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Travelling Waves for a Density Dependent Diffusion Nagumo Equation over the Real Line

Robert A. Van Gorder*

Department of Mathematics, University of Central Florida, P.O. Box 161364, Orlando, FL 32816-1364, USA

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Abstract *We consider the density dependent diffusion Nagumo equation, where the diffusion coefficient is a simple power function. This equation is used in modelling electrical pulse propagation in nerve axons and in population genetics (amongst other areas). In the present paper, the δ-expansion method is applied to a travelling wave reduction of the problem, so that we may obtain globally valid perturbation solutions (in the sense that the perturbation solutions are valid over the entire infinite domain, not just locally; hence the results are a generalization of the local solutions considered recently in the literature). The resulting boundary value problem is solved on the real line subject to conditions at $z \to \pm\infty$. Whenever a perturbative method is applied, it is important to discuss the accuracy and convergence properties of the resulting perturbation expansions. We compare our results with those of two different numerical methods (designed for initial and boundary value problems, respectively) and deduce that the perturbation expansions agree with the numerical results after a reasonable number of iterations. Finally, we are able to discuss the influence of the wave speed c and the asymptotic concentration value α on the obtained solutions. Upon recasting the density dependent diffusion Nagumo equation as a two-dimensional dynamical system, we are also able to discuss the influence of the nonlinear density dependence (which is governed by a power-law parameter m) on oscillations of the travelling wave solutions.*

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Key words: density dependent diffusion equation, nonlinear partial differential equation, perturbation method, δ-expansion method

1 Introduction

The Nagumo (or, Fitzhugh–Nagumo) equation has various applications in the fields of logistic population growth, flame propagation, neurophysiology, autocatalytic chemical reaction, branching Brownian motion process and nuclear reactor theory; see, e.g. Refs. [1–3]. Recently, the method of homotopy analysis has been applied to the Nagumo equation[4−5] in order to construct approximate solutions. There is also a density dependent diffusion Nagumo equation, where the diffusion coefficient is a simple power function. This equation is used in modeling electrical pulse propagation in nerve axons and in population genetics; see Refs. [6]–[9] and the references therein. We consider the density dependent diffusion Nagumo equation, where the diffusion coefficient is a simple power function. This equation is used in modeling electrical pulse propagation in nerve axons and in population genetics; see Refs. [6]–[9] and the references therein. Mathematical analysis for this equation, and the related traveling wave ordinary differential equation, and related generalizations, have recently been considered in Refs. [10]–[12].

The density dependent diffusion Nagumo equation reads

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(u^m \frac{\partial u}{\partial x}\Big) + u(1-u)(u-\alpha)\,, \tag{1}$$

where $\alpha \in (0,1)$, $m \geq 1$, and $u = u(x,t)$ is a real-valued function. We shall be interested in traveling wave solutions $u(x,t) = f(z)$ where $z = x - ct$ and $c \in \mathbb{R}$ is the wave speed. Under such a transformation, Eq. (1) becomes

$$(f^m f')' + cf' + f(1-f)(f-\alpha) = 0\,, \tag{2}$$

where prime denotes differentiation with respect to z. Natural boundary conditions are

$$\lim_{z\to-\infty} f(z) = A_- \quad \text{and} \quad \lim_{z\to+\infty} f(z) = A_+\,, \tag{3}$$

where $A_\pm \in \{0, \alpha, 1\}$ and $A_- \neq A_+$. If $A_- = A_+$, then $f(z) = A_+$ is a constant solution. We then have six possible boundary conditions, all of which we shall discuss later.

In the present paper, we apply the δ-expansion perturbation method, pioneered in the late 1980's by Bender *et al.* (see Refs. [13]–[15]; among a number of many other good references in this area, these are great references to start with) in order to obtain perturbation solutions to the nonlinear boundary value problem (2)–(3). Instead of rephrasing (2)–(3) as an initial value problem (see Refs. [10] and [12]), we are actually able to obtain analytical solutions globally (that is, for all $z \in \mathbb{R}$, as opposed to some strict subset of the real line normally required when solving the system numerically, since we are able to enforce the boundary conditions (3)). It is in

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*E-mail: rav@knights.ucf.edu

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that sense that we say the obtained perturbation solutions are "globally valid". While the δ-expansion method may not always lead to convergent solutions, particularly for higher order approximations, the present author has had some recent luck in applying the δ-expansion method to multiple physical models. See, for instance,[16–17] (where convergent perturbation solutions were obtained for the Blasius problem and the Lane–Emden equation of the second kind, respectively),[18] (where a combined δ-expansion and numerical method permitted accurate solutions for nonlinear stochastic differential equations describing wave propagation in a random medium),[19] (where formal perturbation expansions were obtained for a stochastic generalized KdV model). A general method for obtaining solutions to nonlinear rational differential equations was presented in Ref. [20].

We shall compare the present results with numerical simulations (the accuracy of which we can easily control) in order to demonstrate the relative accuracy of the perturbation method. In particular, we compare our results with those of two different numerical methods (designed for initial and boundary value problems, respectively) and find that the perturbation expansions agree with the numerical results after a reasonable number of iterations. Finally, we are able to discuss the influence of the wave speed c and the asymptotic concentration value α on the obtained solutions. Upon recasting the density dependent diffusion Nagumo equation as a two-dimensional dynamical system, we are also able to discuss the influence of the nonlinear density dependence (which is governed by a power-law parameter m) on oscillations of the travelling wave solutions.

2 Application of the Perturbation Method to the Density Dependent Diffusion Nagumo Equation

In order to most accurately capture the qualitative behavior of the solutions to the density dependent diffusion Nagumo equation (2)–(3), we split the problem on the real live into two problems, one for the negative real axis ($z<0$) and one for the positive real axis ($z>0$). The reason for this lies in the cubic term $f(1-f)(f-\alpha)$ present in Eq. (2). We find that in order to obtain solutions which agree best with the nonlinear ordinary differential equation (2) as $z\to\pm\infty$, one should maintain the factor which vanishes naturally while one should remove the remaining two factors which persist. For instance, if we desire a solution satisfying the boundary conditions $A_-=0$ and $A_+=1$, then for $z<0$ we should consider the modified cubic term

$$f(1-f)^{\delta}(f-\alpha)^{\delta}\,, \tag{4}$$

while for $z>0$ we should consider the modified cubic term

$$f^{\delta}(1-f)(f-\alpha)^{\delta}\,. \tag{5}$$

We would then obtain two branches, $f_-(z)$ for $z<0$ and $f_+(z)$ for $z>0$. In order to obtain a globally valid solution $f(z)\in C^2(\mathbb{R})$, we require that $f_-(z)\in C^2((-\infty,0))$, $f_+(z)\in C^2((0,+\infty))$, along with the matching conditions

$$f_-(0)=f_+(0),\ \ f_-'(0)=f_+'(0),\ \text{ and }\ f_-''(0)=f_+''(0). \tag{6}$$

Then, the solution $f(z)$ is given by

$$f(z)=\begin{cases} f_-(z)\,, & z<0\,,\\ f_-(0)=f_+(0)\,, & z=0\,,\\ f_+(z)\,, & z>0\,. \end{cases} \tag{7}$$

In order to employ the δ-expansion method, we assume perturbation solutions of the form

$$f_-(z)=(f_-)_0+(f_-)_1\delta+(f_-)_2\delta^2+\cdots \tag{8}$$

$$f_+(z)=(f_+)_0+(f_+)_1\delta+(f_+)_2\delta^2+\cdots \tag{9}$$

We require that

$$\lim_{z\to-\infty}(f_-)_0(z)=A_- \quad\text{and}\quad \lim_{z\to+\infty}(f_+)_0(z)=A_+\,, \tag{10}$$

while

$$\lim_{z\to-\infty}(f_-)_k(z)=0 \quad\text{and}\quad \lim_{z\to+\infty}(f_+)_k(z)=0 \tag{11}$$

for all $k\geq 1$. In what follows, we consider the possible cases mentioned above. Note that, in some cases, one or more solution branch $f_\pm(z)$ is identically constant, in which case matching is not possible, so no desired perturbation solution, which is in $C^2(\mathbb{R})$, is found. As we shall see, this break down of the method depends on the sign of the wave speed, c, in some of the cases considered.

In the literature, the cases of most importance to the physical applications are those corresponding to $f(-\infty)=1$. In these two cases, we find that perturbation solutions can exist for $c>0$. However, for $c<0$, there are drastic discontinuities at $z=0$ as one of the two solution branches will be identically constant (at order zero) in order to satisfy the relevant boundary condition, rendering matching of the order zero solutions impossible. Luckily, the physically interesting cases correspond to $c>0$, and our approach is valid in the physical regime $c>0$.

The only other possible scenarios occur when the boundary condition $f(-\infty)$ takes the value either α or zero and the other condition is $f(+\infty)=1$. In these cases, when $c>0$ we obtain no solutions (due to the same matching problems mentioned above). However, when $c<0$, the conditions do allow for a perturbation solution of the type considered here. Note that these solutions are essentially the same as those discussed in the previous paragraph, once one notices the direction of the wave propagation. As such, the perturbation solutions corresponding to $f(-\infty)=1$ may be relabeled to include these cases, so we need not consider them separately.

Note that there are no perturbation solutions for the case in which either $f(-\infty)=\alpha$ and $f(+\infty)=0$ or

$f(-\infty)=0$ and $f(+\infty)=\alpha$ for any c, as at least one solution branch will always be constant at order zero, again rendering matching impossible.

We proceed to consider the specific cases of (i) $f(-\infty)=1$ and $f(+\infty)=\alpha$ and (ii) $f(-\infty)=1$ and $f(+\infty)=0$, below. Reasonable values for c are $c\in(0,1]$.

2.1 *The Case of* $f(-\infty)=1$ *and* $f(+\infty)=\alpha$

In the case where $f(-\infty)=1$ and $f(+\infty)=\alpha$, we consider the boundary value problems

$$(f_-^{\delta m}f_-')'+cf_-'+f_-^{\delta}(1-f_-)(f_--\alpha)^{\delta}=0\,,\qquad \lim_{z\to-\infty}f(z)=1\,, \tag{12}$$

$$(f_+^{\delta m}f_+')'+cf_+'+f_+^{\delta}(1-f_+)^{\delta}(f_+-\alpha)=0\,,\qquad \lim_{z\to+\infty}f(z)=\alpha\,, \tag{13}$$

for $f_-(z)$ and $f_+(z)$, respectively. As mentioned above, we take $c>0$. Assuming solutions of the form (8) and (9), we find that the order zero contributions are governed by

$$(f_-)_0''+c(f_-)_0'-(f_-)_0=-1\,,\quad \lim_{z\to-\infty}(f_-)_0(z)=1\,, \tag{14}$$

$$(f_+)_0''+c(f_+)_0'+(f_-)_0=\alpha\,,\quad \lim_{z\to-\infty}(f_+)_0(z)=\alpha\,. \tag{15}$$

The higher order contributions are given by

$$\begin{aligned}(f_-)_1''+c(f_-)_1'-(f_-)_1 &=(1-(f_-)_0+m(f_-)_0'')\ln((f_-)_0)+\frac{m((f_-)_0')^2}{(f_-)_0}\\ &\quad+(1-(f_-)_0)\ln((f_-)_0-\alpha)=(F_-)_1((f_-)_0)\,,\end{aligned} \tag{16}$$

$$\lim_{z\to-\infty}(f_-)_1(z)=0\,, \tag{17}$$

$$\begin{aligned}(f_+)_1''+c(f_+)_1'+(f_+)_1 &=\frac{m((f_+)_0')^2}{(f_+)_0}+(\ln(1-(f_+)_0)+\ln((f_+)_0))((f_+)_0-\alpha)\\ &\quad+m\ln((f_+)_0)(f_+)_0''=(F_+)_1((f_+)_0)\,,\end{aligned} \tag{18}$$

$$\lim_{z\to-\infty}(f_+)_1(z)=0\,, \tag{19}$$

etc. In order for matching, we require

$$\begin{aligned}&(f_-)_k(0)=(f_+)_k(0)\,,\quad (f_-)_k'(0)=(f_+)_k'(0)\,,\\ &(f_-)_k''(0)=(f_+)_k''(0)\,,\end{aligned} \tag{20}$$

for all $k\geq 0$.

Let us take a look at the order zero contributions. Solving the order zero equations subject to the relevant boundary conditions, we obtain

$$(f_-)_0(z)=1+c_2^-\exp\Big(-\frac{z}{2}\{c-\sqrt{c^2+4}\}\Big)\,, \tag{21}$$

$$\begin{aligned}(f_+)_0(z)&=\alpha+c_1^+\exp\Big(-\frac{z}{2}\{c+\sqrt{c^2-4}\}\Big)\\ &\quad+c_2^+\exp\Big(-\frac{z}{2}\{c-\sqrt{c^2-4}\}\Big)\,.\end{aligned} \tag{22}$$

Employing the matching conditions, we find that

$$c_2^-+1=c_1^++c_2^++\alpha\,, \tag{23}$$

$$(c-\sqrt{c^2+4})c_2^-=(c+\sqrt{c^2-4})c_1^++(c-\sqrt{c^2-4})c_2^+\,, \tag{24}$$

$$\begin{aligned}(c-\sqrt{c^2+4})^2c_2^-&=(c+\sqrt{c^2-4})^2c_1^+\\ &\quad+(c-\sqrt{c^2-4})^2c_2^+\,,\end{aligned} \tag{25}$$

so

$$c_2^-=-\frac{(1-\alpha)}{2}\,, \tag{26}$$

$$\begin{aligned}c_1^+=\Big(\frac{1-\alpha}{4(c^2-4)}\Big)\Big(&c^2-4-2c\sqrt{c^2-4}\\ &+\sqrt{c^2+4}\sqrt{c^2-4}\Big)\,,\end{aligned} \tag{27}$$

$$\begin{aligned}c_2^+=\Big(\frac{1-\alpha}{4(c^2-4)}\Big)\Big(&c^2-4+2c\sqrt{c^2-4}\\ &-\sqrt{c^2+4}\sqrt{c^2-4}\Big)\,.\end{aligned} \tag{28}$$

The order zero solutions then become

$$(f_-)_0(z)=1-\frac{(1-\alpha)}{2}\exp\Big(-\frac{z}{2}\{c-\sqrt{c^2+4}\}\Big)\,, \tag{29}$$

$$\begin{aligned}(f_+)_0(z)&=\alpha+\frac{(1-\alpha)}{2}\,\mathrm{e}^{-cz/2}\Big\{\cosh\Big(\frac{\sqrt{c^2-4}}{2}z\Big)\\ &\quad+\Big(\frac{2c-\sqrt{c^2+4}}{\sqrt{c^2-4}}\Big)\sinh\Big(\frac{\sqrt{c^2-4}}{2}z\Big)\Big\}\,.\end{aligned} \tag{30}$$

Computation of the higher order terms proceeds in a similar manner, the primary difference being that all higher order corrections vanish at the appropriate boundary $z=\pm\infty$. That is to say, only the order zero terms will have boundary contributions. To illustrate this, we compute the order one terms. From the governing equations (16)–(19), we find that

$$\begin{aligned}(f_-)_1(z)&=d_2^-\,\mathrm{e}^{\rho_+z}+\frac{1}{\sqrt{c^2+4}}\int_{-\infty}^{z}(F_-)_1((f_-)_0(t))\\ &\quad\times\{\mathrm{e}^{-\rho_+t}\,\mathrm{e}^{-(\rho_-+c)z}-\mathrm{e}^{-\rho_-t}\,\mathrm{e}^{-(\rho_++c)z}\}\mathrm{d}t\\ &=d_2^-\,\mathrm{e}^{\rho_+z}+(G_-)_1(z;\alpha,c,m)\,,\end{aligned} \tag{31}$$

$$\begin{aligned}(f_+)_1(z)&=d_1^+\,\mathrm{e}^{\sigma_+z}+d_2^+\,\mathrm{e}^{\sigma_-z}-\frac{1}{\sqrt{c^2-4}}\int_{z}^{\infty}(F_+)_1((f_+)_0(t))\\ &\quad\times\{\mathrm{e}^{-\sigma_+t}\,\mathrm{e}^{-(\sigma_-+c)z}-\mathrm{e}^{-\sigma_-t}\,\mathrm{e}^{-(\sigma_++c)z}\}\mathrm{d}t\\ &=d_1^+\,\mathrm{e}^{\sigma_+z}+d_2^+\,\mathrm{e}^{\sigma_-z}+(G_+)_1(z;\alpha,c,m)\,,\end{aligned} \tag{32}$$

where

$$\rho_\pm=-\frac{1}{2}c\pm\frac{1}{2}\sqrt{c^2+4}\,,$$

$$\sigma_\pm=-\frac{1}{2}c\pm\frac{1}{2}\sqrt{c^2-4}\,,$$

and d_2^-, d_1^+ and d_2^+ are constants to be determined by the matching requirements. Employing the matching conditions, we find that

$$d_2^-+(G_-)_1(0;\alpha,c,m)=d_1^++d_2^++(G_+)_1(0;\alpha,c,m)\,, \tag{33}$$

$$\rho_+d_2^- + \frac{\mathrm{d}(G_-)_1}{\mathrm{d}z}(0;\alpha,c,m) = \sigma_+d_1^+ + \sigma_-d_2^+ + \frac{\mathrm{d}(G_+)_1}{\mathrm{d}z}(0;\alpha,c,m)\,, \tag{34}$$

$$(\rho_+)^2d_2^- + \frac{\mathrm{d}^2(G_-)_1}{\mathrm{d}z^2}(0;\alpha,c,m) = (\sigma_+)^2d_1^+ + (\sigma_-)^2d_2^+ + \frac{\mathrm{d}^2(G_+)_1}{\mathrm{d}z^2}(0;\alpha,c,m)\,, \tag{35}$$

so in general we have that the d's are complicated functions of α, c and m; we write $d_2^- = d_2^-(\alpha,c,m)$, $d_1^+ = d_1^+(\alpha,c,m)$, $d_2^+ = d_2^+(\alpha,c,m)$. Hence, the order one solutions are

$$(f_-)_1(z) = d_2^-(\alpha,c,m)\,\mathrm{e}^{\rho_+z} + (G_-)_1(z;\alpha,c,m)\,, \tag{36}$$

$$(f_+)_1(z) = d_1^+(\alpha,c,m)\,\mathrm{e}^{\sigma_+z} + d_2^+(\alpha,c,m)\,\mathrm{e}^{\sigma_-z} + (G_+)_1(z;\alpha,c,m)\,, \tag{37}$$

where the $G_\pm$ terms are the corrections due to the order zero term which result from the nonlinearity, and which in general depend on α, c and m.

2.2 *The Case of $f(-\infty) = 1$ and $f(+\infty) = 0$*

In the case where $f(-\infty) = 1$ and $f(+\infty) = 0$, the mathematical analysis is more or less the same as in the $f(+\infty) = 0$ case. The order zero solutions are

$$(f_-)_0(z) = 1 - \frac{1}{2}\exp\Big(-\frac{z}{2}\{c - \sqrt{c^2+4}\}\Big)\,, \tag{38}$$

$$(f_+)_0(z) = \frac{1}{2}\,\mathrm{e}^{-cz/2}\Big\{\cosh\Big(\frac{\sqrt{c^2-4}}{2}z\Big) + \Big(\frac{2c-\sqrt{c^2+4}}{\sqrt{c^2-4}}\Big)\sinh\Big(\frac{\sqrt{c^2-4}}{2}z\Big)\Big\}\,, \tag{39}$$

and the process of obtaining the higher order terms proceeds in a similar manner.

3 Convergence of Solutions

Note that the expansions considered here are in an artificial exponent, and the convergence of the expansions is predicated on convergence of this exponent at unity. As the number of terms taken goes to infinity, it is difficult to ascertain convergence of the solution. However, as we have seen here and elsewhere,[13−19] the low-order perturbation solutions appear to be quite accurate. Without a formal convergence result, however, this will naturally be doubt about the usefulness of such expansions. As of present, the only reasonable way to ascertain the ability of the perturbation method to approximate any true solution lies in comparison with another solution. Exact solutions are not forthcoming, so it seems best to compare the results with numerical simulations.

Interestingly, the global results here agree well with the local results obtained previously in Van Gorder and Vajravelu,[10] where the traveling wave solutions for the density dependent diffusion Nagumo equation were recast as an initial value problem, in order to make the problem more reasonable for analytical and numerical computation near $z = 0$. We applied the Runge–Kutta–Fehlberg 4–5 method[21] directly to the original equation (2), with the initial conditions obtained for the perturbation solutions (that is, we consider the initial value problem (2), $f(0) = A$, $f'(0) = B$, where $A = f_{\mathrm{pert}}(0)$ and $B = f'_{\mathrm{pert}}(0)$, and solve the initial value problem via the Runge–Kutta–Fehlberg 4–5 method). The number of terms one needs to include for the perturbation expansion to match the numerical solution will in general increase as a function of m. We find that such solutions are in good agreement with the perturbation solutions for $m = 0$, when the order of expansion is greater than or equal to three, while ten terms may be required to match the numerical solutions when $m = 2$. Such results are plotted in Fig. 4. In particular, note that there are oscillations present in the solutions for finite values of $z > 0$; see Fig. 1. In effect, the system over-corrects as it tries to achieve the equilibrium $f \to \alpha$ as $z \to +\infty$. Also, as in the results of Van Gorder and Vajravelu,[10] the oscillatory behavior is tied to the wavespeed $c > 0$, as is evidenced in Fig. 2. When we include the higher order terms, we are able to pick up the contribution due to the nonlinear density dependence. As observed previously via numerical results, the power m presents in the nonlinear density dependence serves to smooth out the solutions to the density dependent diffusion Nagumo equation. Indeed, the oscillations present in the solutions to the standard Nagumo equation (the $m = 0$ case) are smoothed in the $m = 1$ case, and for some parameter values these oscillations dissapear for $m \geq 2$. Thus, for $m \geq 2$, we observe monotone solutions for some parameter values considered; see Fig. 3.

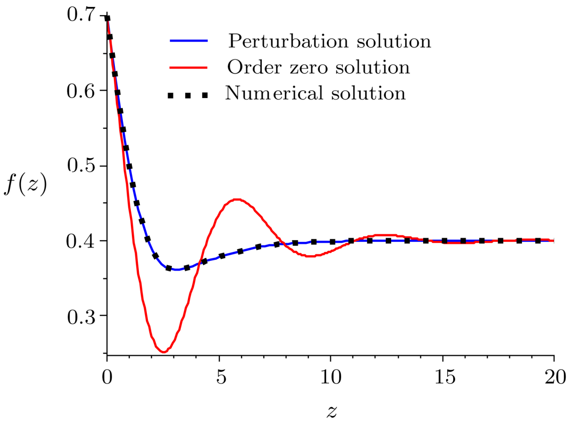

Fig. 1 Comparison of the perturbation solution (solid line) and the order zero solution (dashed line) to the density dependent diffusion Nagumo equation with numerical solution (dotted line) when $f(-\infty) = 1$ and $f(+\infty) = \alpha$ for $\alpha = 0.4$, wavespeed $c = 0.6$, and nonlinearity corresponding to $m = 1$.

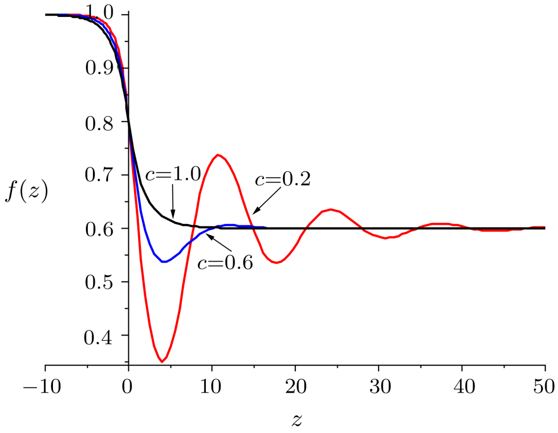

Fig. 2 Plot of the perturbation solutions to the density dependent diffusion Nagumo equation when $f(-\infty) = 1$ and $f(+\infty) = \alpha$ for various values of the wavespeed c. Considered are $c = 0.2,\ 0.6,$ and 1, while $m = 0$ and $\alpha = 0.6$.

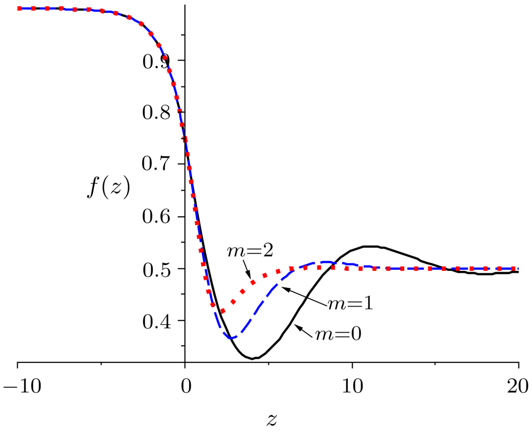

Fig. 3 Plot of the perturbation solutions to the density dependent diffusion Nagumo equation when $f(-\infty) = 1$ and $f(+\infty) = \alpha$ for $m = 0,\ 1,$ and 2. The wavespeed is fixed at $c = 0.4$, while $\alpha = 0.5$. Larger values of m have a smoothing effect on the solutions.

In Fig. 4, we demonstrate agreement of the perturbation solutions with another numerical method, a boundary value problem solver in *Maple* 12 (using dsolve; see Refs. [22]–[23]) which allows us to control the error. The benefit to comparing the results to numerical solutions obtained via a boundary value problem solver, as opposed to an initial value problem solver, is that the initial conditions need not be specified *a priori* (which is more natural for the problem we are trying to solve). We plot the order zero perturbation expansion as well as a higher order expansion (we take up to third-order terms, with the aid of the computer algebra program *Maple*). These are compared to the solutions obtained with the boundary problem solver (which we have used to obtain solutions to a relative error bound of 10^{-5}), shown as a dotted line. A drawback of the numerical method (and a strength of the perturbation method) is that we must specify finite bounds when applying the numerical boundary value problem solver. Here, we have needed to restrict ourselves to the domain,[10,20] whereas such restrictions are not needed for the perturbation method. At each finite value, we fix $f(-10) = 1$ and $f(20) = \alpha$. The size of the domain was picked so that error could be properly constrained. If the regain was made smaller, the solutions would be unreasonable, and the error would be artificially large. As evident from Fig. 4, the third-order expansions agree with the numerical solutions up to plotting accuracy.

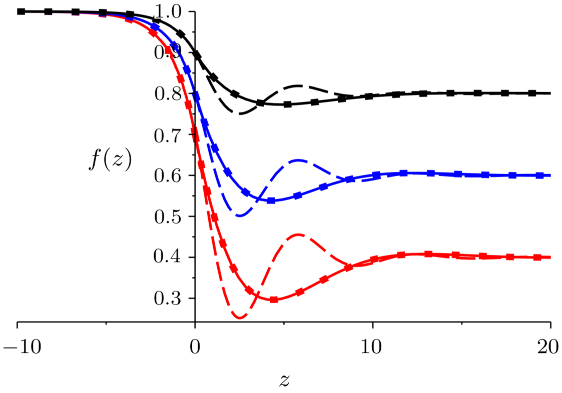

Fig. 4 Plot of the perturbation solutions (solid lines; order 3 perturbation solutions shown), the order zero solutions (dashed lines), and numerical solutions obtained via a boundary value problem solver (dotted lines; relative error constrained to within 10^{-5}) to the density dependent diffusion Nagumo equation when $f(-\infty) = 1$ and $f(+\infty) = \alpha$ for $\alpha = 0.4,\ 0.6,$ and 0.8. The wavespeed is fixed at $c = 0.6$, while $m = 0$. The order zero solutions capture the asyptotic behavior nicely, but need to be corrected in the small - z regime.

We hypothesize that the δ-expansion method works nicely for the present problem due to the fact that the solutions to the problem naturally fall within the bound $|f| \leq 1$. For problems with unbounded, or bounded yet fairly large solutions, the agreement with the perturbation method and the numerical results might not be as good.

4 Results and Discussion

Even after only the zeroth order iteration is considered, the δ-expansion method provides qualitatively good approximate solutions over the entire domain for the traveling wave solutions to the density dependent diffusion Nagumo equation. The order zero term primarily accounts for the cubic nonlinearity $f(1-f)(f-\alpha)$ (where $\alpha \in (0,1)$), so it is best to see these as approximations to the regular diffusion Nagumo equation. The first order approximations then hold information on the density dependent term $(f^m f')'$, and upon solving these one obtains solutions relevant for $m > 0$. Note that, as one increases m, one will in general require more terms in the approximation furnished by the δ-expansion method to attain a desired accuracy in the solutions. This is, of course, to be expected: in general, it is "harder" to obtain solutions to differential equations with greater nonlinearity.

An interesting thing to take note of here is that through a matching of solution branches we were effectively able to obtain perturbation solutions over the entire domain $z \in \mathbb{R}$, hence our solutions are global approximations, rather than local approximations. Interestingly, the global results here agree well with the local results obtained previously in Van Gorder and Vajravelu,[10] where the traveling wave solutions for the density dependent diffusion Nagumo equation were recast as an initial value problem, in order to make the problem more reasonable for analytical and numerical computation near $z = 0$. In particular, note that there are oscillations present in the solutions for finite values of $z > 0$; see Fig. 1. In effect, the system over-corrects as it tries to achieve the equilibrium $f \to \alpha$ as $z \to +\infty$. Also, as in the results of Van Gorder and Vajravelu,[10] the oscillatory behavior is tied to the wavespeed $c > 0$, as is evidenced in Fig. 2. When we include the higher order terms, we are able to pick up the contribution due to the nonlinear density dependence.

As observed previously via numerical results, the power m present in the nonlinear density dependence serves to smooth out the solutions to the density dependent diffusion Nagumo equation. Indeed, the oscillations present in the solutions to the standard Nagumo equation (the $m = 0$ case) are smoothed in the $m = 1$ case, and for some parameter values these oscillations dissapear for $m \geq 2$. Thus, for $m \geq 2$, we observe monotone solutions for some parameter values considered; see Fig. 3. In order to see why this behavior occurs, we must look past the perturbation expansions and numerical solutions. If we recast the travelling wave ordinary differential equation (2) as a two-dimensional dynamical system, viz.,

$$y_1' = y_2 \,, \quad y_2' = -\Big(m\frac{y_2^2}{y_1} + \frac{y_2}{y_1^m} + \frac{(1-y_1)(y_1-\alpha)}{y_1^{m-1}}\Big) \,, \tag{40}$$

subject to the asymptotic conditions $y_1 \to A_+$, $y_2 \to 0$ as $z \to +\infty$ (which define an equilibrium $(y_1^*, y_2^*) = (A_+, 0)$ that is always approached). Here we have considered the case $m \geq 1$. In order to deduce stability and asymptotic behavior, we linearize about the equilibrium $(y_1^*, y_2^*) = (\alpha, 0)$ (here $\alpha \in (0,1]$) which results in the Jacobian

$$J = \begin{bmatrix} 0 & 1 \\ -(1-\alpha)/\alpha^{m-1} & -c/\alpha^m \end{bmatrix} \,. \tag{41}$$

The eigenvalues for J, which determine the linear stability of Eq. (40), are then

$$\lambda_\pm = -\frac{c}{2} \pm \frac{1}{2\alpha^m}\sqrt{c^2 - 4\alpha^{m+1}(1-\alpha)} \ . \tag{42}$$

As $4\alpha^{m+1}(1-\alpha) > 0$ for $\alpha \in (0,1)$, the system is always stable when $\alpha \in (0,1)$ as $\lambda_\pm < 0$ in such a case. Furthermore, when $c^2 < 4\alpha^{m+1}(1-\alpha)$, $\lambda_\pm$ become complex conjugates, and this leads to the oscillations observed in the perturbation and numerical solutions discussed above. For fixed $\alpha \in (0,1)$, $\alpha > \alpha^2 > \cdots > \alpha^m$, and hence the term $4\alpha^{m+1}(1-\alpha)$ is decreasing with an increase in m. As this term decreases to such an extent that $c^2 \geq 4\alpha^{m+1}(1-\alpha)$, the oscillations are no longer apparent. Note that, for any fixed $m \geq 1$, the term $4\alpha^{m+1}(1-\alpha)$ is maximized when α takes the critical value $\alpha_m = (m+1)/(m+2)$. This value thus maximizes the oscillatory behavior of the solutions.

Note that the boundary value problem we have solved implies that there should be fixed values of the invariants $f(0)$ and $f'(0)$, and that these should depend on the parameters α and c. In all Figures provided (and in all other numerical simulations run), the order zero approximations agree very well with the higher order approximations near and the left of the origin ($z \leq 0$). Hence, the initial conditions $f(0)$ and $f'(0)$ may be approximated reasonable well by

$$f(0) = \frac{1+\alpha}{2} \,, \tag{43}$$

$$f'(0) = -\frac{(1-\alpha)}{4}(\sqrt{c^2+4} - c) \,, \tag{44}$$

which we obtain from the order-zero approximations. The contribution of m (the parameter due to the power-law nonlinearity) is held in higher order terms, and hence is not influential in this crude approximation. With a sufficient understanding of the initial values $f(0)$ and $f'(0)$, one may study the local solutions near $z = 0$ by recasting the nonlinear boundary value problem to a nonlinear initial value problem, which naturally lends itself more accessible to numerical computation, as we considered in Refs. [10]–[12].

Furthermore, if we wish to maintain the boundary conditions (particularly "at infinity"), applying numerical methods is not always useful. Indeed, to invoke a condition at $z \to \pm\infty$ numerically, one must consider a finite numerical infinity, z_∞, so that the conditions can be accounted for. In our case, one would solve the problem over the domain $z \in [-z_\infty, z_\infty]$ with the boundary conditions $f(-z_\infty) = A_-$ and $f(z_\infty) = A_+$. Unfortunately, we find that for even large finite values of z_∞ the behavior near $z = 0$ is quite sensitive to the value of z_∞ taken. Of course, as one increases z_∞, the numerical method becomes much more computationally expensive. Thus, the linearization method given here can be a useful alternative to solving the original density dependent diffusion Nagumo equation directly via a numerical boundary value solver. First linearizing the density dependent diffusion Nagumo equation into a system of linear equations via the δ-expansion method, one may then apply numerical methods in order to solve the resulting linear equations.

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