Evolutionary Game Dynamics in a Fitness-Dependent Wright–Fisher Process with Noise*

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(Received March 15, 2011, revised manuscript received May 3, 2011)

Abstract *Evolutionary game dynamics in finite size populations can be described by a fitness-dependent Wright-Fisher process. We consider symmetric 2×2 games in a well-mixed population. In our model, two parameters to describe the level of player's rationality and noise intensity in environment are introduced. In contrast with the fixation probability method that used in a noiseless case, the introducing of the noise intensity parameter makes the process an ergodic Markov process and based on the limit distribution of the process, we can analysis the evolutionary stable strategy (ESS) of the games. We illustrate the effects of the two parameters on the ESS of games using the Prisoner's dilemma games (PDG) and the snowdrift games (SG). We also compare the ESS of our model with that of the replicator dynamics in infinite size populations. The results are determined by simulation experiments.*

PACS numbers: 02.50.Le, 87.23.Kg, 05.45.Pq
Key words: evolutionary games, Wright–Fisher process, evolutionary stable strategy, noise

1 Introduction

Evolutionary game theory is in the framework of bounded rationality and based on the population of individuals. The bounded rationality in it is expressed as individual knowledge of learning the situation of games and thus the evolutionary rules of strategies of the players. The dynamic stable equilibrium of the system is exactly the limit outcome of the strategy updating process of the population. There have been a number of analytical frameworks concerning the strategy evolutionary rules in the games. For example, the replicator dynamics,[1−3] Bayesian learning,[4−5] reinforcement learning,[6−7] belief learning[8−9] and so on. These models are described for very large, unstructured populations. In realistic systems, the population is finite and subjected to fluctuations. So it is not clear under which circumstances these dynamics are a good approximation of the dynamics in realistic systems. To answer this question, some evolutionary models in finite populations using stochastic process have also been proposed.[10−13]

In these years, two other processes: Moran process[14] and Wright–Fisher process[15] which are known in population genetics have been transferred to evolutionary game theory in finite size and well-mixed populations.[16−18] These processes can be seen as a study model of bounded rational players based on Darwin's Natural selection. Recent papers that were based their models on and dealt with the Moran process include Refs. [19–26].

In contrast with the Moran process, literatures based their models on the Wright–Fisher process is very few. As we know, Imhof *et al.*[18] pioneered the study of evolutionary games using the Wright-Fisher process. They used the fixation probability method to evaluate which strategy would be favored for symmetric 2×2 games in a noiseless environment.

However, the existence of noise induced by various factors can have great effects on the equilibrium state of the system.[27−31] Perc[27] studied the noise effects induced by external stochastic influence on the evolution of cooperation in the spatial prison's dilemma games (SCPG). Tao *et al.*[29] studied intrinsic noise induced by random interactions between discrete individuals in the evolutionary games with finite population size. Helbing *et al.*[30] studied noise induced by migration in the SCPG, finding out breaking of cooperation because of noise. Jiang *et al.*[31] studied adaptive migration rules in promoting cooperation in spatial games also considering the effect of environmental noise.

In this paper, we set up a dynamic evolutionary model for symmetric 2×2 games in a well-mixed finite size population using the fitness-dependent Wright-Fisher process. We introduce two parameters to describe the level of player's bounded rationality and noise intensity in environment. In contrast with the fixation probability method that used in a noiseless case, the introducing of the noise parameter makes the process an ergodic Markov process. Based on the limit distribution of process, we can analysis the evolutionary stable strategy (ESS) of games with bounded rational players and finite size populations in a noisy environment.

*Supported by the National Natural Science Foundation of China under Grant Nos. 71071119 and 60574071
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The rest of the letter is organized as follows. In the next section, we describe the model of the evolutionary games. The simulation results and discussions are given in Sec. 3. And the letter is concluded by the last section.

2 Model

Consider a finite population with size N playing symmetric 2×2 games. The strategy set of the game is $\{\mathrm{A},\mathrm{B}\}$ and the payoff matrix is $\left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right)$, which means an A player will obtain $a_{11}$when playing against another A or a_{12} when playing against B. Choosing strategy B results in either obtaining a_{21} (against A) or a_{22} (against B).

Assume that the population is well mixed. Based on the payoff matrix, when the number of individuals choosing A is i, the expected payoffs of choosing strategy A and B are respectively:

$$\pi_A^i = \frac{a_{11}(i-1)+a_{12}(N-i)}{N-1}\,, \quad i\in\{1,2,\ldots,N\}\,, \tag{1}$$

$$\pi_B^i = \frac{a_{21}i+a_{22}(N-1-i)}{N-1}\,, \quad i\in\{0,1,\ldots,N-1\}\,. \tag{2}$$

Individual fitness f is a function of its payoff π from the game. We use the exponential function in this letter,

$$f=\exp(\beta\pi)\,. \tag{3}$$

The parameter β ($\beta>0$) denotes selection strength, which can be seen as the level of player's rationality.

In each generation, every individual reproduces several offspring to form an offspring pool. The number of offspring one reproduces is proportional to its fitness. The offspring uses the same strategy of its mother with probability $1-\varepsilon$, and uses a different strategy with probability ε. As parameter $\varepsilon\,(0<\varepsilon<1/2)$ describes the ratio of mutation in the population because of environmental noise, we call it noise intensity directly. Individuals of the next generation are chosen uniformly from the offspring pool. Suppose that the number of the population is fixed and the choosing process is with replacement. If the number of individuals choosing strategy A in the last generation is i, then the number of individuals choosing A in the next generation obeys the binomial distribution with parameters (N,p_i), where

$$p_i = \frac{i\cdot f_A^i\cdot(1-\varepsilon)+(N-i)\cdot f_B^i\cdot\varepsilon}{if_A^i+(N-i)f_B^i} \quad (1\le i\le N-1)$$
$$p_0=\varepsilon\,,\quad p_N=1-\varepsilon\,. \tag{4}$$

Let the reproduction period corresponding to the game period. Let T denote the set of game phase. Introduce stochastic process $z(t)$ that denote the number of the individuals choosing strategy A at time t ($t\in T$), so $z(t)$ is a discrete time Markov chain with state space $S=\{0,1,\ldots,N\}$. Let $P(t)=\{p_{ij}(t)\}_{i,j\in S}$ be the t-step probability transition matrix. Because of homogeneous of the process,

$$p_{ij}(t)=p\{z(s+t)=j|z(s)=i\}\,,\quad \forall s\in T\,. \tag{5}$$

The one-step probability transition matrix is $P(1)=\{p_{ij}(1)\}_{(N+1)\times(N+1)}$, where

$$p_{ij}(1)=\binom{N}{j}p_i^j(1-p_i)^{N-j}\,,\quad 0\le i,j\le N\,. \tag{6}$$

$\forall 0\le i,j\le N$, $p_{ij}(1)>0$, so the Markov chain is ergodic which means the system can reach to an equilibrium state though long run transition and the limit distribution is independent of the initial state. Namely, $\forall i\in S$, $\lim_{t\to\infty}p_{ij}(t)$ exists and all of them are equal, denote it v_j. Then $(v_0,v_1,\ldots,v_N)$ is the limit distribution of the Markov chain. By using Chapman–Kolmogorov equation, we can obtain the n-step Markov transition probability matrix

$$P(n)=P(1)^n\,. \tag{7}$$

This is also the Markov transition probability matrix of n-time-repeat evolutionary games. Obviously,

$$P(n)\xrightarrow{n\to\infty}\begin{pmatrix} v_0 & v_1 & \cdots & v_N\\ v_0 & v_1 & \cdots & v_N\\ \vdots & \vdots & \ddots & \vdots\\ v_0 & v_1 & \cdots & v_N\end{pmatrix}. \tag{8}$$

If the process has the limit distribution, which is independent of its initial state, then the game has an ESS.

3 Simulation Results and Discussion

Two illustrative examples of the prisoner's dilemma games (PDG) and the snowdrift games (SG) are present to explain our model. The payoff matrixes are given as

$$\begin{array}{cc} & \begin{matrix} C & D\end{matrix}\\ \begin{matrix} C\\ D\end{matrix} & \begin{bmatrix} 1 & 0\\ b & 0\end{bmatrix}\end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{matrix} C & D\end{matrix}\\ \begin{matrix} C\\ D\end{matrix} & \begin{bmatrix} 1 & 1-r\\ 1+r & 0\end{bmatrix}\end{array}$$

respectively. Each game is thus controlled by a single parameter: the temptation to defect $b\,(1<b\le 2)$ for PDG and the cost to cooperate $r\,(0<r<1)$ for SG.[31] Based on the replicator equations in the infinite size population without noise, the state of "all the individuals choosing defection (D)" is the only ESS in PDG and the mixed strategy of $(1-r,r)$ is the only ESS in SG. We give the simulation results of our model which describe the effects of the noise intensity and rational level of the players on the ESS of the two case games in the following.

Our simulations start from a random initial condition where individuals randomly select C or D at the beginning. First, we study the effect of population size N on the ESS of the two games. Figure 1 gives the results that after $n\to\infty$ repeat games, when the system reaches the stationary state, the expected population frequency of choosing D (denote it f_D) as a function of the rational level β for different values of ε and N given fixed $b=2$ in PDG. Two values of $\varepsilon=0.01, 0.1$ and four values of $N=10,20,50,100$ are considered respectively. Simulation results show that increasing N lower the value of f_D no matter what values of ε, but the effect appears nonlinear. That is, when N is small, i.e., $N=10$, the increasing of N leads to a notable change of f_D, but when N reaches a certain value and continues to increase, the increasing of N has a neglectable effect on f_D. We also test the case of $N=1000$, in which case the curve is almost the same as the case of $N=100$. Figure 2 gives the results of f_D as a function of the noise intensity ε for different values of β and N given fixed $b=2$ in PDG. The conclusion is the same as in Fig. 1: when N is large enough, i.e., $N \geq 100$, the value of N has little effect on f_D. So we choose fixed $N = 100$ in the subsequent simulations in PDG.

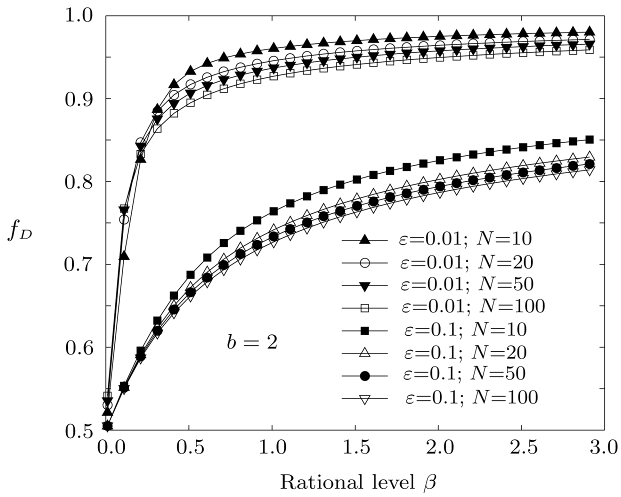

Fig. 1 Defection frequency f_D vs. β in PDG with different values of ε and N, give fixed $b = 2$.

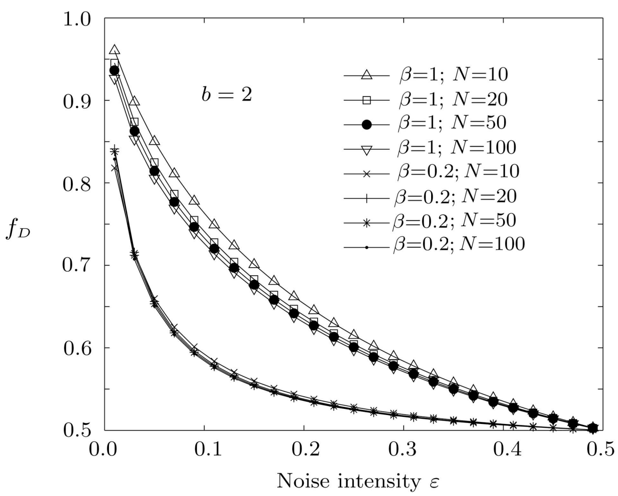

Fig. 2 Defection frequency f_D vs. ε in PDG with different values of β and N, give fixed $b = 2$.

The effect of size N on f_D in SG is shown by Figs. 3 and 4. In Fig. 3, different values of $\varepsilon = 0.01, 0.1$, $r = 0.1, 0.9$, and $N = 10, 50, 100$ are considered for f_D as a function of β. In Fig. 4, different values of $\beta = 0.2, 1$, $r = 0.1, 0.9$, and $N = 10, 50, 100$ are considered for f_D as a function of ε. The two figures also confirm that when N is large enough, i.e., $N \geq 100$, the value of N has little effect on f_D no matter what other parameters are. So we can choose fixed $N = 100$ in the subsequent simulations in SG.

Figure 5 gives f_D as a function of parameter b in PDG for different values of ε and β given fixed $N = 100$. All cures are increasing which implies that a larger temptation to defect will lead to a larger defector frequency f_D. This result is a common sense. We also note that when $\beta = 1$ is fixed, the curve corresponding to $\varepsilon = 0.01$ is above the curve corresponding to $\varepsilon = 0.02$, which is above the curve corresponding to $\varepsilon = 0.03$, and is then above the curve corresponding to $\varepsilon = 0.04$ no matter what values of b. This implies that the increasing of noise intensity can lower the defector frequency f_D in PDG. When $\varepsilon = 0.02$ is fixed, the increasing of rational level can enhance the defector frequency f_D in PDG.

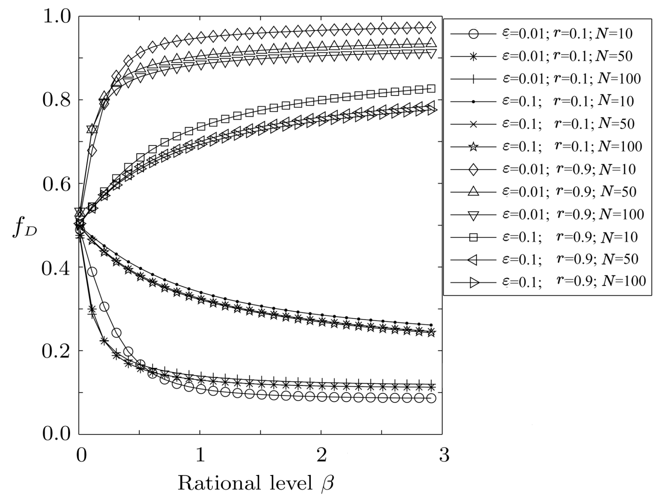

Fig. 3 Defection frequency f_D vs. β in SG with different values of ε, r, and N.

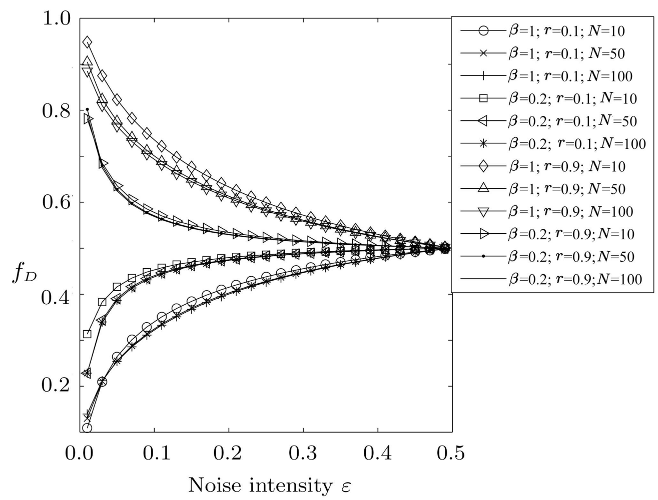

Fig. 4 Defection frequency f_D vs. ε in SG with different values of β, r, and N.

The effect of r on f_D in SG is more subtle. Figure 6 gives f_D as a function of parameter r in SG for different values of ε and β given fixed $N = 100$. All cures are increasing which implies that a larger cost for cooperation will lead to a larger defector frequency f_D. This result is a common sense. Simulation results also show that when $0 < r < 0.5$, the increasing of ε will enhance f_D, and the increasing of β will inhibit f_D; but when $0.5 < r < 1$, the situation is reversed, that is, the increasing of ε will inhibit f_D, and the increasing of β will enhance f_D.

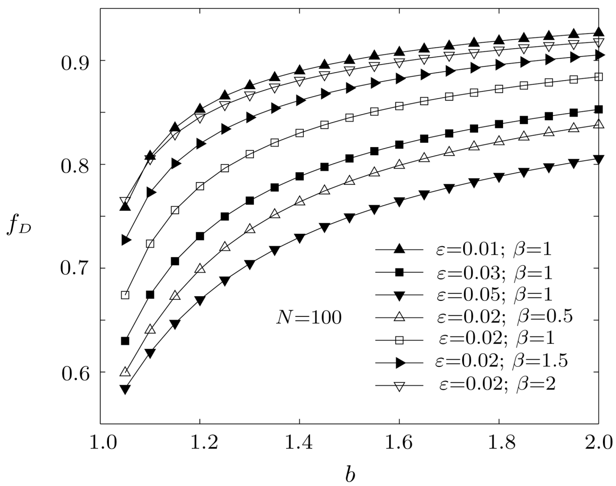

Fig. 5 Defection frequency f_D vs. b in PDG with different values of ε and β, give fixed $N = 100$.

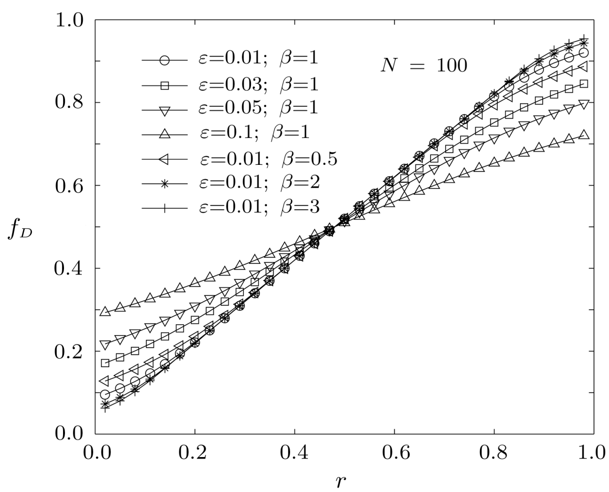

Fig. 6 Defection frequency f_D vs. r in SG with different values of ε and β, give fixed $N = 100$.

Figure 7 gives f_D as a function of ε in PDG for different values of β given fixed $b = 2$ and $N = 100$. For all values of $\beta = 0.2, 0.5, 0.8, 1, 2, 3$, the noise intensity ε has a dominant influence on the system's stationary state. That is, when $\varepsilon \to 0^+$ and β is large enough, f_D is close to 1. As ε increases, f_D decreases; but the effect appears nonlinear. The smaller the value of ε, the more sensitive the value of f_D is when ε changes. When ε reaches to 0.5, f_D also reaches to 0.5, which means that individuals choose their strategies almost randomly. Figure 8 gives f_D as a function of β in PDG for different values of ε given fixed $b = 2$ and $N = 100$. Simulation results show that when $\beta \to 0^+$, then $f_D \to 0.5$ for any values of ε; but when β is large enough and ε is small enough, f_D is close to 1. Figures 9 and 10 show the according cases in SG, and the conclusions are more subtle. The equilibrium state of $(1-r, r)$ in replicator dynamics also can be approached in our model; when r is small, i.e., $r = 0.1$, the increase of ε leads to the increase of f_D; when r is large, i.e., $r = 0.9$, the increase of ε leads to the decrease of f_D; when ε is too large or β is too small, individuals will choose their strategies almost randomly.

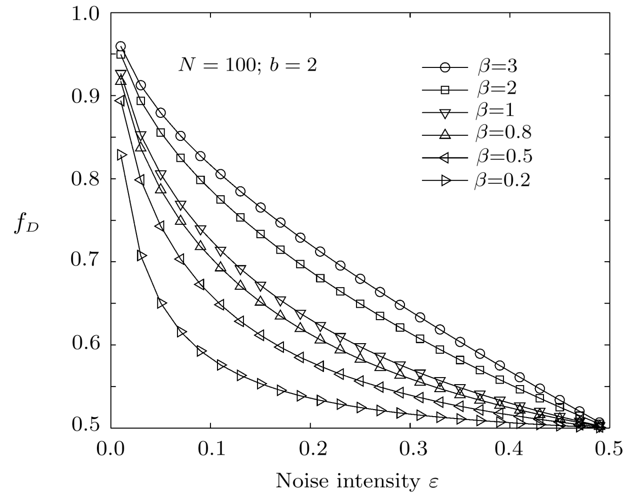

Fig. 7 Defection frequency f_D vs. ε in PDG with different values of β, give fixed $b = 2$ and $N = 100$.

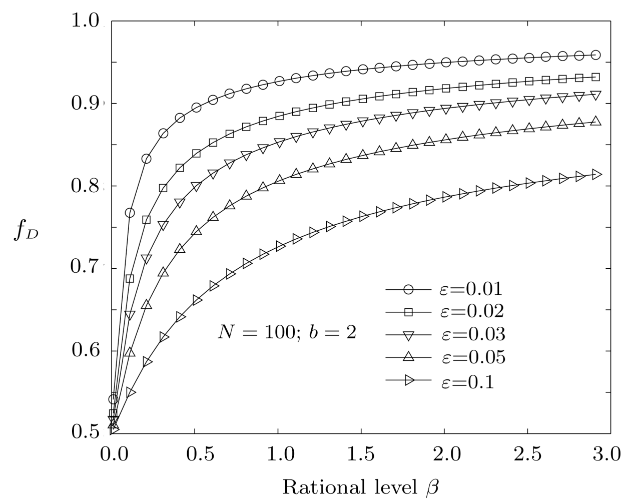

Fig. 8 Defection frequency f_D vs. β in PDG with different values of ε, give fixed $b = 2$ and $N = 100$.

Thus, we can conclude that either too low the rational level or too large the noise intensity can lead the system to get into a state of chaos; but when the noise intensity is small enough and the rational level is large enough, the equilibrium state of the system will be close to the one that predicted by the replicator dynamics in the infinite size population.

To get more impression, we also give the three-dimensional graphs of f_D as a function of ε and β given fixed $N = 100$. Two values of $b = 1.2, 2$ for PDG and two values of $r = 0.1, 0.9$ for SG are shown respectively by Figs. 11 and 12.

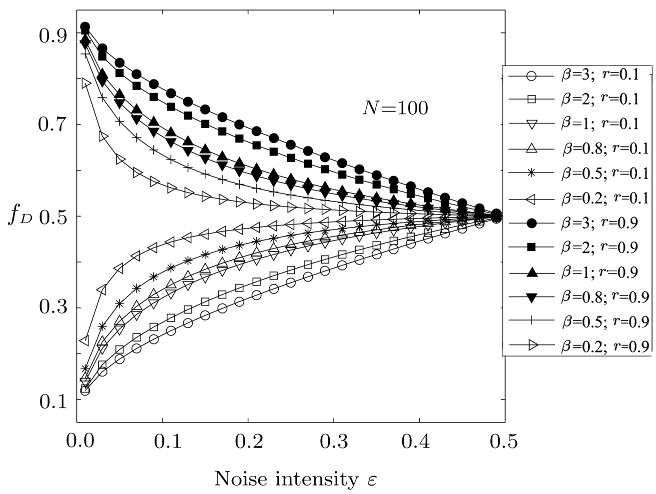

Fig. 9 Defection frequency f_D vs. ε in SG with different values of β and r, give fixed $N = 100$.

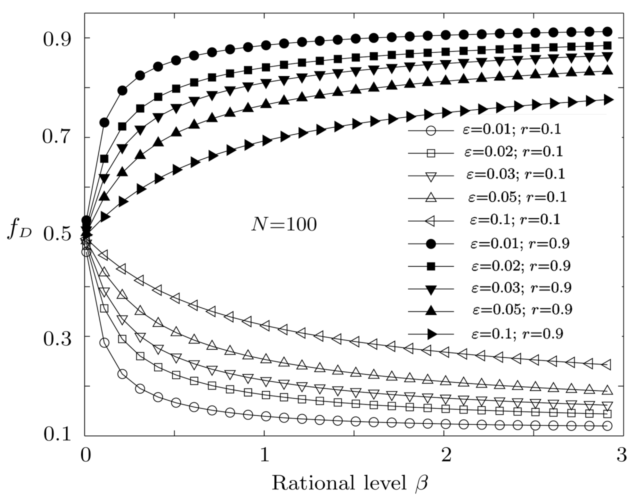

Fig. 10 Defection frequency f_D vs. β in SG with different values of ε and r, give fixed $N = 100$.

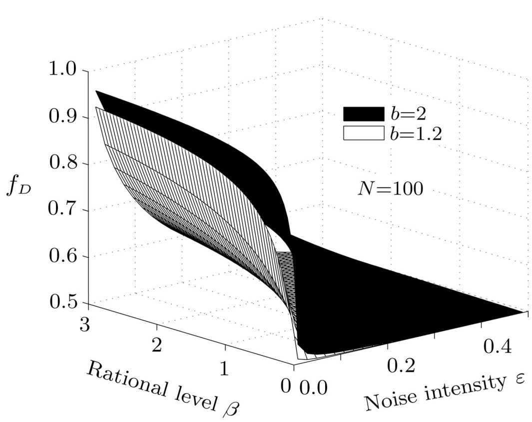

Fig. 11 Defection frequency f_D vs. (ε, β) in PDG with different values of b, give fixed $N = 100$.

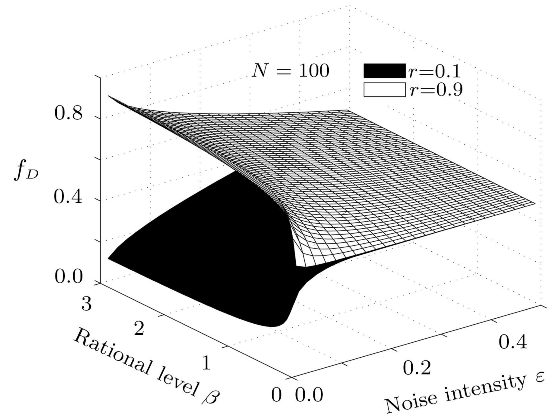

Fig. 12 Defection frequency f_D vs. (ε, β) in SG with different values of r, give fixed $N = 100$.

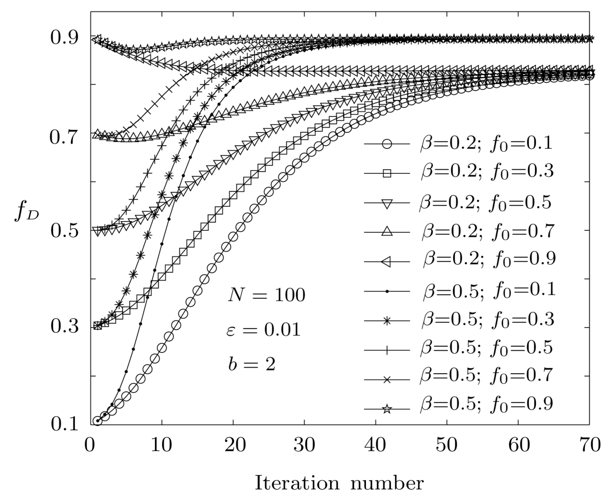

Fig. 13 Evolutionary processes in PDG with different initial states f_0 and β, give fixed $\varepsilon = 0.01$, $b = 2$, and $N = 100$.

In the last, we want to test whether the initial state of the system has no effect on the equilibrium state of the system. To do this, we choose five initial states of $f_0 = 0.1, 0.3, 0.5, 0.7, 0.9$, which value denotes the defector frequency at the beginning. Figure 13 gives the evolutionary processes in PDG for different values of β, given fixed $\varepsilon = 0.01$ and $b = 2$. Figure 14 gives the evolutionary processes in PDG for different values of ε, given fixed $\beta = 0.5$ and $b = 2$. In the SG, we choose random values of f_0 for different α and β given fixed $r = 0.1$. In each case, we choose two random values of f_0. The simulation results of SG are shown in Fig. 15. All these simulations show that the initial state of the system has no effects at all on the equilibrium state of the system no matter what other parameters are. This is consistent with the theoretical results in Sec. 2.

Figures 13–15 also imply that when ε is fixed, the greater the β, the less the iteration number required till the system reaches the stationary state; when β is fixed, the greater the ε, the less the iteration number required till the system reaches the stationary state.

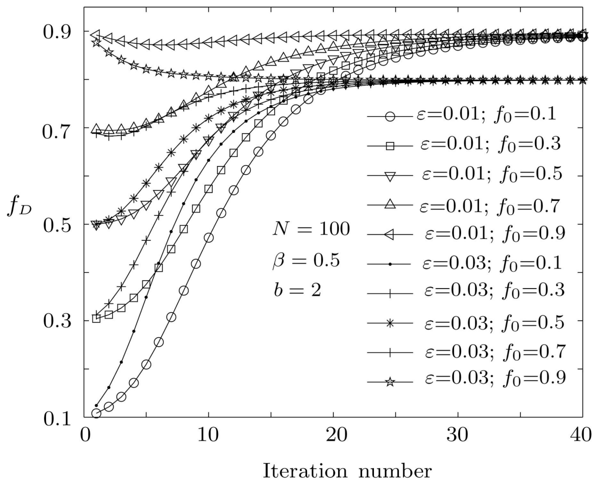

Fig. 14 Evolutionary processes in PDG with different values of ε and initial states f_0, give fixed $\beta = 0.5$, $b = 2$, and $N = 100$.

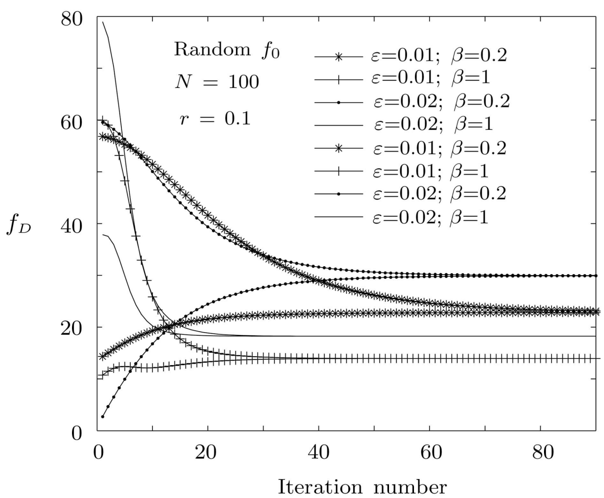

Fig. 15 Evolutionary processes in SG with different values of ε, β and two random initial states f_0, give fixed $r = 0.1$ and $N = 100$.

4 Conclusions

In summary, a dynamic evolutionary model for symmetric 2×2 games in a well-mixed population with finite size is proposed in this letter. We use the fitness-dependent Wright-Fisher process to describe the strategy updating process of the players in the system. Unlike the standard model used in Ref. [18], we introduce two parameters to describe the level of player's bounded rationality and noise intensity in environment. Simulation results for PDG and SG show that both the two parameters have dramatic impact on the system's stable equilibrium.

In the PDG, "All D" is the only ESS based on replicator dynamics. In our model, when noise intensity $\varepsilon \to 0^+$ and rationality level β reaches a certain value, the system will also converge to the "All D" equilibrium state. Otherwise, the system will reach an intermediate state that "coexistence of C and D" will become the equilibrium state. Increasing ε or decreasing β will lower the defector frequency; especially, when ε is too large or β is too small, the system will be chaotic which means that individuals choose their strategies almost randomly. Experimental results also show that smaller ε and β require more time till the system reaches the steady state.

In the SG, the results are more instructive. The game has only one mixed ESS based on replicator dynamics. In our model, simulation results show that the same ESS can also be approached when ε is under some small values and β is under some large values; but when ε is too large or β is too small, the system will be chaotic. We also find that for small values of r, i.e., $r = 0.1$, the increasing of ε or the decreasing of β will enhance the defector frequency; but when r is large ($r = 0.9$), the increasing of ε or the decreasing of β will lower the defector frequency. The same as PDG, smaller ε and β require more time till the system reaches the steady state.

Introducing network structure to describe heterogeneous of the population in some social systems, Moran-process-based strategy evolutionary rules can lead to the evolution of cooperation.[32] The Moran process describes a biological population with asynchronous replication. In contrast, the Wight–Fisher process describes a biological population with overlapping generations, which is more general in population genetics.[18] How to extend our model to the heterogeneous mixed populations and how network parameters affect the cooperation ratio of the system are the subjects of our future works. Obviously, these can enhance our understanding of the emergence of cooperation in some self-organized complex systems.

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