Relativistic Oscillators in a Noncommutative Space and in a Magnetic Field

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Abstract *In this work, we study the relativistic oscillators in a noncommutative space and in a magnetic field. It is shown that the effect of the magnetic field may compete with that of the noncommutative space and that is able to vanish the effect of the noncommutative space.*

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1 Introduction

Theories of noncommutative space have been studied extensively over the past few years (for a review see Ref. [1]). Noncommutative field theories are related to M-theory compactification,[2] string theory in nontrivial backgrounds,[3] and quantum Hall effect.[4] Inclusion of noncommutativity in quantum field theory can be achieved in either of two different ways: via Moyal $\star$-product on the space of ordinary functions, or by defining the field theory on a coordinate operator space which is intrinsically noncommutative.[1,5] The equivalence between the two approaches has been nicely described in Ref. [6]. A simple insight on the role of noncommutativity in field theory can be obtained by studying the one particle sector, which prompted an interest in the study of noncommutative quantum mechanics (NCQM).[7−14] In these studies, attention was paid to the two-dimensional NCQM and its relation to the Landau problem. It has been shown that the equation of motion of a harmonic oscillator in a noncommutative space is similar to that of a particle in a constant magnetic field and in the lowest Landau level.[13,15] We generalize these relations to the relativistic quantum mechanics. In particular, it is shown that the Dirac and Klein–Gordon oscillators in a noncommutative space have similar behavior to the Landau problem in a commutative space although an exact map does not exist. However, for the Dirac oscillator there is a new term which is spin-dependent. The noncommutative spaces can be realized as spaces where the coordinate operator $\hat{x}^\mu$ satisfies the commutation relations

$$[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}\,, \tag{1}$$

where, $\theta^{\mu\nu}$ is an antisymmetric tensor of space dimension $(\text{length})^2$. We note that space-time noncommutativity, $\theta^{0i} \neq 0$, may lead to some problems with unitarity and causality. Such problems do not occur for the quantum mechanics on a noncommutative space with a usual commutative time coordinate. The noncommutative models specified by Eq. (1) can be realized in terms of a $\star$-product: the commutative algebra of functions with the usual product $f(x)g(x)$ is replaced by the $\star$-product Moyal algebra:

$$(f\star g)(x) = \exp\Big[\frac{\mathrm{i}}{2}\theta_{\mu\nu}\partial_{x_\mu}\partial_{y_\nu}\Big] f(x)g(y)|_{x=y}\,. \tag{2}$$

In the case when $[\hat{p}_i, \hat{p}_j] = 0$, the noncommutative quantum mechanics

$$H(\hat{p},\hat{x})\star\psi(\hat{x}) = E\psi(\hat{x}) \tag{3}$$

reduces to the usual one described by,[7−14]

$$H(\hat{p},\hat{x})\psi(x) = E\psi(x)\,, \tag{4}$$

where

$$\hat{x}_i = x_i - \frac{1}{2\hbar}\theta_{ij}p_j\,, \quad \hat{p}_i = p_i\,. \tag{5}$$

The new variables satisfy the usual canonical commutation relations:

$$[x_i, x_j] = 0, \quad [p_i, p_j] = 0, \quad [x_i, p_j] = \mathrm{i}\hbar\delta_{ij}\,. \tag{6}$$

This paper is organized as follows. In Sec. 2, the Klein–Gordon oscillator in a noncommutative space is investigated and its map to the Landau problem in a commutative space is given. In Sec. 3, the Dirac oscillator in a noncommutative space is defined and its relation to the Landau problem is clarified. Finally, in Sec. 4, we explore the Kemmer oscillator in both a noncommutative space and a magnetic field.

2 Klein–Gordon Oscillator in a Noncommutative Space

The similarity of the behavior of the relativistic Klein–Gordon harmonic oscillator in a noncommutative space to that of the same oscillator in a constant magnetic field has been previously studied in [15]. Drawing upon this similarity, in this section we investigate this oscillator in a noncommutative space and in a constant magnetic field.

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The Klein–Gordon equation in a constant magnetic field is defined by the following equation,

$$c^2\Big(\vec{p}-\frac{e}{2c}\vec{B}\times\vec{r}\Big)\cdot\Big(\vec{p}-\frac{e}{2c}\vec{B}\times\vec{r}\Big)\psi=(E^2-m^2c^4)\psi\,. \quad (7)$$

Working in the Coulomb gauge, it yields

$$c^2\Big[(p_x^2+p_y^2)+\Big(\frac{e^2B^2}{4c^2}\Big)(x^2+y^2)-\frac{eB}{c}L_z\Big]\psi = (E^2-m^2c^4)\psi\,. \quad (8)$$

The Klein–Gordon oscillator is defined as follows:

$$c^2(\vec{p}+\mathrm{i}mw\vec{r})\cdot(\vec{p}-\mathrm{i}mw\vec{r})\psi=(E^2-m^2c^4)\psi\,, \quad (9)$$

which can be rewritten in the following form ((2+1)-dimensional space):

$$c^2[(p_x^2+p_y^2)+m^2w^2(x^2+y^2)]\psi = (E^2-m^2c^4+2mc^2\hbar w)\psi\,. \quad (10)$$

Based on (10) and (8), it is clear that the Klein–Gordon equation in an external magnetic field is similar to the Klein–Gordon oscillator except for the term $-(eB/c)L_z$.

The Klein–Gordon oscillator in a two-dimensional ((2+1)-dimensional space-time) commutative space and in a magnetic field has the following form:

$$c^2\Big(\vec{p}-\frac{e}{2c}\vec{B}\times\vec{r}+\mathrm{i}mw\vec{r}\Big)\cdot\Big(\vec{p}-\frac{e}{2c}\vec{B}\times\vec{r}-\mathrm{i}mw\vec{r}\Big)\psi=(E^2-m^2c^4)\psi\,. \quad (11)$$

In a noncommutative space, it is described by the following equation:

$$c^2\Big(\vec{p}-\frac{e}{2c}\vec{B}\times\vec{r}+\mathrm{i}mw\vec{r}\Big)\cdot\Big(\vec{p}-\frac{e}{2c}\vec{B}\times\vec{r}-\mathrm{i}mw\vec{r}\Big)\star\psi=(E^2-m^2c^4)\psi\,. \quad (12)$$

Using the definition of the Moyal product, we can rewrite this equation in the commutative space ($\theta_{ij}=\epsilon_{ijk}\theta_k$)

$$c^2\Big[\vec{p}-\frac{e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)+\mathrm{i}mw\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)\Big]\cdot\Big[\vec{p}-\frac{e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)-\mathrm{i}mw\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)\Big]\psi=(E^2-m^2c^4)\psi\,. \quad (13)$$

By rotating the coordinate so that the $\vec{\theta}$ aligns with the z axis and in a constant magnetic field $\vec{A}=(B/2)(-y\hat{i}+x\hat{j})$, after straightforward calculation, we exactly obtain the following equation

$$c^2\Big[\Big(1+\frac{eB\theta}{2c\hbar}+\frac{e^2B^2\theta^2}{16c^2\hbar^2}+\frac{m^2w^2\theta^2}{4\hbar^2}\Big)(p_x^2+p_y^2)+\Big(m^2w^2+\frac{e^2B^2}{4c^2}\Big)(x^2+y^2)-\Big(\frac{2eB}{c}+\frac{e^2B^2\theta}{4c^2\hbar}+\frac{m^2w^2\theta}{\hbar}\Big)L_z\Big]\psi = (E^2-m^2c^4+2mc^2\hbar w+cmweB\theta)\psi\,. \quad (14)$$

As pointed out in [15], the energy eigenvalue of such an oscillator is similar to the normal Zeeman effect. By comparing Eq. (14) with Eq. (8), we may conclude that the effect of the magnetic field is able to counteract the effect of the noncommutative space so that a critical magnetic field is obtained when the coefficient of L_Z is put equal to zero. A consistent solution ($\theta\simeq 0\Longrightarrow B\simeq 0$) is given by

$$B=\frac{-4\hbar c}{e\theta}\Big[1-\sqrt{1-\frac{m^2w^2\theta^2}{4\hbar^2}}\,\Big]\,. \quad (15)$$

By considering $m^2w^2\theta^2/4\hbar^2\ll 1$, and expanding the second term, we obtain

$$B=-\frac{m^2w^2c\theta}{2e\hbar}+O(\theta^3)\,. \quad (16)$$

The negative sign means that the magnetic momentum is in the opposite direction of the vector $\vec{\theta}$. If we substitute the parameter θ in Eq. (14), it leads to an oscillator in a commutative space with new constants:

$$c^2\Big[(p_x^2+p_y^2)+\Big(m^2w^2+\frac{e^2B^2}{4c^2}\Big)(x^2+y^2)\Big]\psi=\Big(E^2-m^2c^4+2mc^2\hbar w-\frac{2e^2B^2\hbar}{mw}\Big)\psi\,. \quad (17)$$

3 Dirac Oscillator in a Noncommutative Space and in a Magnetic Field

Dirac oscillator is defined as:[15–20]

$$[c\vec{\alpha}\cdot(\vec{p}-\mathrm{i}mw\beta\vec{r})+\beta mc^2]\psi(\vec{r})=E\psi(\vec{r})\,. \quad (18)$$

in which, σ_i designates the Pauli matrices and m and w, respectively, are the oscillator's mass and frequency and

$$\psi(\vec{r})=\begin{pmatrix}\psi_a(\vec{r})\\ \psi_b(\vec{r})\end{pmatrix}\,.$$

Equation (18) could be written in the following form in terms of its components

$$c\vec{\sigma}\cdot(\vec{p}+\mathrm{i}mw\vec{r})\psi_b(\vec{r})+mc^2\psi_a(\vec{r})=E\psi_a(\vec{r})\,,$$
$$c\vec{\sigma}\cdot(\vec{p}-\mathrm{i}mw\vec{r})\psi_a(\vec{r})-mc^2\psi_b(\vec{r})=E\psi_b(\vec{r})\,. \quad (19)$$

These two equations can be used to eliminate ψ_b in favour of ψ_a, so that one can have ((2+1)-space):

$$c^2\Big[(p_x^2+p_y^2)+m^2\omega^2(x^2+y^2)-\frac{4m\omega}{\hbar}S_zL_z+2m\omega\hbar\Big]\psi_a=(E^2-m^2c^4)\psi_a\,. \tag{20}$$

In a constant magnetic field $\vec{A}=(B/2)(-y\hat{\imath}+x\hat{\jmath})$, it becomes:

$$c^2\Big[(p_x^2+p_y^2)+\Big(m^2\omega^2+\frac{e^2B^2}{4c^2}\Big)(x^2+y^2)-\frac{4m\omega}{\hbar}S_zL_z-\frac{eB}{c}(L_z+2S_Z)+2m\omega\hbar\Big]\psi_a=(E^2-m^2c^4)\psi_a\,, \tag{21}$$

which is comparable to the following equation, i.e. Dirac oscillator in a noncommutative (2+1)-dimensional space $\vec{\theta}=\theta\hat{k}$:

$$\begin{aligned}c^2\Big[\Big(1+\frac{m^2\omega^2\theta^2}{4\hbar^2}\Big)(p_x^2+p_y^2)+m^2\omega^2(x^2+y^2)-\frac{4m\omega}{\hbar}S_zL_z-\frac{m^2\omega^2\theta}{\hbar}(L_z+2S_z)\\+\frac{2m\omega\theta}{\hbar^2}S_zp^2+2m\omega\hbar\Big]\psi_a=(E^2-m^2c^4)\psi_a\,.\end{aligned} \tag{22}$$

These equations are similar except for the additional term in the noncommutative space which depends on the non-commutative parameter θ, spin, and the momentum operator. The frequencies are connected to each other by the following relation:

$$\omega_1=1\pm\frac{m\omega\theta}{\hbar}\,. \tag{23}$$

From this similarity, one may deduce that the effect of the noncommutative space can be counteracted in presence of a constant magnetic field. The Dirac oscillator in a noncommutative space and in a constant magnetic field is given by:

$$\begin{aligned}&c\vec{\sigma}\cdot\Big[\vec{p}-\frac{e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)+\mathrm{i}mw\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)\Big]\psi_b(\vec{r})+mc^2\psi_a(\vec{r})=E\psi_a(\vec{r})\,,\\&c\vec{\sigma}\cdot\Big[\vec{p}-\frac{e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)-\mathrm{i}mw\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)\Big]\psi_a(\vec{r})-mc^2\psi_b(\vec{r})=E\psi_b(\vec{r})\,.\end{aligned} \tag{24}$$

These two equations can be used to eliminate ψ_b in favour of ψ_a, so that one can have:

$$\Big(c^2\vec{\sigma}\cdot\Big[\vec{p}-\frac{e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)-\mathrm{i}mw\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)\Big]\vec{\sigma}\cdot\Big[\vec{p}-\frac{e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)+\mathrm{i}mw\Big(\vec{r}+\frac{\vec{\theta}\times\vec{p}}{2\hbar}\Big)\Big]\Big)\psi_a=(E^2-mc^4)\psi_a\,. \tag{25}$$

According to the following relations:

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b})=\vec{a}\cdot\vec{b}+\mathrm{i}\vec{\sigma}\cdot(\vec{a}\times\vec{b})\,,$$

different terms in Eq. (25) can be written in a very simple way; as before ($\vec{\theta}=\theta\hat{k}$) and $\vec{A}=(B/2)(-y\hat{\imath}+x\hat{\jmath})$:

$$\begin{aligned}&c^2\Big[\Big(1+\frac{eB\theta}{2\hbar c}+\frac{e^2B^2\theta^2}{16\hbar^2c^2}+\frac{m^2\omega^2\theta^2}{4\hbar^2}\Big)(p_x^2+p_y^2)+\Big(m^2\omega^2+\frac{e^2B^2}{4c^2}\Big)(x^2+y^2)-\Big(\frac{4m\omega}{\hbar}+\frac{2m\omega eB\theta}{\hbar^2c}\Big)S_zL_z\\&\quad-\Big(\frac{eB}{c}+\frac{m^2\omega^2\theta}{\hbar}+\frac{e^2B^2\theta}{4\hbar c^2}\Big)(L_z+2S_z)+\Big(\frac{2m\omega\theta}{\hbar^2}+\frac{m\omega eB\theta^2}{2\hbar^3c}\Big)S_zp^2-\Big(\frac{m\omega eB\theta}{c}\Big)+2m\omega\hbar\Big]\psi_a\\&=(E^2-m^2c^4)\psi_a\,.\end{aligned} \tag{26}$$

In the above equation if we put the coefficient of (L_z+2S_z) equal to zero it becomes similar to Eq. (20) but with new coefficients. So the critical magnetic field is determined by

$$\frac{eB}{c}+\frac{m^2\omega^2\theta}{\hbar}+\frac{e^2B^2\theta}{4\hbar c^2}=0\,. \tag{27}$$

A consistent solution ($\theta\simeq0\Longrightarrow B\simeq0$) is given by

$$B=\frac{-2\hbar c}{e\theta}\Big[1-\sqrt{1-\frac{m^2w^2\theta^2}{4\hbar^2}}\Big]\,. \tag{28}$$

By considering $m^2w^2\theta^2/4\hbar^2\ll1$, and expanding the second term, we obtain

$$B=-\frac{m^2w^2c\theta}{e\hbar}+O(\theta^3)\,. \tag{29}$$

The minus sign in Eq. (29) indicates that $\vec{B}$ and $\vec{\theta}$ have inverse directions ($e>0$). It is interesting that the critical value of B for bosons (Eq. (15)) is two times of that for fermions (Eq. (28)).

4 Kemmer Oscillator in a Noncommutative Space and in a Magnetic Field

Kemmer equation is a Dirac-like relativistic equation for spin-1 particles:[21]

$$(c\beta^\mu p_\mu-mc^2)\Psi_K=0\,, \tag{30}$$

in which, m is the total mass of two identical spin-1/2 particles. The β^μ matrices satisfy the following relation

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=g^{\mu\nu}\beta^\lambda+g^{\lambda\nu}\beta^\mu\,. \tag{31}$$

For recent studies on the Kemmer equation see [22–26]. We consider a 10×10 representation of β^μ matrices which is defined as follows[27−30]

$$\beta^0=\begin{pmatrix}0 & \bar{0} & \bar{0} & \bar{0}\\ \bar{0}^{\rm T} & 0 & \boldsymbol{I} & 0\\ \bar{0}^{\rm T} & \boldsymbol{I} & 0 & 0\\ \bar{0}^{\rm T} & 0 & 0 & 0\end{pmatrix},$$

$$\beta^i=\begin{pmatrix}0 & \bar{0} & e_i & \bar{0}\\ \bar{0}^{\rm T} & \boldsymbol{0} & \boldsymbol{0} & -{\rm i}s_i\\ -{e_i}^{\rm T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0}\\ \bar{0}^{\rm T} & -{\rm i}s_i & \boldsymbol{0} & \boldsymbol{0}\end{pmatrix}, \tag{32}$$

where s_i are the usual 3×3 matrices and

$$\bar{0}=(0\ 0\ 0),\qquad e_1=(1\ 0\ 0),$$
$$e_2=(0\ 1\ 0),\qquad e_3=(0\ 0\ 1), \tag{33}$$

while $\boldsymbol{I}$ and $\boldsymbol{0}$ designate the 3×3 identity and zero matrices, respectively. The stationary solution of Eq. (30), has the form $\Psi_K(r,t)={\rm e}^{-{\rm i}Et}\psi(\vec{r})$. In the case of spin one, $\psi(\vec{r})$ is a vector with ten components that can be written as

$$\psi(\vec{r})^{\rm T}=({\rm i}\varphi,\boldsymbol{A}(\vec{r}),\boldsymbol{B}(\vec{r}),\boldsymbol{C}(\vec{r})), \tag{34}$$

where any one of $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ has three components. By substituting the momentum operator $\boldsymbol{p}$ in the Kemmer equation (30) for $\boldsymbol{p}-{\rm i}m\omega g\vec{r}$, we can define the Kemmer oscillator as:

$$[c\vec{\beta}\cdot(\boldsymbol{p}-{\rm i}m\omega g\vec{r})+mc^2]\psi(\vec{r})=E\beta^0\psi(\vec{r}), \tag{35}$$

where ω is the frequency and $g=2(\beta^0)^2-1$. In a noncommutative space and constant magnetic field, the Kemmer oscillator becomes:

$$\Big\{c\vec{\beta}\cdot\Big[\Big(\boldsymbol{p}-\frac{e}{2c}\vec{B}\times\vec{r}\Big)-{\rm i}m\omega g\vec{r}\Big]+mc^2\Big\}\star\psi(\vec{r})=E\beta^0\psi(\vec{r}). \tag{36}$$

The star product is equal to replacing $\vec{r}$ with $\vec{r}+(\vec{\theta}\times p)/2\hbar$. Equation (36) and the definition of $\psi(\vec{r})$ (Eq. (34)) lead to the following equations:

$$mc^2\varphi=-c\boldsymbol{p}^-\cdot\boldsymbol{B}, \tag{37}$$
$$mc^2\boldsymbol{A}=E\boldsymbol{B}-c\boldsymbol{p}^+\times\boldsymbol{C}, \tag{38}$$
$$mc^2\boldsymbol{B}=E\boldsymbol{A}+c\boldsymbol{p}^+\varphi, \tag{39}$$
$$mc^2\boldsymbol{C}=-c\boldsymbol{p}^-\times\boldsymbol{A}, \tag{40}$$

where

$$\boldsymbol{p}^+=\boldsymbol{p}'+{\rm i}m\omega\Big(\vec{r}+\frac{\vec{\theta}\times\boldsymbol{p}}{2\hbar}\Big),$$
$$\boldsymbol{p}^-=\boldsymbol{p}'-{\rm i}m\omega\Big(\vec{r}+\frac{\vec{\theta}\times\boldsymbol{p}}{2\hbar}\Big), \tag{41}$$

and $\boldsymbol{p}'$ is defined as follows

$$\boldsymbol{p}'=\boldsymbol{p}-\frac{\rm e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\boldsymbol{p}}{2\hbar}\Big). \tag{42}$$

After solving four equations (37) to (40) in favor of $\boldsymbol{A}$ in a (2+1)-dimensional space and by assuming $\vec{A}=(B/2)(-y\hat{i}+x\hat{j})$, we get:[30]

$$(E^2-m^2c^4)\boldsymbol{A}=-c^2\boldsymbol{p}^+\times(\boldsymbol{p}^-\times\boldsymbol{A})+c^2\boldsymbol{p}^+(\boldsymbol{p}^-\cdot\boldsymbol{A})-\frac{1}{m^2}\boldsymbol{p}^+\{\boldsymbol{p}^-\cdot[\boldsymbol{p}^+\times(\boldsymbol{p}^-\times\boldsymbol{A})]\}. \tag{43}$$

A direct calculation for the first terms in Eq. (43) gives:

$$\begin{aligned}\boldsymbol{p}^+\times(\boldsymbol{p}^-\times\boldsymbol{A})&=\boldsymbol{p}'(\boldsymbol{p}'\cdot\boldsymbol{A})-\boldsymbol{p}'^2\boldsymbol{A}+m^2\omega^2[\vec{r}\,(\vec{r}\cdot\boldsymbol{A})-r^2\boldsymbol{A}]+m\omega\Big(\hbar+\frac{1}{\hbar}\vec{L}\cdot\vec{S}\Big)\boldsymbol{A}\\&\quad-\frac{m^2\omega^2}{4\hbar^2}\{\boldsymbol{p}\times[\vec{\theta}(\vec{\theta}\cdot[\boldsymbol{p}\times\boldsymbol{A}])]+\theta^2(p_x^2+p_y^2)\boldsymbol{A}\}\\&\quad-\frac{m^2\omega^2}{2\hbar}[-2\theta(L_z+S_z)\boldsymbol{A}-\vec{r}(\vec{\theta}\cdot[\boldsymbol{p}\times\boldsymbol{A}])+\boldsymbol{p}\times(\vec{\theta}\,[\vec{r}\cdot\boldsymbol{A}])]-\frac{m\omega}{2\hbar^2}[(\vec{S}\times\boldsymbol{p}')\cdot(\vec{\theta}\times\boldsymbol{p})]\boldsymbol{A},\end{aligned} \tag{44}$$

$$\begin{aligned}\boldsymbol{p}^+(\boldsymbol{p}^-\cdot\boldsymbol{A})&=\boldsymbol{p}'(\boldsymbol{p}'\cdot\boldsymbol{A})+m^2\omega^2\vec{r}(\vec{r}\cdot\boldsymbol{A})-m\omega\Big(\hbar+\frac{1}{\hbar}\vec{L}\cdot\vec{S}\Big)\boldsymbol{A}-\frac{m^2\omega^2}{4\hbar^2}\{\boldsymbol{p}\times[\vec{\theta}\,(\vec{\theta}\cdot[\boldsymbol{p}\times\boldsymbol{A}])]\}\\&\quad-\frac{m^2\omega^2}{2\hbar}\{-\vec{r}[\vec{\theta}\cdot(\boldsymbol{p}\times\boldsymbol{A})]+\boldsymbol{p}\times[\vec{\theta}(\vec{r}\cdot\boldsymbol{A})]\}+\frac{m\omega}{2\hbar^2}[(\vec{S}\times\boldsymbol{p}')\cdot(\vec{\theta}\times\boldsymbol{p})]\boldsymbol{A}.\end{aligned} \tag{45}$$

By substituting the last terms in Eq. (43), we obtain

$$\begin{aligned}(E^2-m^2c^4)\boldsymbol{A}&=c^2\Big\{\Big[\boldsymbol{p}-\frac{e}{2c}\vec{B}\times\Big(\vec{r}+\frac{\vec{\theta}\times\boldsymbol{p}}{2\hbar}\Big)\Big]^2+m^2\omega^2r^2+\frac{m^2\omega^2}{4\hbar^2}\theta^2(p_x^2+p_y^2)-\frac{m^2\omega^2}{\hbar}\theta(L_z+S_z)-\frac{2m\omega}{\hbar}\vec{L}\cdot\vec{S}\\&\quad-2m\omega\hbar+\frac{m\omega}{\hbar^2}(\vec{S}\times\boldsymbol{p}')\cdot(\vec{\theta}\times\boldsymbol{p})\Big\}\boldsymbol{A}+\frac{1}{m^2}\boldsymbol{p}^+\{\boldsymbol{p}^-\cdot[\boldsymbol{p}^+\times(\boldsymbol{p}^-\times\boldsymbol{A})]\}.\end{aligned} \tag{46}$$

Within the nonrelativistic limit the last term is negligible, and finally we have:

$$\begin{aligned}&4c^2\Big[\Big(1+\frac{eB\theta}{2\hbar c}+\frac{e^2B^2\theta^2}{16\hbar^2c^2}+\frac{m^2\omega^2\theta^2}{4\hbar^2}\Big)(p_x^2+p_y^2)+\Big(m^2\omega^2+\frac{e^2B^2}{4c^2}\Big)(x^2+y^2)-\Big(\frac{4m\omega}{\hbar}+\frac{2m\omega eB\theta}{\hbar^2c}\Big)S_zL_z\\&\quad-\Big(\frac{eB}{c}+\frac{m^2\omega^2\theta}{\hbar}+\frac{e^2B^2\theta}{4\hbar c^2}\Big)(L_z+2S_z)+\Big(\frac{2m\omega\theta}{\hbar^2}+\frac{m\omega eB\theta^2}{2\hbar^3c}\Big)S_zp^2-\Big(\frac{m\omega eB\theta}{c}\Big)+2m\omega\hbar\Big]\psi_1\\&=(E^2-m^2c^4)\psi_1.\end{aligned} \tag{47}$$

So, the critical magnetic field is determined by:

$$\frac{eB}{c}+\frac{m^2\omega^2\theta}{\hbar}+\frac{e^2B^2\theta}{4\hbar c^2}=0\,, \tag{48}$$

and a consistent solution ($\theta\simeq 0\Longrightarrow B\simeq 0$) is given by

$$B=\frac{-2\hbar c}{e\theta}\Big[1-\sqrt{1-\frac{m^2w^2\theta^2}{4\hbar^2}}\Big]\,. \tag{49}$$

For small values of θ or ($m^2w^2\theta^2/4\hbar^2\ll 1$) we get

$$B=-\frac{m^2w^2c\theta}{e\hbar}+O(\theta^3)\,. \tag{50}$$

Here again, the minus sign indicates that for $e>0$, $\vec{B}$, and $\vec{\theta}$ have inverse directions. It is interesting that Eq. (49) is equal to the bosonic result in Eq. (15) and equals to half of the magnetic field for fermions in Eq. (28).

5 Conclusion

The Klein–Gordon, Dirac and Kemmer oscilators reveal some interesting relations between the noncommutativity of space and presence of a constant magnetic field. It will be interesting to see how much of these results remain for a quantum field theory in a background magnetic field and in a noncommutative space.

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