

## Exact Solutions of the Harry–Dym Equation

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**Abstract** *The aim of this paper is to generate exact travelling wave solutions of the Harry–Dym equation through the methods of Adomian decomposition, He’s variational iteration, direct integration, and power series. We show that the two later methods are more successful than the two former to obtain more solutions of the equation.*

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**Key words:** Harry–Dym equation, exact travelling wave solution, Adomian decomposition method, variational iteration method, direct integration method, power series method

### 1 Introduction

Recently, finding exact travelling wave solutions of nonlinear evolution equations (NLEEs), which plays an important role in nonlinear sciences has received much concern among many brilliant scientists in several fields of particular Physics and Mathematics.

Bäcklund transformation,<sup>[1–2]</sup> inverse scattering transform,<sup>[3]</sup> f-expansion method,<sup>[4]</sup> tanh-function method,<sup>[5]</sup> homogeneous balance method,<sup>[6]</sup> Hirota bilinear method,<sup>[7]</sup> Jacobi elliptic function method,<sup>[8]</sup> variational iteration method,<sup>[9–10]</sup> homotopy perturbation method<sup>[11–12]</sup> and exp-function method<sup>[13]</sup> are several powerful methods which have been employed to obtain exact travelling wave solutions of NLEEs.

The Harry–Dym (HD) equation  $u_t = u^3 u_{3x}$  that has nonlinearity and dispersion coupled together was discovered by H. Dym in 1973–1974 while its first appearance in the literature occurred in a 1975 paper of Kruskal,<sup>[14]</sup> where it was named after its discoverer. It arises, e.g., in the analysis of the Saffman–Taylor problem with surface tension.<sup>[15]</sup>

The HD equation belongs to a class of NLEEs known as completely integrable systems, whose members share a number of remarkable properties including soliton solutions, Bäcklund transformations, the Painlevé property and infinitely many conservation laws.<sup>[16–18]</sup> An implicit cusp-type single solitary wave solution of the HD had been constructed via a direct method in Ref. [19].

In this paper, we intend to establish some exact travelling wave solutions of the HD equation. For this purpose the well-known methods of Adomian decomposition, He’s variational iteration, direct integration, and power series are applied.

The rest of the paper is organized as follows. Direct integration method is explored in Sec. 2. Methods of Adomian decomposition and He’s variational iteration are briefly described in Secs. 3 and 4, respectively. Sec-

tion 5 is devoted to explanation of power series method. The paper is concluded with a capsulized conclusion.

### 2 Direct Integration Method

Using the wave variable  $\xi = x + ct$  and  $u(x, t) = v(\xi)$ , the Harry–Dym equation

$$u_t = u^3 u_{xxx} \quad (1)$$

changes to

$$cv' = v^3 v''' \quad (2)$$

or

$$v''' + \left(\frac{c}{2v^2}\right)' = 0. \quad (3)$$

A first integration with respect to  $\xi$ , yields

$$v'' + \frac{c}{2v^2} = \frac{c_1}{2}. \quad (4)$$

Multiplication (4) by  $v'$ , followed by a second integration with respect to  $\xi$ , yields

$$v'^2 = \frac{c}{v} + c_1 v + c_2. \quad (5)$$

Separation of variables and integration then leads to

$$\xi = \pm \int \sqrt{\frac{v}{c_1 v^2 + c_2 v + c}} dv + c_3. \quad (6)$$

After that, it can be obtained

$$\xi = \frac{\pm i ABCD}{c_1 G} (\text{ES}(i \sinh^{-1}(G), H) - \text{EF}(i \sinh^{-1}(G), H)) + c_3, \quad (7)$$

where

$$A = -c_2 + \sqrt{-4cc_1 + c_2^2},$$

$$B = \sqrt{\frac{c_2 + \sqrt{-4cc_1 + c_2^2} + 2c_1 v}{c_2 + \sqrt{-4cc_1 + c_2^2}}},$$

$$C = \sqrt{1 + \frac{2c_1 v}{c_2 - \sqrt{-4cc_1 + c_2^2}}}, \quad D = \sqrt{\frac{v}{c + v(c_2 + c_1 v)}},$$

$$G = \sqrt{\frac{2c_1 v}{c_2 + \sqrt{-4cc_1 + c_2^2}}}, \quad H = \frac{c_2 + \sqrt{-4cc_1 + c_2^2}}{c_2 - \sqrt{-4cc_1 + c_2^2}},$$

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and  $EF(\phi, m) = \int_0^\phi (1 - m \sin^2 \theta)^{-1/2} d\theta$  and  $ES(\phi, m) = \int_0^\phi (1 - m \sin^2 \theta)^{1/2} d\theta$  are the elliptic integrals of the first and second kind, respectively. Although, solution (7) is obtained by a computer algebra system (CAS), such as *Mathematica*, one cannot apply it readily neither analytically nor numerically. Therefore, we shall find some simpler solutions. Some interesting solutions are obtained directly from Eq. (6) as follows

- Setting  $c_1 = c_2 = 0$  leads to

$$\xi = \pm \frac{2v^{3/2}}{3\sqrt{c}} + c_3,$$

which is equivalent to

$$v = \left( c_3 \pm \frac{3\sqrt{c}}{2} \xi \right)^{2/3}, \tag{8}$$

- Assuming  $c_2 \neq 0$ , setting  $c_1 = 0$  leads to

$$\xi = \pm \frac{\sqrt{c_2 v(c_2 v + c)} - c \ln(2(\sqrt{c_2 v} + \sqrt{c_2 v + c}))}{c_2^{3/2}} + c_3, \tag{9}$$

- Setting  $c_2 = 2\sqrt{cc_1}$  leads to

$$\xi = \pm \frac{2(c_1^{1/4} \sqrt{v} - c^{1/4} \tan^{-1}(c_1^{1/4} \sqrt{v}/c^{1/4}))}{c_1^{3/4}} + c_3,$$

which by putting  $c_1 = c$  reduces to

$$\xi = \pm \frac{2}{\sqrt{c}} (\sqrt{v} - \tan^{-1} \sqrt{v}) + c_3, \tag{10}$$

- Setting  $c_2 = -2\sqrt{cc_1}$  leads to

$$\xi = \pm \frac{2(c_1^{1/4} \sqrt{v} - c^{1/4} \tanh^{-1}(c_1^{1/4} \sqrt{v}/c^{1/4}))}{c_1^{3/4}} + c_3,$$

which by putting  $c_1 = c$  reduces to

$$\xi = \pm \frac{2}{\sqrt{c}} (\sqrt{v} - \tanh^{-1} \sqrt{v}) + c_3. \tag{11}$$

In the hope of getting  $v$  with respect to  $\xi$  explicitly, we interchange the role of  $v$  and  $\xi$  and change Eq. (2) as

$$c\xi'^4 = 3v^3\xi''^2 - v^3\xi'\xi''', \tag{12}$$

where  $\xi' = d\xi/dv$ . After doing some manipulation, we derive

$$\left( \ln \frac{\xi'^3}{\xi''} \right)' = \frac{c}{v^3} \frac{\xi'^3}{\xi''}. \tag{13}$$

A first integration with respect to  $v$ , yields

$$\frac{\xi''}{\xi'^3} = \frac{c}{2v^2} - \frac{c_1}{2}. \tag{14}$$

A second integration with respect to  $v$ , yields Eq. (5). Therefore, this approach fails to obtain  $v$  with respect to  $\xi$  explicitly. Hence, it seems that Eq. (8) is the only explicit solution of Eq. (1).

### 3 Adomian Decomposition Method

In the Adomian decomposition method, at first we convert (1) to

$$u(x, t) = f(x) + \int_0^t u^3(x, t) u_{xxx}(x, t) dt, \tag{15}$$

where  $f(x) = u(x, 0)$ . Then, we substitute a series solution as follows

$$u = \sum_{n=0}^{\infty} v_n,$$

in Eq. (15) to obtain

$$\sum_{n=0}^{\infty} v_n = f(x) + \sum_{n=0}^{\infty} \int_0^t A_n(v_0, \dots, v_n) dt, \tag{16}$$

where  $A_0, A_1, \dots$  are known as Adomian polynomials and determined as

$$A_n(v_0, \dots, v_n) = \sum_{m=0}^n \sum_{l=0}^{n-m} \sum_{k=0}^{n-m-l} v_{n-m-l-k} v_l v_{m+k} v_{m+k}.$$

Immediately with regard to Eq. (16), it can be deduced that

$$\begin{cases} v_n = \int_0^t A_{n-1}(v_0, \dots, v_{n-1}) dt, & n = 1, 2, \dots, \\ v_0 = f(x). \end{cases}$$

Finally, approximate solutions of problem (1) are generated as follows

$$u_n = \sum_{i=0}^n v_i, \quad n = 0, 1, \dots$$

For example, by setting

$$f(x) = \left( c_3 - \frac{3\sqrt{c}}{2} x \right)^{2/3}, \tag{17}$$

it can be deduced that  $\{u_n\}_{n=0}^{\infty}$  converges to

$$u(x, t) = \left( c_3 - \frac{3\sqrt{c}}{2} (x + ct) \right)^{2/3}. \tag{18}$$

Taylor expansion of  $u_3, u_4$ , and  $u$  reported in Tables 1, 2, and 3, respectively, confirm this claim. We are not able to obtain implicit solutions (9)–(11) with the aid of Adomian decomposition method.

**Table 1** Coefficients of powers of  $x$  in Taylor expansion of  $u_3$ .

$n$	Coefficient of $x^n$			
0	$c_3^{2/3}$	$-\frac{c^{3/2}t}{c_3^{1/3}}$	$-\frac{c^3t^2}{4c_3^{4/3}}$	$-\frac{c^9/2t^3}{6c_3^{7/3}}$
1	$-\frac{\sqrt{c}}{c_3^{1/3}}$	$-\frac{c^2t}{2c_3^{4/3}}$	$-\frac{c^7/2t^2}{2c_3^{7/3}}$	$-\frac{7c^5t^3}{12c_3^{10/3}}$
2	$-\frac{c}{4c_3^{4/3}}$	$-\frac{c^5/2t}{2c_3^{7/3}}$	$-\frac{7c^4t^2}{8c_3^{10/3}}$	$-\frac{35c^{11/2}t^3}{24c_3^{13/3}}$
3	$-\frac{c^{3/2}}{6c_3^{7/3}}$	$-\frac{7c^3t}{12c_3^{10/3}}$	$-\frac{35c^9/2t^2}{24c_3^{13/3}}$	$-\frac{455c^6t^3}{144c_3^{16/3}}$
4	$-\frac{7c^2}{48c_3^{10/3}}$	$-\frac{35c^7/2t}{48c_3^{13/3}}$	$-\frac{455c^5t^2}{192c_3^{16/3}}$	$-\frac{455c^{13/2}t^3}{72c_3^{19/3}}$
5	$-\frac{7c^{5/2}}{48c_3^{13/3}}$	$-\frac{91c^4t}{96c_3^{16/3}}$	$-\frac{91c^{11/2}t^2}{24c_3^{19/3}}$	$-\frac{1729c^7t^3}{144c_3^{22/3}}$

**Table 2** Coefficients of powers of  $x$  in Taylor expansion of  $u_4$ .

$n$	Coefficient of $x^n$				
0	$c_3^{2/3}$	$\frac{c^3/2t}{c_3^{1/3}}$	$\frac{c^3t^2}{4c_3^{4/3}}$	$\frac{c^9/2t^3}{6c_3^{7/3}}$	$\frac{7c^6t^4}{48c_3^{10/3}}$
1	$-\frac{\sqrt{c}}{c_3^{1/3}}$	$\frac{c^2t}{2c_3^{4/3}}$	$\frac{c^7/2t^2}{2c_3^{7/3}}$	$\frac{7c^5t^3}{12c_3^{10/3}}$	$\frac{35c^{13/2}t^4}{48c_3^{13/3}}$
2	$\frac{c}{4c_3^{4/3}}$	$\frac{c^5/2t}{2c_3^{7/3}}$	$\frac{7c^4t^2}{8c_3^{10/3}}$	$\frac{35c^{11/2}t^3}{24c_3^{13/3}}$	$\frac{455c^7t^4}{192c_3^{16/3}}$
3	$-\frac{c^{3/2}}{6c_3^{7/3}}$	$\frac{7c^3t}{12c_3^{10/3}}$	$\frac{35c^9/2t^2}{24c_3^{13/3}}$	$\frac{455c^6t^3}{144c_3^{16/3}}$	$\frac{455c^{15/2}t^4}{72c_3^{19/3}}$
4	$\frac{7c^2}{48c_3^{10/3}}$	$\frac{35c^7/2t}{48c_3^{13/3}}$	$\frac{455c^5t^2}{192c_3^{16/3}}$	$\frac{455c^{13/2}t^3}{72c_3^{19/3}}$	$\frac{8645c^8t^4}{576c_3^{22/3}}$
5	$-\frac{7c^{5/2}}{48c_3^{13/3}}$	$\frac{91c^4t}{96c_3^{16/3}}$	$\frac{91c^{11/2}t^2}{24c_3^{19/3}}$	$\frac{1729c^7t^3}{144c_3^{22/3}}$	$\frac{19019c^{17/2}t^4}{576c_3^{25/3}}$

**4 Variational Iteration Method**

In the variational iteration method, the approximate solutions of Eq. (1) are generated sequentially as

$$U_{n+1}(x, t) = U_n(x, 0) + \int_0^t U_n^3(x, t)U_{nxxx}(x, t)dt, \quad n = 0, 1, \dots \tag{19}$$

For example, by selecting  $U_0(x, t) = f(x)$  ( $f(x)$  given in Eq. (17)), it can be deduced that  $\{U_n\}_{n=0}^\infty$  converges to  $u(x, t)$  given in Eq. (18). This claim will be confirmed by comparing results of Tables 3, 4, and 5. It must be pointed out that we are not able to obtain implicit solutions (9)–(11) with the aid of variational iteration method.

**Table 3** Coefficients of powers of  $x$  in Taylor expansion of  $u$ .

$n$	Coefficient of $x^n$				
0	$c_3^{2/3}$	$\frac{c^3/2t}{c_3^{1/3}}$	$\frac{c^3t^2}{4c_3^{4/3}}$	$\frac{c^9/2t^3}{6c_3^{7/3}}$	$\frac{7c^6t^4}{48c_3^{10/3}} + O[t^5]$
1	$-\frac{\sqrt{c}}{c_3^{1/3}}$	$\frac{c^2t}{2c_3^{4/3}}$	$\frac{c^7/2t^2}{2c_3^{7/3}}$	$\frac{7c^5t^3}{12c_3^{10/3}}$	$\frac{35c^{13/2}t^4}{48c_3^{13/3}} + O[t^5]$
2	$\frac{c}{4c_3^{4/3}}$	$\frac{c^5/2t}{2c_3^{7/3}}$	$\frac{7c^4t^2}{8c_3^{10/3}}$	$\frac{35c^{11/2}t^3}{24c_3^{13/3}}$	$\frac{455c^7t^4}{192c_3^{16/3}} + O[t^5]$
3	$-\frac{c^{3/2}}{6c_3^{7/3}}$	$\frac{7c^3t}{12c_3^{10/3}}$	$\frac{35c^9/2t^2}{24c_3^{13/3}}$	$\frac{455c^6t^3}{144c_3^{16/3}}$	$\frac{455c^{15/2}t^4}{72c_3^{19/3}} + O[t^5]$
4	$\frac{7c^2}{48c_3^{10/3}}$	$\frac{35c^7/2t}{48c_3^{13/3}}$	$\frac{455c^5t^2}{192c_3^{16/3}}$	$\frac{455c^{13/2}t^3}{72c_3^{19/3}}$	$\frac{8645c^8t^4}{576c_3^{22/3}} + O[t^5]$
5	$-\frac{7c^{5/2}}{48c_3^{13/3}}$	$\frac{91c^4t}{96c_3^{16/3}}$	$\frac{91c^{11/2}t^2}{24c_3^{19/3}}$	$\frac{1729c^7t^3}{144c_3^{22/3}}$	$\frac{19019c^{17/2}t^4}{576c_3^{25/3}} + O[t^5]$

**Table 4** Coefficients of powers of  $x$  in Taylor expansion of  $U_3$ .

$n$	Coefficient of $x^n$				
0	$c_3^{2/3}$	$\frac{c^3/2t}{c_3^{1/3}}$	$\frac{c^3t^2}{4c_3^{4/3}}$	$\frac{c^9/2t^3}{6c_3^{7/3}}$	$\frac{1179c^6t^4}{16c_3^{10/3}} + O[t^5]$
1	$-\frac{\sqrt{c}}{c_3^{1/3}}$	$\frac{c^2t}{2c_3^{4/3}}$	$\frac{c^7/2t^2}{2c_3^{7/3}}$	$\frac{7c^5t^3}{12c_3^{10/3}}$	$\frac{5895c^{13/2}t^4}{16c_3^{13/3}} + O[t^5]$
2	$\frac{c}{4c_3^{4/3}}$	$\frac{c^5/2t}{2c_3^{7/3}}$	$\frac{7c^4t^2}{8c_3^{10/3}}$	$\frac{35c^{11/2}t^3}{24c_3^{13/3}}$	$\frac{76635c^7t^4}{64c_3^{16/3}} + O[t^5]$
3	$-\frac{c^{3/2}}{6c_3^{7/3}}$	$\frac{7c^3t}{12c_3^{10/3}}$	$\frac{35c^9/2t^2}{24c_3^{13/3}}$	$\frac{455c^6t^3}{144c_3^{16/3}}$	$\frac{25545c^{15/2}t^4}{8c_3^{19/3}} + O[t^5]$
4	$\frac{7c^2}{48c_3^{10/3}}$	$\frac{35c^7/2t}{48c_3^{13/3}}$	$\frac{455c^5t^2}{192c_3^{16/3}}$	$\frac{455c^{13/2}t^3}{72c_3^{19/3}}$	$\frac{485355c^8t^4}{64c_3^{22/3}} + O[t^5]$
5	$-\frac{7c^{5/2}}{48c_3^{13/3}}$	$\frac{91c^4t}{96c_3^{16/3}}$	$\frac{91c^{11/2}t^2}{24c_3^{19/3}}$	$\frac{1729c^7t^3}{144c_3^{22/3}}$	$\frac{1067781c^{17/2}t^4}{64c_3^{25/3}} + O[t^5]$

**Table 5** Coefficients of powers of  $x$  in Taylor expansion of  $U_4$ .

$n$	Coefficient of $x^n$
0	$c_3^{2/3} - \frac{c^{3/2}t}{c_3^{1/3}} - \frac{c^3t^2}{4c_3^{4/3}} - \frac{c^9/2t^3}{6c_3^{7/3}} - \frac{7c^6t^4}{48c_3^{10/3}} + O[t^5]$
1	$-\frac{\sqrt{c}}{c_3^{1/3}} - \frac{c^2t}{2c_3^{4/3}} - \frac{c^{7/2}t^2}{2c_3^{7/3}} - \frac{12c_3^{10/3}}{7c^5t^3} - \frac{35c^{13/2}t^4}{48c_3^{13/3}} + O[t^5]$
2	$-\frac{4c_3^{4/3}}{c} - \frac{2c_3^{7/3}}{c^{5/2}t} - \frac{8c_3^{10/3}}{7c^4t^2} - \frac{24c_3^{13/3}}{35c^{11/2}t^3} - \frac{192c_3^{16/3}}{455c^7t^4} + O[t^5]$
3	$-\frac{c^{3/2}}{6c_3^{7/3}} - \frac{12c_3^{10/3}}{7c^2} - \frac{24c_3^{13/3}}{35c^9/2t^2} - \frac{144c_3^{16/3}}{455c^6t^3} - \frac{72c_3^{19/3}}{455c^{15/2}t^4} + O[t^5]$
4	$-\frac{48c_3^{10/3}}{7c^2} - \frac{48c_3^{13/3}}{35c^{7/2}t} - \frac{192c_3^{16/3}}{455c^5t^2} - \frac{72c_3^{19/3}}{455c^{13/2}t^3} - \frac{576c_3^{22/3}}{8645c^8t^4} + O[t^5]$
5	$-\frac{7c^{5/2}}{48c_3^{13/3}} - \frac{91c^4t}{96c_3^{16/3}} - \frac{91c^{11/2}t^2}{24c_3^{19/3}} - \frac{1729c^7t^3}{144c_3^{22/3}} - \frac{19019c^{17/2}t^4}{576c_3^{25/3}} + O[t^5]$

### 5 Power Series Method

In this section we show that with the aid of power series method we are able not only to obtain the explicit solution (8) and the implicit solutions (9)–(11) but also to establish another implicit solutions for Eq. (1). Equation (5) can be considered as

$$vv'^2 = c_1v^2 + c_2v + c. \tag{20}$$

Substituting a power series solution as

$$v = \sum_{n=0}^{\infty} a_n \xi^n \tag{21}$$

in Eq. (20) deduces that

$$\sum_{n=0}^{\infty} \left( \sum_{l=0}^n \left( \sum_{k=0}^l (k+1)(l-k+1)a_{k+1}a_{l-k+1} - c_1a_l \right) a_{n-l} - c_2a_n \right) \xi^n - c = 0. \tag{22}$$

Equating to zero the coefficients of all powers of  $\xi^n$  yields

$$\begin{aligned} a_1 &= \pm \sqrt{\frac{c + c_2a_0 + c_1a_0^2}{a_0}}, \quad a_2 = \frac{c_2 + 2c_1a_0 - a_1^2}{4a_0}, \quad a_3 = \frac{c_1a_1^2 + c_2a_2 + 2c_1a_0a_2 - 5a_1^2a_2 - 4a_0a_2^2}{6a_0a_1}, \\ a_4 &= \frac{2c_1a_1a_2 - 8a_1a_2^2 + c_2a_3 + 2c_1a_0a_3 - 7a_1^2a_3 - 12a_0a_2a_3}{8a_0a_1}, \\ a_5 &= \frac{c_1a_2^2 - 4a_2^3 + 2c_1a_1a_3 - 22a_1a_2a_3 - 9a_0a_3^2 + c_2a_4 + 2c_1a_0a_4 - 9a_1^2a_4 - 16a_0a_2a_4}{10a_0a_1}, \\ &\vdots \end{aligned} \tag{23}$$

Setting  $c_1 = c_2 = 0$  and  $a_0 = c_3^{2/3}$  leads to the following solution

$$v = c_3^{2/3} \pm \frac{c^{1/2}}{c_3^{1/3}} \xi - \frac{c}{4c_3^{4/3}} \xi^2 \pm \frac{c^{3/2}}{6c_3^{7/3}} \xi^3 - \frac{7c^2}{48c_3^{10/3}} \xi^4 \pm \frac{7c^{5/2}}{48c_3^{13/3}} \xi^5 - \pm \dots, \tag{24}$$

which is the Taylor series of Eq. (8). Obviously, another setting of  $c_1, c_2$ , and  $a_0$  leads to another different solution.

Since it can not be established the solutions (9), (10), and (11) with the aid of power series (21), we seek the solutions of Eq. (12), which have the following form

$$\xi = b_0 + \sum_{n=1}^{\infty} b_n v^{(2n-1)/2}. \tag{25}$$

Substituting this guest in Eq. (12), we find immediately

that  $b_1 = 0$ . Hence, we change (25) as follows

$$\xi = a_0 + \sum_{n=1}^{\infty} a_n v^{(2n+1)/2}, \tag{26}$$

and substitute it in Eq. (12). Therefore, we have

$$\sum_{n=2}^{\infty} C_n v^n = 0, \tag{27}$$

where

$$C_2 = -\frac{9}{4}a_1^2 + \frac{81}{16}ca_1^4,$$

$$\begin{aligned}
C_3 &= -15a_1a_2 + \frac{135}{4}ca_1^3a_2, & \pm \frac{3c_2^2}{28c^{5/2}}v^{5/2} \mp \frac{5c_2^3}{72c^{7/2}}v^{9/2} \dots, & (31) \\
C_4 &= -\frac{75}{2}a_2^2 + \frac{675}{8}ca_1^2a_2^2 - 21a_1a_3 + \frac{189}{4}ca_1^3a_3, & & \\
C_5 &= \frac{375}{4}ca_1a_2^3 - \frac{315}{2}a_2a_3 + \frac{945}{4}ca_1^2a_2a_3 \\
&\quad - \frac{27}{2}a_1a_4 + \frac{243}{4}ca_1^3a_4, \\
&\vdots & (28)
\end{aligned}$$

Equating to zero  $C_2$  yields

$$a_1 = 0 \text{ or } a_1 = \pm \frac{2}{3\sqrt{c}}. \quad (29)$$

Selecting  $a_1 = 0$  leads to a constant solution but choosing  $a_0 = \pm 2/3\sqrt{c}$  derives  $C_3 = C_4 = 0$  and by equating to zero  $C_5, C_6, \dots$ , it can be deduced that

$$\begin{aligned}
a_4 &= -\frac{5}{18}\sqrt{c}(25\sqrt{c}a_2^3 \mp 21a_2a_3), \\
a_5 &= \frac{7}{176}\sqrt{c}(\mp 625ca_2^4 + 300\sqrt{c}a_2^2a_3 \pm 84a_3^2), \\
a_6 &= \frac{25}{416}\sqrt{c}(-375c^{3/2}a_2^5 \mp 700ca_2^3a_3 + 588\sqrt{c}a_2a_3^2), \\
a_7 &= \frac{1}{96}\sqrt{c}(\pm 18750c^2a_2^6 - 39375c^{3/2}a_2^4a_3 \\
&\quad \pm 14700ca_2^2a_3^2 + 1372\sqrt{c}a_3^3), \\
&\vdots & (30)
\end{aligned}$$

- Setting  $a_0 = c_3$ ,  $a_1 = \pm 2/3\sqrt{c}$ , and  $a_2 = a_3 = 0$  leads to a solution, which is equivalent to Eq. (8),
- Setting  $a_0 = c_3 \mp c \ln(2\sqrt{c})/c_2^{3/2}$ ,  $a_1 = \pm 2/3\sqrt{c}$ ,  $a_2 = \mp c_2/5c^{3/2}$ , and  $a_3 = \pm 3c_2^2/28c^{5/2}$  leads to the following solution

$$\xi = c_3 \mp \frac{c \ln(2\sqrt{c})}{c_2^{3/2}} \pm \frac{2}{3\sqrt{c}}v^{3/2} \mp \frac{c_2}{5c^{3/2}}v^{5/2}$$

which is equivalent to Eq. (9),

- Setting  $a_0 = c_3$ ,  $a_1 = \pm 2/3\sqrt{c}$ ,  $a_2 = \mp 2/5\sqrt{c}$ , and  $a_3 = \pm 2/7\sqrt{c}$  leads to the following solution

$$\begin{aligned}
\xi &= c_3 \pm \frac{2}{3\sqrt{c}}v^{3/2} \mp \frac{2}{5\sqrt{c}}v^{5/2} \pm \frac{2}{7\sqrt{c}}v^{7/2} \\
&\quad \mp \frac{2}{9\sqrt{c}}v^{9/2} \pm \frac{2}{11\sqrt{c}}v^{11/2} \dots, & (32)
\end{aligned}$$

which is equivalent to Eq. (10),

- Setting  $a_0 = c_3$ ,  $a_1 = \mp 2/3\sqrt{c}$ ,  $a_2 = \mp 2/5\sqrt{c}$ , and  $a_3 = \mp 2/7\sqrt{c}$  leads to the following solution

$$\begin{aligned}
\xi &= c_3 \mp \frac{2}{3\sqrt{c}}v^{3/2} \mp \frac{2}{5\sqrt{c}}v^{5/2} \mp \frac{2}{7\sqrt{c}}v^{7/2} \\
&\quad \mp \frac{2}{9\sqrt{c}}v^{9/2} \mp \frac{2}{11\sqrt{c}}v^{11/2} \dots, & (33)
\end{aligned}$$

which is equivalent to Eq. (11).

Obviously, another setting of  $a_0$ ,  $a_2$ , and  $a_3$  leads to another different solution.

## 6 Conclusion

In this work, the Adomian decomposition, He's variational iteration, direct integration, and power series methods extend to establish readily some exact travelling wave solutions of the HD equation. It should be pointed out that the Adomian decomposition and He's variational iteration methods fail to obtain implicit solutions of the HD equation while the traditional methods of direct integration and power series success to get more implicit solutions of the HD equation. The tedious computations associated with the algebraic calculations are facilitated using a CAS such as *Mathematica*.

## References

- [1] M. Wadati, J. Phys. Soc. Jpn. **38** (1975) 673.
- [2] M. Wadati, J. Phys. Soc. Jpn. **38** (1975) 681.
- [3] M.J. Ablowitz and H. Segur, *Solitons and the Inverse Spectral Transform*, Philadelphia SIAM (1981).
- [4] J. Liu and K. Yang, Chaos, Solitons and Fractals **22(1)** (2004) 111.
- [5] E.J. Parkers and B.R. Duffy, Comput. Phys. Commun. **98** (1996) 288.
- [6] E. Fan and H. Zhang, Phys. Lett. A **246** (1998) 403.
- [7] R. Hirota, Phys. Rev. Lett. **27** (1971) 1192.
- [8] Z.T. Fu, S.K. Liu, S.D. Liu, and Q. Zhao, Phys. Lett. A **290** (2001) 72.
- [9] J.H. He, Commun. Nonlinear Sci. Numer. Simul. **2(4)** (1997) 230.
- [10] J.H. He, Commun. Nonlinear Sci. Numer. Simul. **2(4)** (1997) 235.
- [11] J.H. He, Int. J. Non-Linear Mech. **35(1)** (2000) 37.
- [12] J.H. He, J. Nonlinear Sci. Numer. Simul. **(2)** (2005) 207.
- [13] J.H. He and X.H. Wu, Chaos, Solitons and Fractals **30(3)** (2006) 700.
- [14] M.D. Kruskal and J. Moser (ed.), *Dynamical Systems, Theory and Applications, Lec. Notes Phys.*, Springer, Berlin (1975) 38.
- [15] L.P. Kadanoff, Phys. Rev. Lett. **65(24)** (1990) 2986.
- [16] A.C. Newell, *Solitons in Mathematics and Physics*, CBMS-NSF SIAM (1985) 48.
- [17] R.S. Palais, Bull. Amer. Math. Soc. **34(4)** (1997) 339.
- [18] M. Wadati, Y.H. Ichikawa, and T. Shimizu, Prog. Theor. Phys. **64** (1980) 1959.
- [19] W. Hereman, P.P. Banerjee, and M. Chatterjee, J. Phys. A: Math. **22(3)** (1989) 241.