear Schrödinger equation with radial symmetry. With the help of symbolic computation, the one-, two-, and N-soliton solutions are obtained through the bilinear method. Bäcklund transformation in the bilinear form is presented, through which a new solution is constructed. Graphically, we have found that the solitons are symmetric about $x = 0$, while the soliton pulse width and amplitude will change along with the distance and time during the propagation.*

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1 Introduction

Existence of the bright solitons in the lossless fibers has been theoretically demonstrated,[1] and the bright solitons have been experimentally observed.[2] Since then, optical solitons have generated interest both in theoretical and experimental studies because of their potential applications in the optical communication systems.[3] Among the nonlinear evolution equations (NLEEs),[3–5] the nonlinear Schrödinger equation (NLSE) is a model to describe several phenomena of the optical solitons in the optical fibers.[1,3] Generally speaking, propagation of the optical solitons is governed by the NLSE and associated with the group velocity dispersive and nonlinear effects of the fibers.[3] Besides, the NLSE plays a key role in such the various fields of physical science as the fluid dynamics, plasma physics, and Bose–Einstein condensation.[3,6] The standard NLSE is as follows:[3]

$$i\, q_t + q_{xx} + 2\,|\,q\,|^2\, q = 0\,, \qquad (1)$$

where $i = \sqrt{-1}$, q is the complex envelope of the electrical field in the comoving frame with x as the propagation distance and t as the retarded time. The subscripts x and t denote partial derivatives.

Seen in the prevenient studies, Eq. (1) has been investigated in such aspects as the Painlevé-property,[7–8] Lax pair,[8] Bäcklund transformation (BT)[9] and Darboux transformation (DT).[10] However, Eq. (1) ignores some terms, which can not be negligible in a lot of physically relevant cases.[11] Moreover, the inhomogeneous medium in the fibers has an influence on the transmission of the solitons.[12] Thus, it is significant and advisable to investigate the generalized NLSE with more complex terms.[12–16] For example, the generalized variable-coefficient NLSE can be investigated when the varying dispersion, nonlinearity and gain/loss are considered;[3,12–13] for the optical pulses in the femtosecond regime, when considering the experimental requirements on the higher power and ultrashort pulse propagation as seen in the soliton compression and frequency shift, Ref. [14] has done some studies on the generalized variable-coefficient higher-order NLSE with the higher-order effects such as the third-order dispersion, self-steepening, and decayed nonlinear response; for a realistic fiber, when considering the effects of the averaged random birefringence on an orthogonally polarized pulse, Ref. [15] has presented the two-coupled NLSEs of the Manakov type. The derivative NLSE and N-coupled generalized NLSEs can be obtained as well.[15–16]

Besides, Ref. [17] has presented the following type of the generalized NLSE:

$$i\, q_t + q_{xx} + |\,q\,|^2\, q + \frac{n-1}{x}\, q_x = 0\,, \qquad (2)$$

where the nonnegative integer n is the number of space dimensions,[17] which can describe the one-dimensional

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waves in the n space dimensions in some physical situations including some phenomena in the plasma physics and nonlinear optics.[17]

In this paper, we will focus on the generalized NLSE with radial symmetry[18−19]

$$i\,q_t + q_{xx} + 2\,|\,q\,|^2\,q + \frac{n-1}{x}\,q_x - \frac{n-1}{x^2}\,q + 4\,(n-1)\,q\int_0^x \frac{|\,q\,|^2}{x'}\,dx' = 0\,, \tag{3}$$

where n and q are the same as before. Furthermore, Eq. (3) is more general than Eq. (2) in Ref. [17], which is also our purpose of investigating Eq. (3). Equation (3) has been shown to possess the Painlevé-property when $n = 1$ and $n = 2$.[18] For $n = 1$, Eq. (3) is reduced to the standard NLSE; for $n = 2$, Eq. (3) becomes the following form that we will investigate here:

$$i\,q_t + q_{xx} + 2\,|\,q\,|^2\,q + \frac{q_x}{x} - \frac{q}{x^2} + 4\,q\int_0^x \frac{|\,q\,|^2}{x'}\,dx' = 0\,. \tag{4}$$

Reference [18] has derived the Lax pair in the shape of the matrices, BT and one-soliton solution for Eq. (4).

Generally, in order to understand some phenomena described by the NLEEs better, it is necessary to explore the analytic solutions for those equations.[3−5] Therefore, various methods have been presented with the rapid development of the soliton theory in the last few years.[20−23] For example, the NLEEs can be solved by virtue of the BT, DT, inverse scattering transformation[8] and bilinear method.[24−27] In this paper, on the basis of the bilinear method, we will first transform Eq. (4) into a new form which is different from that in Ref. [18], and obtain the bilinear form, N-soliton solutions and BT in the bilinear form for Eq. (4) with symbolic computation.[3−5] Furthermore, we can analyze and explain the soliton properties and features via the obtained analytic solutions graphically.[28] Relevant issues can be seen, e.g., in Refs. [31].

The process of this paper will be as follows. In Sec. 2, making use of the truncated Painlevé expansion, the bilinear form for Eq. (4) will be obtained. At the end of this section, the one-, two-, and N-soliton solutions will be derived from the bilinear form with the help of symbolic computation. In Sec. 3, the BT in the bilinear form for Eq. (4) and the nontrivial solution from the BT and trivial solution will be presented. In Sec. 4, some conclusions and discussions will be given.

2 Bilinear Form and Soliton Solutions

To deal with the integral term in Eq. (4), we can transform it into the following form:

$$i\,q_t + q_{xx} + 2\,|\,q\,|^2\,q + \frac{q_x}{x} - \frac{q}{x^2} + 4\,q\,r = 0\,, \tag{5}$$

$$r_x - \frac{|\,q\,|^2}{x} = 0\,, \tag{6}$$

where r is a real function of variables x and t. Equations (5) and (6) can be turned into the form as follows:

$$i\,q_t + q_{xx} + 2\,q^2\,p + \frac{q_x}{x} - \frac{q}{x^2} + 4\,qr = 0\,, \tag{7}$$

$$-\,i\,p_t + p_{xx} + 2\,p^2\,q + \frac{p_x}{x} - \frac{p}{x^2} + 4\,p\,r = 0\,, \tag{8}$$

$$r_x - \frac{q\,p}{x} = 0\,, \tag{9}$$

where p is the complex conjugate of q.

According to the Painlevé analysis,[29] q, p, and r are assumed to be

$$q \sim q_0\,\phi^{\alpha}(x,t), \quad p \sim p_0\,\phi^{\beta}(x,t)\,, \quad r \sim r_0\,\phi^{\gamma}(x,t)\,, \tag{10}$$

where q_0, p_0, and r_0 are the analytic functions of variables x and t, and $\phi(x,t)$ is the movable singularity manifold, with α, β, and γ as the integers. Substituting expressions (10) into Eqs. (7)–(9), we can get

$$\alpha = \beta = \gamma = -1\,. \tag{11}$$

By the truncated Painlevé expansions, we assume

$$q = q_0\,\phi^{-1}(x,t) + q_1\,, \tag{12}$$

$$p = p_0\,\phi^{-1}(x,t) + p_1\,, \tag{13}$$

$$r = r_0\,\phi^{-1}(x,t) + r_1\,, \tag{14}$$

where q_1, p_1, and r_1 are the analytic functions of variables x and t. Substituting expressions (12)–(14) into Eqs. (7)–(9), we can yield

$$q_0\,p_0 = -\phi_x^2(x,t), \quad r_0 = \frac{\phi_x(x,t)}{x}\,. \tag{15}$$

To obtain the bilinear form for Eq. (4), the dependent variable transformations are made as follows:

$$q = x\,\frac{g(x,t)}{f(x,t)}, \quad r = \frac{[\ln f(x,t)]_x}{x}\,, \tag{16}$$

where $g(x,t)$ is a complex differentiable function and $f(x,t)$ is a real one. Substituting transformations (16) into Eqs. (5) and (6), we obtain the bilinear form for Eq. (4) as follows:

$$[\,i\,x\,D_t + 3\,D_x + x\,D_x^2](g\cdot f) = -2\,g\,f_x\,, \tag{17}$$

$$D_x^2(f\cdot f) = 2\,x^2\,g\,g^* + \frac{2}{x}\,f\,f_x\,, \tag{18}$$

where $*$ stands for the complex conjugate, D_x and D_t are the bilinear derivative operators[24] defined by

$$D_x^m\,D_t^n\,a(x,t)\cdot b(x,t) = \Big(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\Big)^m\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)^n \times a(x,t)\,b(x',t')|_{x'=x,t'=t}\,.$$

First of all, we expand $g(x,t)$ and $f(x,t)$, with respect to a small parameter ε as follows:[24]

$$g(x,t) = \varepsilon\,g_1(x,t) + \varepsilon^3\,g_3(x,t) + \varepsilon^5\,g_5(x,t) + \cdots\,, \tag{19}$$

$$f(x,t) = 1 + \varepsilon^2\,f_2(x,t) + \varepsilon^4\,f_4(x,t) + \varepsilon^6\,f_6(x,t) + \cdots\,. \tag{20}$$

Next, substituting expansions (19) and (20) into Eqs. (17) and (18) and collecting the coefficients of each order of ε, we can derive

$$\varepsilon^0 :\ D_x^2(1\cdot 1)=0\,, \tag{21}$$

$$\varepsilon^1 :\ [\,i\,x\,D_t+3\,D_x+x\,D_x^2](g_1\cdot 1)=0\,, \tag{22}$$

$$\varepsilon^2 :\ D_x^2(f_2\cdot 1+1\cdot f_2)=2\,x^2\,g_1\,g_1^*\,, \tag{23}$$

$$\varepsilon^3 :\ [\,i\,x\,D_t+3\,D_x+x\,D_x^2](g_1\cdot f_2+g_3\cdot 1) = -2\,g_1\,f_{2\,x}\,, \tag{24}$$

$$\varepsilon^4 :\ D_x^2(f_4\cdot 1+f_2\cdot f_2+1\cdot f_4) = 2\,x^2(g_1\,g_3^*+g_3\,g_1^*)+\frac{2}{x}\,f_2\,f_{2\,x}\,, \tag{25}$$

$$\vdots$$

With the benefit of expressions (21)–(25) and symbolic computation, we can obtain the one-, two-, and N-soliton solutions for Eq. (4).

Now, to get the one-soliton solution, we assume that expression (19) is truncated to $g_1(x,t)$ and expression (20) is truncated to $f_2(x,t)$, that is to say, while $g_j(x,t)=0$ $(j=3,5,7,\ldots)$ and $f_k(x,t)=0$ $(k=4,6,8,\ldots)$. $g_1(x,t)$ can be considered as

$$g_1(x,t)=e^{\eta}\quad \text{with}\quad \eta=\varphi_1(t)\,x^2+\varphi_2(t)\,, \tag{26}$$

where $\varphi_1(t)$ and $\varphi_2(t)$ are both the complex differentiable functions to be determined. Substituting expression (26) into expressions (21)–(25), we have

$$g_1(x,t)=\frac{\exp[\sigma\,-x^2/(c+4\,i\,t)]}{(c+4\,i\,t)^2}\,, \tag{27}$$

$$f_2(x,t)=\frac{\exp[\sigma+\sigma^*-\frac{(c+c^*)\,x^2}{(c+4\,i\,t)(c^*-4\,i\,t)}]}{4\,(c+c^*)^2}\,, \tag{28}$$

where σ and c are both the complex constants. Without loss of generality, we set $\varepsilon=1$ and obtain

$$g(x,t)=\frac{\exp[\sigma\,-x^2/(c+4\,i\,t)]}{(c+4\,i\,t)^2}\,, \tag{29}$$

$$f(x,t)=1+\frac{\exp[\sigma+\sigma^*\,-\,\frac{(c+c^*)\,x^2}{(c+4\,i\,t)(c^*-4\,i\,t)}]}{4\,(c+c^*)^2}\,. \tag{30}$$

Substituting expressions (29) and (30) into transformations (16), we obtain the one-soliton solution for Eq. (4)

$$q(x,t)=\frac{x}{2\,(c+4\,i\,t)^2}\,\mathrm{sech}\Big(\frac{\theta+\theta^*+\theta_0}{2}\Big)\,e^{(\theta-\theta^*-\theta_0)/2}\,, \tag{31}$$

where $\theta=\sigma-x^2/(c+4\,i\,t)$ and $\theta_0=-2\ln|2\,(c+c^*)|$.

Similarly, to obtain the two-soliton solution, $g_1(x,t)$ can be assumed to be

$$g_1(x,t)=e^{\eta_1}+e^{\eta_2}\quad \text{with}\quad \eta_k=\varphi_{k1}(t)\,x^2+\varphi_{k2}(t)\quad (k=1,2)\,, \tag{32}$$

where $\varphi_{k1}(t)$ and $\varphi_{k2}(t)$ are both the complex differentiable functions to be determined.

Taking the advantage of expressions (21)–(25) and setting $\varepsilon=1$, we obtain the two-soliton solution for Eq. (4)

$$q(x,t)=x\,\frac{g_1(x,t)+g_3(x,t)}{1+f_2(x,t)+f_4(x,t)}\,, \tag{33}$$

with

$$g_1(x,t)=\frac{e^{\theta_1}}{(c_1+4\,i\,t)^2}+\frac{e^{\theta_2}}{(c_2+4\,i\,t)^2}\,,$$

$$f_2(x,t)=\frac{e^{\theta_1+\theta_1^*}}{4\,(c_1+c_1^*)^2}+\frac{e^{\theta_1+\theta_2^*}}{4\,(c_1+c_2^*)^2}+\frac{e^{\theta_2+\theta_1^*}}{4\,(c_2+c_1^*)^2}+\frac{e^{\theta_2+\theta_2^*}}{4\,(c_2+c_2^*)^2}\,,$$

$$g_3(x,t)=\frac{(c_1^*-4\,i\,t)^2\,(c_1-c_2)^2\,e^{\theta_1+\theta_2+\theta_1^*}}{4\,(c_1+4\,i\,t)^2\,(c_2+4\,i\,t)^2\,(c_1+c_1^*)^2\,(c_2+c_1^*)^2} + \frac{(c_2^*-4\,i\,t)^2\,(c_1-c_2)^2\,e^{\theta_1+\theta_2+\theta_2^*}}{4\,(c_1+4\,i\,t)^2\,(c_2+4\,i\,t)^2\,(c_1+c_2^*)^2\,(c_2+c_2^*)^2}\,,$$

$$f_4(x,t)=\frac{(c_1-c_2)^2\,(c_1^*-c_2^*)^2\,e^{\theta_1+\theta_2+\theta_1^*+\theta_2^*}}{16\,(c_1+c_1^*)^2\,(c_1+c_2^*)^2\,(c_2+c_1^*)^2\,(c_2+c_2^*)^2}\,,$$

where $\theta_l=\sigma_l-x^2/(c_l+4\,i\,t)$ with σ_l and c_l both as the complex constants $(l=1,2)$.

More generally, we can obtain the N-soliton solution for Eq. (4) in the sense of Refs. [27, 30] as follows:

$$q(x,t)=x\,\frac{g(x,t)}{f(x,t)}\,, \tag{34}$$

where

$$f(x,t)=\sum_{\mu=0,1}{}'\exp\left[\sum_{k=1}^{2N}\mu_k\,\chi_k+\sum_{k<j}^{2N}\psi(k,j)\mu_k\mu_j\right]\,,$$

$$g(x,t) = \sum_{\mu=0,1}{}'' \exp\left[\sum_{k=1}^{2N}\mu_k\,\chi_k + \sum_{k<j}^{2N}\psi(k,j)\mu_k\mu_j\right],$$

$$g^*(x,t) = \sum_{\mu=0,1}{}''' \exp\left[\sum_{k=1}^{2N}\mu_k\,\chi_k + \sum_{k<j}^{2N}\psi(k,j)\mu_k\mu_j\right],$$

with

$$\chi_k = \xi_k(t)\,x^2 + \delta_k(t), \quad \xi_k(t) = -\frac{1}{c_k + 4\,i\,t}, \quad \delta_k(t) = \sigma_k - 2\,\ln|\xi_k(t)|, \quad \text{for} \quad k = 1\,,2\,,3\,,\ldots,2N\,,$$

$$\chi_{k+N} = \chi_k^*, \quad \xi_{k+N} = \xi_k^*, \quad \delta_{k+N} = \delta_k^*, \quad \text{for} \quad k = 1\,,2\,,3\,,\ldots,N\,,$$

$$\psi(k,j) = -2\ln|2\,[\xi_k(t) + \xi_j(t)]|, \quad \text{for} \quad k = 1\,,\ldots,N \quad \text{and} \quad j = N+1\,,\ldots,2N\,,$$

$$\text{or} \quad k = N+1\,,\ldots,2N \quad \text{and} \quad j = 1\,,\ldots,N\,,$$

$$\psi(k,j) = 2\ln|2\,[\xi_k(t) - \xi_j(t)]|, \quad \text{for} \quad k = 1\,,\ldots,N \quad \text{and} \quad j = 1\,,\ldots,N\,,$$

$$\text{or} \quad k = N+1\,,\ldots,2N \quad \text{and} \quad j = N+1\,,\ldots,2N\,,$$

where c_k and σ_k are the complex constants ($k = 1, 2, \ldots, 2N$), $\sum_{k<j}^{2N}$ stands for the summation over all possible combinations taken from $2N$ elements only if $k < j$, while $\sum_{\mu=0,1}{}'$, $\sum_{\mu=0,1}{}''$, and $\sum_{\mu=0,1}{}'''$ denote the summations over all possible combinations of $\mu_k = 0, 1$ ($k = 1,\, 2, \ldots, 2N$), satisfing the relationships as follows:

$$\sum_{k=1}^{N}{}'\mu_k = \sum_{k=1}^{N}{}'\mu_{k+N}\,, \quad \sum_{k=1}^{N}{}''\mu_k = 1 + \sum_{k=1}^{N}{}''\mu_{k+N}\,, \quad \sum_{k=1}^{N}{}'''\mu_{k+N} = 1 + \sum_{k=1}^{N}{}'''\mu_k\,.$$

3 BT in Bilinear Form

It is known that the BT is a method structuring the new solution from a known one, especially a trivial solution.[9] In this section, our main purpose is to obtain the BT in the bilinear form by virtue of Eqs. (17) and (18).

Hence, we consider

$$P \equiv f^{\,2}[\,i\,x\,D_t + 3\,D_x + x\,D_x^2](g'\cdot f') - f'^{\,2}[\,i\,x\,D_t + 3\,D_x + x\,D_x^2](g\cdot f) = 0\,, \tag{35}$$

where (f', g') and (f, g) both satisfy Eqs. (17) and (18).

Taking the advantage of

$$D_x(g'\cdot f + f'\cdot g) = \kappa(t)\,x\,(g'\,f - f'\,g)\,, \tag{36}$$

where $\kappa(t)$ is the complex differential function of variable t, and using some exchange formulas of the bilinear operator[24−27] and Eq. (18), we can rewrite Eq. (35) as

$$\begin{aligned} P \equiv\; & f'\,f\,\{[\,i\,x\,D_t + 3\,D_x](g'\cdot f + f'\cdot g) - x\,D_x^2(g'\cdot f - f'\cdot g) + 6\,(g'\,f_x - g\,f'_x)\} \\ & - (g'\,f + f'\,g)\{[\,i\,x\,D_t +\,3\,D_x](f'\cdot f) - x^3(g'\,g^* - g\,g'^*) + 3\,(f'\,f_x - f\,f'_x)\} \\ & + (g'\,f - f'\,g)[x\,D_x^2(f'\cdot f) - x^3(g'\,g^* + g\,g'^*) - (f'\,f_x + f\,f'_x)] \\ & + 2\,x\,D_x[D_x(g'\cdot f + f'\cdot g)\cdot f'\,f] \end{aligned} \tag{37}$$

$$\begin{aligned} \equiv\; & f'\,f\,\{[\,i\,x\,D_t + 5\,D_x](g'\cdot f + f'\cdot g) - x\,D_x^2(g'\cdot f - f'\cdot g) + 6\,(g'\,f_x - g\,f'_x)\} \\ & - (g'\,f + f'\,g)\{[\,i\,x\,D_t + 3\,D_x + 2\,\kappa(t)\,x^2 D_x](f'\cdot f) - 2\,i\,x^3\,\mathrm{Im}(g'\,g^*) + 3\,(f'\,f_x - f\,f'_x)\} \\ & + (g'\,f - f'\,g)\{[\,x\,D_x^2 + 2\,\kappa^2(t)\,x^3](f'\cdot f) - 2\,x^3\,\mathrm{Re}(g'\,g^*) - (f'\,f_x + f\,f'_x)\}\,, \end{aligned} \tag{38}$$

where $\mathrm{Re}(g'\,g^*)$ and $\mathrm{Im}(g'\,g^*)$ denote the real part and the imaginary part for $g'\,g^*$, respectively.

Decoupling Eq. (38), we derive the BT in the bilinear form as follows:

$$[\,i\,x\,D_t + 5\,D_x](g'\cdot f + f'\cdot g) - [\,x\,D_x^2 + \lambda(t)\,x^3](g'\cdot f - f'\cdot g) + 6\,(g'\,f_x - g\,f'_x) = 0\,, \tag{39}$$

$$[\,i\,x\,D_t + 3\,D_x + 2\,\kappa(t)\,x^2 D_x](f'\cdot f) - 2\,i\,x^3\,\mathrm{Im}(g'\,g^*) + 3\,(f'\,f_x - f\,f'_x) = 0\,, \tag{40}$$

$$[\,x\,D_x^2 + 2\,\kappa^2(t)\,x^3 + \lambda(t)\,x^3](f'\cdot f) - 2\,x^3\,\mathrm{Re}(g'\,g^*) - (f'\,f_x + f\,f'_x) = 0\,, \tag{41}$$

$$D_x(g'\cdot f + f'\cdot g) = \kappa(t)\,x\,(g'\,f - f'\,g)\,, \tag{42}$$

where $\lambda(t)$ is the complex differential function of variable t.

In order to get another solution for Eq. (4) from the trivial solution $q(x,t)=0$, we can make use of Eqs. (39)–(42). Setting

$$g(x,t)=0, \quad f(x,t)=1\,, \tag{43}$$

and substituting expressions (43) to Eqs. (39)–(42), we have

$$i\,x\,g'_t+5\,g'_x-x\,g'_{x\,x}-\lambda(t)\,x^3\,g'=0\,, \tag{44}$$
$$i\,x\,f'_t+2\,\kappa(t)\,x^2\,f'_x=0\,, \tag{45}$$
$$x\,f'_{x\,x}+2\,\kappa^2(t)\,x^3\,f'+\lambda(t)\,x^3\,f'-f'_x=0\,, \tag{46}$$
$$g'_x-\kappa(t)\,x\,g'=0\,. \tag{47}$$

Via Eqs. (44)–(47), we can derive

$$g'=e^{(1/2)\,x^2\,\kappa(t)+\omega(t)}\,,$$
$$f'=e^{a(t)\,x^2+b(t)}+\tau\,e^{-a(t)\,x^2-b(t)}\,, \tag{48}$$

with

$$a(t)=\frac{4\,\rho_3}{256+\rho_3^2\,(\rho_4+t)^2}, \quad b(t)=\rho_2\,,$$
$$\omega(t)=\rho_1+\ln\Big[\frac{4\,\rho_3}{256+\rho_3^2\,(\rho_4+t)^2}\Big]\,,$$
$$\kappa(t)=\frac{i\,\rho_3^2\,(\rho_4+t)}{2\,[256+\rho_3^2\,(\rho_4+t)^2]}\,,$$
$$\lambda(t)=\frac{\rho_3^2\,[-128+\rho_3^2\,(\rho_4+t)^2]}{2\,[256+\rho_3^2\,(\rho_4+t)^2]^2}\,,$$
$$\tau=\frac{\rho_1+\rho_1^*}{16}\,,$$

where ρ_1 is a complex constant with ρ_2, ρ_3, and ρ_4 as the real constants. Through the transformation

$$q'(x,t)=x\,\frac{g'(x,t)}{f'(x,t)}\,, \tag{49}$$

we can obtain $q'(x,t)$ as the new solution for Eq. (4).

4 Discussions and Conclusions

In this section, we will show some figures of the solutions for Eq. (4) and illustrate them.

As seen in Figs. 1–2, in common, that the solitons in the nonlinear optics are symmetric about $x=0$, which results from that Eq. (4) possesses the nature of radial symmetry. Meanwhile, the soliton pulse width and amplitude will change along with the distance and time during the propagation, which is different from those where the soliton is stable, that is, the soliton pulse width and amplitude are unchangeable during the propagation in such classical equations as the Korteweg-de Vries equation and standard NLSE.[20] In Fig. 1, the soliton amplitude grows higher when the pulse width becomes narrower, but the amplitude begins to drop when it obtains the highest point, meanwhile the pulse width spreads. Figure 2 describes the interaction of the two solitons in the nonlinear optics and the changing tendency is similar to the one soliton in Fig. 1.

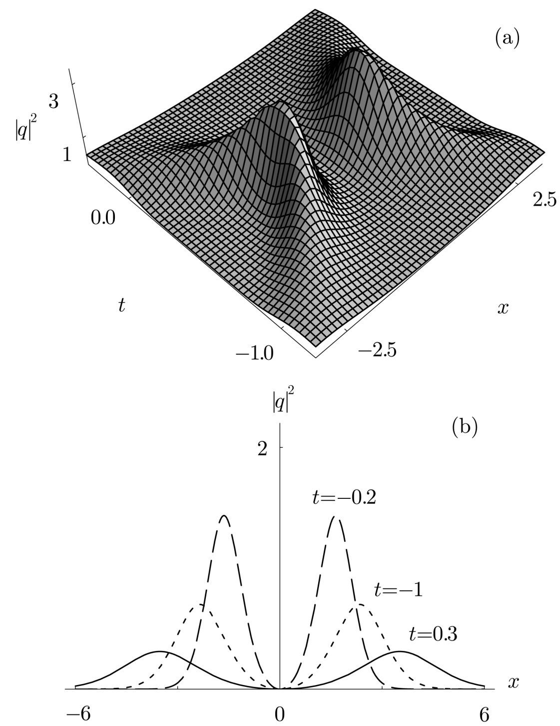

Fig. 1 Evolution of the one soliton via expression (31) and section graphics at $t=-1$, $t=-0.2$, and $t=0.3$ for $|q|^2$ with parameters: $\sigma=2-i$ and $c=1+2\,i$.

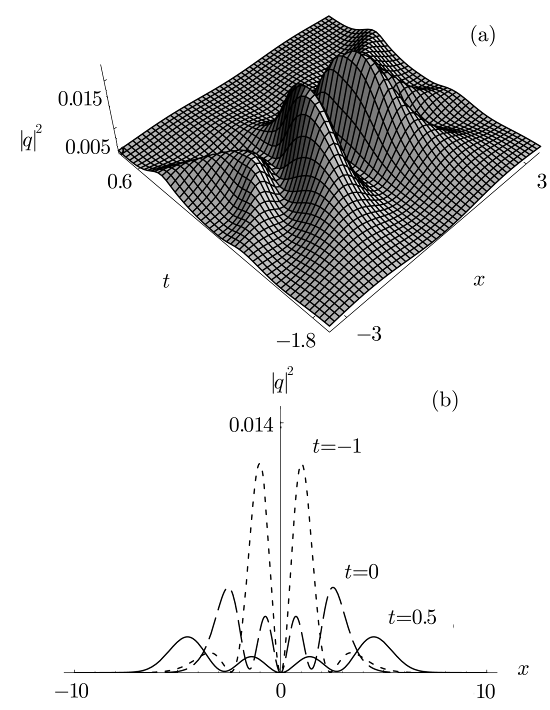

Fig. 2 Evolution of the two solitons via expression (33) and section graphics at $t=-1$, $t=0$, and $t=0.5$ for $|q|^2$ with parameters: $\sigma_1=-1$, $\sigma_2=4\,i$, $c_1=1+i$, and $c_2=2+3\,i$.

In conclusion, we have got the bilinear form and N-

soliton solution [see expression (34)] for Eq. (4). Through the graphical analysis, we have obtained the following: (1) the solitons are symmetric abourt $x = 0$ due of the nature of radial symmetry; (2) the soliton amplitude grows higher when the pulse width becomes narrower, but the amplitude begins to drop when it obtains the highest point, meanwhile the pulse width spreads, as seen in Figs. 1 and 2. Besides, we have obtained the BT in the bilinear form [see expressions (39)–(42)] and new solution [see expression (49)].

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