Light Propagation in the Second post-Newtonian Approximation of Scalar-Tensor Theory of Gravity*

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Abstract *In this paper, we use the metric coefficients and the equation of motion obtained in the second post-Newtonian approximation of scalar-tensor theory to derive the second-order light propagation equation and the light deflection angle and compare it with previous works. These results are useful for precision astrometry missions like ASTROD, GAIA, Darwin and SIM which aim at astrometry with micro-arcsecond and nano-arcsecond accuracies, and need for the second post-Newtonian framework and ephemeris for observations to determine the stellar and spacecraft positions.*

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1 Introduction

Gravitational theories have flourished since the issues of dark matter and dark energy in cosmology were intensively discussed.[1] In addition to laboratory,[2−3] astrophysical[4−6] and cosmological[7] tests, many of these theories can be tested by the precision measurements/observations in our solar system. Various theories have already been ruled out from solar system measurements.[8−9] With development of technology, dedicated missions like ASTROD I (Astrodynamical Space Test of Relativity using Optical Devices I),[10] ASTROD,[11] LATOR (Laser Astrometric Test Of Relativity),[12] GAUGE (GrAnd Unification and Gravity Explorer),[13] EGE (Einstein Gravity Explorer),[14] and SAGAS (Quantum Physics Exploring Gravity in the Outer Solar System: The Sagas Project)[15] have been proposed to test relativistic gravity with 3–5 orders of improvements. These tests include light deflection, Shapiro retardation, and solar dynamics. Missions mainly for other purposes like Bepi–Colombo[16] and GAIA[17] will also have a saying in the improvement scenario. The relativistic light deflection passing near the solar rim is 1.75 as (arcsec). The first post-Newtonian approximation is valid to about 10^{-6} and the second post-Newtonian (2PN) is valid to about 10^{-12} of relativistic effects such as light deflection and Shapiro time delay in the solar system. For astrometry missions like GAIA, SIMS,[18] and Darwin[19] to measure angles with accuracy in the n as to μ as range, second post-Newtonian approximation (2PNA) of relevant theories of gravity is required both for the angular measurement and spacecraft position determination.

In the context of dark energy and inflation of cosmology, the scalar-tensor theory is widely discussed and used.[20] In order to confront the predictions of these scalar-tensor theory with experiment in the solar system, it is necessary to compute it is 2PNA and associated gravitational effects such as deflection of light, Shapiro time delay and perihelion shift in this approximation. The 2PNA of general scalar-tensor theory for perfect fluid has been derived in Ref. [21]. On the other hand, The 2PN contribution for light ray is readily available from the literature,[22−23] but not the position and velocity of photon as functions of time. The analytical solution for light propagation in the 2PNA is given in Refs. [24] and [25] for the Schwarzschild metric and parameterized post-Newtonian metric respectively where the position and velocity of photon are the functions of time. In Ref. [26], the first-order multiple-system post-Newtonian scheme is extended to the second-order contributions to light propagation, and further the 2PN light ray equation using this formalism is given in Ref. [27]. General Relativistic Theory of Light Propagation in the Field of Radiative Gravitational Multipoles and Lorentz Covariant Theory of Light Propagation in Gravitational Fields of Arbitrary-Moving Bodies are given in Refs. [28] and [29] respectively for the first post-Newtonian approximation. In order to extend these theories to the 2PNA, the analytical formula for light propagation to that order is needed. In additions to this, finite range of the positions of emission source and

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observer should be considered in some space progjects like ASTROD. But the solution obtained in Ref. [25] did not include such terms. Here we use the technique of integration of the equation of light propagation in Ref. [30], and extend it to the second order to get the solutions of light trajectory to the 2PNA, which include the contributions coming from finite range of the positions of emission source and observer. Furthermore we give the deflection angle to the 2PNA.

In what follows, our conventions and notations generally follow those of Ref. [31], the metric signature is $(-,+,+,+)$. Greek indices take the values from 0 to 3, while Latin indices take the values from 1 to 3. Bold letters denote spatial vectors. We work in the global coordinates $(t,\boldsymbol{x})$. The paper is organized as follows: In Sec. 2 we obtain the equations of motion of photon by using the 2PN metric derived in Ref. [21]. In Sec. 3 we integrate the equations of motion of photon to get the light trajectory to the 2PN order by the iterative method used in Ref. [26]. Furthermore we get the deflection angle to the 2PN order and comparing it with previous works. Finally we give the concluding remarks in Sec. 4.

2 Differential Equations of Light Propagation

The purpose of this section is to derive the differential equations of light propagation with 2PN metric by using the geodetic equation. Unlike in Ref. [24], the equation expressing the isotropic geodetic conditions for the light propagation is used as constraint equation here.

2.1 *The Metric Tensor to Order c^{-4}*

The calculation of light propagation to 2PNA requires knowledge of terms in the metric to order c^{-4}. In Ref. [21], the 2PNA of scalar tensor theory were derived with the quasi-harmonic gauge, which can be reduced to the harmonic gauge when the scalar field losses its dynamic properties. On the basis of these results, we get the 2PN metric of rigid rotated sun for light propagation as follows

$$\begin{aligned}
g_{00} &= -1 + 2\frac{M}{r}\Big(1 - \frac{R^2}{r^2}J_2\frac{3\cos^2\theta - 1}{2}\Big) \\
&\quad - 2\beta\frac{M^2}{r^2} + \mathcal{O}(c^{-5})\,, \\
g_{0i} &= (1+\gamma)\frac{\varepsilon_{ijk}J_j x_k}{r^3} + \mathcal{O}(c^{-5})\,, \\
g_{ij} &= \delta_{ij}\Big[1 + 2\gamma\frac{M}{r}\Big(1 - \frac{R^2}{r^2}J_2\frac{3\cos^2\theta - 1}{2}\Big)\Big] \\
&\quad + \varepsilon\Big(\delta_{ij} + \frac{x^i x^j}{r^2}\Big)\frac{M^2}{r^2} + \mathcal{O}(c^{-5})\,, \qquad (1)
\end{aligned}$$

where M, J_2, $\boldsymbol{J}$, and R are the mass, dimensionless quadrupole moment parameter, total angular momentum and diameter of the sun respectively. $r = |\boldsymbol{x}| = \sqrt{\delta_{ij}x^i x^j}$ is the Euclidean absolute value of vector $\boldsymbol{x}$. Parameters γ, β and the post-linear parameter ε are given in Ref. [21] as

$$\begin{aligned}
\gamma &= \frac{\omega_0 + 1}{\omega_0 + 2}\,, \\
\beta &= 1 + \frac{\omega_1}{(2\omega_0+3)(2\omega_0+4)^2}\,, \\
\varepsilon &= 1 + \frac{15}{6}(\gamma - 1) + \frac{4}{3}[(\gamma-1)^2 + (\beta - 1)]\,, \qquad (2)
\end{aligned}$$

where ω_0 and ω_1 are expansion coefficients of the couple function $\theta(\phi)$ around the ground value of scalar field ϕ. Note that γ measures space curvature produced by unit rest mass, β denotes the second order nonlinearity in the superposition law for gravity,[31] and ε is the nonlinear combination of γ and β,[32] which is equivalent to the parameter Λ in Ref. [23] and parameter ϵ in Ref. [22]. There no new parameters appear in the 2PN metric of scalar-tensor theory.

2.2 *Differential Equations of Light Propagation*

In this section we will deduce the equations of light propagation in the global coordinates by using the metric tensor given in Subsec 2.1. First of all, we start with the basic equation of geodesic motion:

$$\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\sigma^2} + \Gamma^\mu_{\nu\rho}\frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\rho}{\mathrm{d}\sigma} = 0\,, \qquad (3)$$

where σ is an affine parameter measured along the trajectory. Normally we can rewrite Eq. (3) using coordinate time $t = x^0$ rather than σ, then the spatial components of Eq. (3) can be rewritten as

$$\frac{\mathrm{d}^2x^i}{\mathrm{d}t^2} = \Big(\Gamma^0_{\nu\rho}\frac{\mathrm{d}x^i}{\mathrm{d}t} - \Gamma^i_{\rho\nu}\Big)\frac{\mathrm{d}x^\nu}{\mathrm{d}t}\frac{\mathrm{d}x^\rho}{\mathrm{d}t}\,. \qquad (4)$$

Substituting the 2PN metric tensor into Eq. (4), we get the differential equation to 2PN precision as follows:

$$\begin{aligned}
\ddot{\boldsymbol{x}} &= 2(1+\gamma)\Big(1 + \frac{3J_2R^2}{2r^2}\Big)\frac{M}{r^3}(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})\dot{\boldsymbol{x}} \\
&\quad - (1+\gamma\dot{\boldsymbol{x}}^2)\Big(1 + \frac{3J_2R^2}{2r^2}\Big)\frac{M}{r^3}\boldsymbol{x} \\
&\quad \pm (1+\gamma)\frac{J}{r^3}\boldsymbol{x} + M^2\Big\{2\varepsilon\frac{(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})^2\boldsymbol{x}}{r^6} \\
&\quad + 2[2(1-\beta) + \varepsilon - 2\gamma^2]\frac{(\dot{\boldsymbol{x}}\cdot\boldsymbol{x})\dot{\boldsymbol{x}}}{r^4} \\
&\quad + 2[\gamma + \beta + (\gamma^2 - \varepsilon)\dot{\boldsymbol{x}}^2]\frac{\boldsymbol{x}}{r^4}\Big\} + \mathcal{O}(c^{-4})\,, \qquad (5)
\end{aligned}$$

where $\ddot{\boldsymbol{x}} = \mathrm{d}^2\boldsymbol{x}/\mathrm{d}t^2$, $\dot{\boldsymbol{x}} = \mathrm{d}\boldsymbol{x}/\mathrm{d}t$, and $J = |\boldsymbol{J}|$. The dot symbol "·" between two spatial vectors denote the Euclidean dot. Here we have simplified the equation by considering light propagating on the equatorial plane of the sun with $\theta = \pi/2$ and $\boldsymbol{J} \perp \boldsymbol{x}$. The sign $\pm$ of in front of the term $(1+\gamma)(J\boldsymbol{x}/r^3)$ comes from two different directions of angular momentum with respect to the incoming photon.

3 Analytical Second Post-Newtonian Solution

Considering a light signal emitted at coordinate time t_0 at a point $\boldsymbol{x}_0$, in an initial direction described by the unit vector $\boldsymbol{n}$, where $\boldsymbol{n}\cdot\boldsymbol{n}=1$. Including the first post-Newtonian correction $\boldsymbol{x}_{1\mathrm{p}}$ and the second post-Newtonian correction $\boldsymbol{x}_{2\mathrm{p}}$, the resulting trajectory of photon has the form

$$\boldsymbol{x}_{\mathrm{pp}}(\boldsymbol{x})=\boldsymbol{x}_{\mathrm{M}}+\boldsymbol{x}_{1\mathrm{p}}(\boldsymbol{x})+\boldsymbol{x}_{2\mathrm{p}}(\boldsymbol{x})\,, \tag{6}$$

where $\boldsymbol{x}_{\mathrm{M}}=\boldsymbol{x}_0+\boldsymbol{n}(t-t_0)$ is the photon trajectory in the Minkowskian spacetime faraway from the gravitational source, and we have imposed the boundary condition $\boldsymbol{x}_{\mathrm{pp}}(\boldsymbol{x}_0)=\boldsymbol{x}_0$ and $\boldsymbol{x}_{1\mathrm{p}}(\boldsymbol{x}_0)=\boldsymbol{x}_{2\mathrm{p}}(\boldsymbol{x}_0)=0$. From Eq. (6), we get the dimensionless vector

$$\boldsymbol{k}_{\mathrm{pp}}(\boldsymbol{x})=\boldsymbol{n}+\boldsymbol{k}_{1\mathrm{p}}(\boldsymbol{x})+\boldsymbol{k}_{2\mathrm{p}}(\boldsymbol{x})\,, \tag{7}$$

where $\boldsymbol{k}=\mathrm{d}\boldsymbol{x}/\mathrm{d}t$. This equation will be used to get the deflection angle. We will obtain the integrals of photon trajectory to the 2PNA by the iterative method in the next subsection.

3.1 *1PN Solutions of Photon Trajectory*

To get the 2PN photon trajectory, we must know the 1PN results first, then by using iterative method to get the 2PN solution. First, we consider the photon trajectory having the form $\boldsymbol{x}=\boldsymbol{x}_{\mathrm{M}}+\boldsymbol{x}_{1\mathrm{p}}$, then substitute it into Eq. (5), we obtain

$$\ddot{\boldsymbol{x}}_{1\mathrm{p}}=-(1+\gamma)M\Big[\frac{\boldsymbol{x}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{x})}{r^3}-\frac{\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{x})}{r^3}\Big]\,, \tag{8}$$

where only the first order contributions are kept and $\boldsymbol{x}$ is kept to the Minkowskian order which is enough for solving the 1PN equation of motion. To make integrating process easy, we introduce impact parameter b and variable λ as follows:

$$\begin{aligned}&\boldsymbol{b}=\boldsymbol{x}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{x})\,,\\ &b=\sqrt{\boldsymbol{x}^2-(\boldsymbol{n}\cdot\boldsymbol{x})^2}\,,\quad \lambda=\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{b}\,,\end{aligned} \tag{9}$$

where the variable $\boldsymbol{x}=\boldsymbol{x}_{\mathrm{M}}$. Note that $\boldsymbol{b}$ is the vector joining the center of the body and the point of closest approach of the unperturbed ray. Then we have the following relations

$$\mathrm{d}t=b\,\mathrm{d}\lambda\,,\quad r=b\sqrt{1+\lambda^2}\,, \tag{10}$$

which will be used in the 1PN and 2PN integration of the photon trajectory. Substituting Eqs. (9) and (10) into Eq. (8), then, integrating Eq. (8) along the unperturbed photon path, we obtain

$$\dot{\boldsymbol{x}}_{\mathrm{p}}(\boldsymbol{x})=\boldsymbol{n}-(1+\gamma)M\frac{\boldsymbol{b}}{b^2}\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}\Big)-(1+\gamma)M\frac{\boldsymbol{n}}{r}\,, \tag{11}$$

where variable λ has been replaced by $\boldsymbol{n}\cdot\boldsymbol{x}$ and r. We have used the constraint equation $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu=0$. Furthermore, integrating Eq. (11) along the unperturbed photon path again, we get the light trajectory to 1PN order as follows:

$$\begin{aligned}\boldsymbol{x}_{\mathrm{p}}(\boldsymbol{x})=\boldsymbol{x}_{\mathrm{M}}&-(1+\gamma)M\frac{\boldsymbol{b}}{b^2}\Big(r-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}\boldsymbol{n}\cdot\boldsymbol{x}\Big)\\ &-(1+\gamma)M\boldsymbol{n}\ln(r+\boldsymbol{n}\cdot\boldsymbol{x})+\boldsymbol{c}_1\,,\end{aligned} \tag{12}$$

with

$$\boldsymbol{c}_1=(1+\gamma)M\Big[\frac{\boldsymbol{b}}{b^2}\Big(r_0-\frac{(\boldsymbol{n}\cdot\boldsymbol{x}_0)^2}{r_0}\Big)+\boldsymbol{n}\ln(r_0+\boldsymbol{n}\cdot\boldsymbol{x}_0)\Big]\,,$$

where the parameter λ has been replaced by $\boldsymbol{x}$ by using Eqs. (9) and (10). The term $\boldsymbol{n}\cdot\boldsymbol{x}_0/r_0$ in Eqs. (11) and (16) determined by the position of the emission source is constant. In practice, this term can be expanded as follows:

$$\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}=-\Big(1-\frac{1}{2}\frac{b^2}{r_0^2}\Big)+\mathcal{O}(r_0^{-3})\,, \tag{13}$$

for the reason of $b\ll r_0$. Moreover the term $(1/2)(b^2/r_0^2)$ has a significant contribution to the 1PN light trajectory, but will be leaved out in the 2PN equation of light ray. Hence we take $\boldsymbol{n}\cdot\boldsymbol{x}_0/r_0=1$ in the next section. Note that the impact parameter $\boldsymbol{b}$ will not be a constant if variable $\boldsymbol{x}$ in Eqs. (9) is replaced by $\boldsymbol{x}_{\mathrm{p}}$, therefore the quantity $\boldsymbol{b}$ and b in Eqs. (11) and (12) will be replaced by $\boldsymbol{x}_{\mathrm{p}}-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{p}})$ and $\sqrt{\boldsymbol{x}_{\mathrm{p}}^2-(\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{p}})^2}$ in the next section.

3.2 *2PN Solutions of Photon Trajectory*

To get the 2PN solutions of light trajectory, one must distinguish $(\mathrm{d}/\mathrm{d}t)[\dot{\boldsymbol{x}}_{\mathrm{p}}(\boldsymbol{x}_{\mathrm{p}})]$ of Eq. (11) from

$$2(1+\gamma)(\dot{\boldsymbol{x}}_{\mathrm{p}}\cdot\boldsymbol{x}_{\mathrm{p}})\dot{\boldsymbol{x}}_{\mathrm{p}}\frac{M}{r_{\mathrm{p}}^3}-(1+\gamma\dot{\boldsymbol{x}}_{\mathrm{p}}^2)\boldsymbol{x}_{\mathrm{p}}\frac{M}{r_{\mathrm{p}}^3}\,, \tag{14}$$

which comes from the first two lines in Eq. (5). Under these considerations, we get the following differential equations for $\boldsymbol{x}_{2\mathrm{p}}$

$$\begin{aligned}\ddot{\boldsymbol{x}}_{2\mathrm{p}}=(1+\gamma)M\Big[\frac{\boldsymbol{n}(\boldsymbol{k}_{1\mathrm{p}}\cdot\boldsymbol{x}_{\mathrm{M}})+2\boldsymbol{k}_{1\mathrm{p}}(\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{M}})}{r_{\mathrm{M}}^3}\Big]&+(1+\gamma)M\Big\{\Big[r_{\mathrm{M}}\boldsymbol{n}\cdot\boldsymbol{k}_{1\mathrm{p}}-2(\boldsymbol{k}_{1\mathrm{p}}\cdot\boldsymbol{x}_{\mathrm{M}})+\frac{(\boldsymbol{k}_{1\mathrm{p}}\cdot\boldsymbol{x}_{\mathrm{M}})(\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{M}})}{r_{\mathrm{M}}}\Big]\\ \times\frac{\boldsymbol{b}}{r_{\mathrm{M}}^2(r_{\mathrm{M}}-\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{M}})^2}+\frac{\boldsymbol{n}\times(\boldsymbol{k}_{1\mathrm{p}}\times\boldsymbol{n})}{r_{\mathrm{M}}(r_{\mathrm{M}}-\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{M}})}\Big\}&+M^2\Big\{2\varepsilon\frac{\boldsymbol{x}_{\mathrm{M}}(\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{M}})^2}{r_{\mathrm{M}}^6}+2[2(1-\beta)+\varepsilon-2\gamma^2]\frac{\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{x}_{\mathrm{M}})}{r_{\mathrm{M}}^4}\\ +2[2\gamma(1+\gamma)+\beta-\varepsilon]\frac{\boldsymbol{x}_{\mathrm{M}}}{r_{\mathrm{M}}^4}\Big\}&\pm(1+\gamma)\frac{J}{r_{\mathrm{M}}^3}\boldsymbol{x}_{\mathrm{M}}+(1+\gamma)\Big(\frac{3J_2R^2}{2r_{\mathrm{M}}^2}\Big)\frac{M}{r_{\mathrm{M}}^3}[2(\dot{\boldsymbol{x}}_{\mathrm{M}}\cdot\boldsymbol{x}_{\mathrm{M}})\dot{\boldsymbol{x}}_{\mathrm{M}}-\boldsymbol{x}_{\mathrm{M}}]\,,\end{aligned} \tag{15}$$

where $r_{\mathrm{M}}=|\boldsymbol{x}_{\mathrm{M}}|$, $\boldsymbol{k}_{1\mathrm{p}}=\dot{\boldsymbol{x}}_{\mathrm{p}}-\boldsymbol{n}$, and $\boldsymbol{k}_{1\mathrm{p}}(\boldsymbol{x})=\boldsymbol{k}_{1\mathrm{p}}(\boldsymbol{x}_{\mathrm{M}})$. Note that variable $\boldsymbol{x}$ appearing in this equation has been substituted by $\boldsymbol{x}_{\mathrm{M}}$ and other higher order terms have been dropped. Following the same method used in Subsec. 3.1 substituting Eqs. (9), (10), and (12) into Eq. (15) and integrating Eq. (15) along the unperturbed photon path, we get

$$\boldsymbol{k}_{2\mathrm{p}}=M^2\Big\{(1+\gamma)^2\Big[\frac{\boldsymbol{b}}{b^2}\Big(2\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r^2}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0^2}\Big)+\Big(\frac{1}{r}-\frac{1}{r_0}\Big)+2\Big(\frac{r+\boldsymbol{n}\cdot\boldsymbol{x}}{b^2}-\frac{r_0+\boldsymbol{n}\cdot\boldsymbol{x}_0}{b^2}\Big)\Big)-\frac{\boldsymbol{n}}{b^2}\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}\Big)\Big]$$

$$+\Big[2\gamma(1+\gamma)+\beta-\frac{1}{2}\varepsilon\Big]\frac{\boldsymbol{n}}{r^2}-\frac{1}{2}\varepsilon\Big[\frac{\boldsymbol{x}(\boldsymbol{n}\cdot\boldsymbol{x})}{r^4}-\frac{\boldsymbol{x}_0(\boldsymbol{n}\cdot\boldsymbol{x}_0)}{r_0^4}\Big]-\Big[2(1+\gamma)-\beta+\frac{3}{4}\varepsilon\Big]\frac{\boldsymbol{b}}{b^2}\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r^2}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0^2}\Big)$$
$$-\Big[2(1+\gamma)-\beta+\frac{3}{4}\varepsilon\Big]\frac{\boldsymbol{b}}{b^3}(\pi-\vartheta)\Big\}+(1+\gamma)\Big[\Big(\frac{J_2MR^2}{b^2}\pm\frac{J}{b}\Big)\frac{\boldsymbol{b}}{b^2}\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}\Big)+\Big(\frac{J}{b}-\frac{J_2MR^2}{r^2}\Big)\frac{\boldsymbol{n}}{r}\Big]\,, \tag{16}$$

where $\vartheta=\arccos(\boldsymbol{n}\cdot\boldsymbol{x}/|\boldsymbol{n}||\boldsymbol{x}|)$, λ has been substituted by $\boldsymbol{x}$, all the higher order terms about J and J_2 like $(J/b)(b/r)$ and $(J_2MR^2/b^3)(b/r)$ have been dropped for the reason that $b\ll r$. By the same way, after some tedious calculations we obtain the second order perturbation of light trajectory as follows:

$$\boldsymbol{x}_{2\mathrm{p}}=(1+\gamma)J_2MR^2\Big[\frac{\boldsymbol{b}}{b^4}(r+\boldsymbol{n}\cdot\boldsymbol{x})-\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r}\boldsymbol{n}\Big]\pm(1+\gamma)\frac{J}{b}\Big[\frac{\boldsymbol{b}}{b^2}(r+\boldsymbol{n}\cdot\boldsymbol{x})+\boldsymbol{n}\ln(r+\boldsymbol{n}\cdot\boldsymbol{x})\Big]$$
$$+\Big\{(1+\gamma)^2\Big[\frac{\boldsymbol{b}}{b^4}(r+\boldsymbol{n}\cdot\boldsymbol{x})^2-\frac{\boldsymbol{n}}{b^2}(r+\boldsymbol{n}\cdot\boldsymbol{x})\Big]-\Big[2(1+\gamma)-\beta+\frac{3}{4}\varepsilon\Big]\Big[\frac{\boldsymbol{n}}{b}\Big(\frac{\pi}{2}-\vartheta\Big)+(\boldsymbol{n}\cdot\boldsymbol{x})\frac{\boldsymbol{b}}{b^3}(\pi-\vartheta)\Big]+\frac{1}{4}\varepsilon\frac{\boldsymbol{x}}{r^2}\Big\}M^2-\boldsymbol{c}_2\,, \tag{17}$$

where λ has been substituted by $\boldsymbol{x}$ and $\boldsymbol{c}_2$ is determined by initial condition $\boldsymbol{x}_{2\mathrm{M}}(\boldsymbol{x}_0)=0$. Thus we have obatined the analytical solutions of the photons trajectories to 2PN order by combining Eqs. (12) and (17). Using these results, one can discuss the boundary problem, Shapiro time delay, bending of light, gravitational shift of frequency/doppler displacement to the 2PN order. As an example, we will get the deflection angle to the 2PN order in the next subsection.

3.3 *Deflection Angle*

With the knowledge of the vector $\boldsymbol{k}$, we can get the deflection angle of light ray passing the sun. We use the formula given in Ref. [28], which is

$$\alpha=|\boldsymbol{n}\times\boldsymbol{k}|\,. \tag{18}$$

Substituting Eqs. (11) and (16) into Eq. (18) we obtain

$$\begin{aligned}\alpha=&(1+\gamma)\frac{M}{b}\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}\Big)\\&+(1+\gamma)\Big(\frac{J}{b^2}+\frac{J_2MR^2}{b^3}\Big)\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}\Big)\\&+\frac{M^2}{b^2}\Big\{\Big[2(1+\gamma)-\beta+\frac{3}{4}\varepsilon\Big]|\pi-\vartheta(\boldsymbol{n},\boldsymbol{x})|\\&-(1+\gamma)^2\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0}\Big)\Big(\frac{r_0+\boldsymbol{n}\cdot\boldsymbol{x}_0}{b}\Big)\\&+\Big[2(1+\gamma)-\beta+\frac{3}{4}\varepsilon\Big]b\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r^2}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0^2}\Big)\\&+\frac{1}{2}\varepsilon b^3\Big(\frac{\boldsymbol{n}\cdot\boldsymbol{x}}{r^4}-\frac{\boldsymbol{n}\cdot\boldsymbol{x}_0}{r_0^4}\Big)\Big\}+\mathcal{O}(c^{-6})\,,\end{aligned} \tag{19}$$

where the term including $r_0+\boldsymbol{n}\cdot\boldsymbol{x}_0$ comes from expansion of $\boldsymbol{k}_{1\mathrm{p}}(\boldsymbol{x}_\mathrm{p})$. In fact, the observer should be located in the trajectory of light ray $\boldsymbol{x}_{\mathrm{pp}}$, so the invariant compact parameter should be defined by

$$\begin{aligned}B&=\lim_{\to-\infty}|\boldsymbol{n}\times(\boldsymbol{x}_{\mathrm{pp}}\times\boldsymbol{n})|\\&=b+(1+\gamma)M\frac{r_0+\boldsymbol{n}\cdot\boldsymbol{x}_0}{b}+\mathcal{O}(c^{-4})\,.\end{aligned} \tag{20}$$

On the other hand, from Eqs. (9) and (10) one gets the relation $\boldsymbol{n}\cdot\boldsymbol{x}=r\sqrt{1-b^2/r^2}$. In practice, $b\ll r$, so one has the following relation

$$\boldsymbol{n}\cdot\boldsymbol{x}=r-\frac{1}{2}\frac{b^2}{r}+\mathcal{O}(r^{-2})\,, \tag{21}$$

by Taylor-expansion of b/r. Then substituting Eqs. (13), (20), and (21) into Eq. (19) and dropping the terms with the factor b/r in the brace, we have

$$\alpha=\alpha_{1-\mathrm{M}}+\alpha_{1-\mathrm{F}}+\alpha_{2-\mathrm{M}^2}+\alpha_{2-\mathrm{J}}+\alpha_{2-\mathrm{J}_2}+\mathcal{O}(c^{-6})\,, \tag{22}$$

with

$$\begin{aligned}\alpha_{1-\mathrm{M}}&=2(1+\gamma)\frac{M}{B}\,,\\\alpha_{1-\mathrm{F}}&=-\frac{1}{2}(1+\gamma)\frac{M}{B}\frac{B^2}{r^2}-\frac{1}{2}(1+\gamma)\frac{M}{B}\frac{B^2}{r_0^2}\,,\\\alpha_{2-\mathrm{M}^2}&=\Big[2(1+\gamma)-\beta+\frac{3}{4}\varepsilon\Big]\frac{M^2}{B^2}\pi\,,\\\alpha_{2-\mathrm{J}}&=\pm2(1+\gamma)\frac{J}{B^2}\,,\\\alpha_{2-\mathrm{J}_2}&=2(1+\gamma)\frac{J_2MR^2}{B^3}\,,\end{aligned} \tag{23}$$

where $\alpha_{1-\mathrm{M}}$, $\alpha_{1-\mathrm{F}}$, $\alpha_{2-\mathrm{M}^2}$, $\alpha_{2-\mathrm{J}}$, and $\alpha_{2-\mathrm{J}_2}$ represent the contributions coming from spherical part of 1PNA, finite range of observer, and emission source, spherical part of 2PNA, angular momentum, and quadruple momentum of the sun. Our results of $\alpha_{1-\mathrm{M}}$ and $\alpha_{2-\mathrm{M}^2}$ are the same with those in Ref. [25]. If we consider the relation

$$B=R\Big[1+(1+\gamma)\frac{M}{R}\Big]+\mathcal{O}(c^{-4})\,, \tag{24}$$

for the light ray just grazing the limb of sun. Our results are compatible with Ref. [23] except the term $-(1/2)(1+\gamma)(M/B)(B^2/r_0^2)$ in Eq. (23), which comes from the finite range of the emission source.

4 Concluding Remarks

The analytical second post-Newtonian solution for light propagation is derived in this paper, which is slight different from the one derived in Ref. [25]. The former can be reduced to the later one by considering the source of light signal being located at infinity. As a consequence, the deflection angle to the 2PN order in our case includes the contributions coming from the finite range of emission source and observer besides the usual contributions. These results would be useful for precision astrometry missions like ASTROD.

On the basis of this work, the Lorentz-covariant theory presented in Ref. [28] can be extended to the event of strong gravitational fields, which will require knowing solutions of the equations of light propagation in the 2PNA of general relativity or alternative theory of gravity. Hence one can expect to find differences between predictions of two gravity theories, which may be used for suggesting new observational test of the theories.

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