

Finite-Time Generalized Outer Synchronization Between Two Different Complex Networks*

SUN Yong-Zheng (孙永征),^{1,†} LI Wang (李望),¹ and RUAN Jiong (阮炯)²

¹School of Science, China University of Mining and Technology, Xuzhou 221008, China

²School of Mathematical Sciences, Research Center for Nonlinear Sciences, Fudan University, Shanghai 200433, China

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Abstract This paper investigates the finite-time generalized outer synchronization between two complex dynamical networks with different dynamical behaviors. The two networks can be undirected or directed, and they may also contain isolated nodes and clusters. By using suitable controllers, sufficient conditions for finite-time generalized outer synchronization are derived based on the finite-time stability theory. Finally, numerical examples are examined to illustrate the effectiveness of the analytical results. The effect of control parameters on the synchronization time is also numerically demonstrated.

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1 Introduction

Recent years have witnessed an increasing interest in the complex networks with different topologies. In general, a complex network is defined as a set of interconnected nodes, in which a node is fundamental unit with specific dynamical contents. Examples of complex networks include the Internet, the World Wide Web, food webs, biological neural networks and various communication networks, etc.^[1] As a significant collective behavior, synchronization of complex networks has attracted more and more attention. This is partly due to its broad applications in many fields ranging from physics to engineering.^[2–3]

In the past decade, the majority of works in network synchronization has focused on the synchronization behavior insider a network, which is regarded as “inner synchronization”.^[4–11] On the other hand, there exists another type of network synchronization, i.e., “outer synchronization” between two or more coupled networks, which means that the corresponding nodes of coupled networks will achieve synchronization regardless of synchronization of the inner networks. Examples of outer synchronization include the spread of diseases, such as AIDS and bird flu, between two communities,^[12] balance between predator-prey networks in ecological communities^[13] and synchronization between two groups of clock neurons in the adult drosophila brain.^[14] These examples show that the study of the outer synchronization between two coupled complex networks is evidently important.

Recently, many researchers have studied the outer synchronization between two coupled complex networks. Li *et al.* pioneered the study of the complete outer synchronization problem between two coupled continuous-time complex networks by using an open-plus-closed-loop (OPCL) controller.^[15] Later on, the outer synchronization between two coupled complex networks was extended to the discrete-time case.^[16] In Refs. [17–20], outer synchronization between two complex networks with identical or nonidentical topological structures was investigated under the application of adaptive controllers. In Ref. [19], Wang *et al.* investigated outer synchronization between two delayed-coupled complex networks with circumstance noise. In Ref. [21], Sun *et al.* investigated the outer synchronization between two coupled complex networks with impulsive controller. In Refs. [22–23], outer synchronization between two complex networks in noisy environments was investigated. In these early papers on outer synchronization, it is usually assumed that the corresponding nodes in two networks manifest completely the same dynamics. However, in reality, nodes in different networks usually have different dynamics, while the two networks may still behavior in a synchronous way. This kind of synchronization is called generalized synchronization, which represents another degree of coherence.^[24] So, the study of outer synchronization synchronization between two networks with different dynamical behaviors is very important to the perspective of control theory and practical application.

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†Corresponding author, E-mail: yzsung@gmail.com; 071018027@fudan.edu.cn

Recently, some theoretical results for the outer synchronization between two complex networks with different dynamics have been obtained.^[24–25] In Ref. [24] Wu *et al.* investigated the generalized outer synchronization between two completely different complex networks. More recently, Wu *et al.* investigated the generalized outer synchronization in master-slave networks with OPCL configurations.^[25] However, all these studies focused on the stability of the synchronized state, i.e., to demonstrate under which conditions on the interaction strengths and topologies coupled networks can become synchronized. The outer synchronization occurs asymptotically and convergence time is unbounded. In some practical situations, it is often required that the synchronization be reached in a finite time. For instance in secure communication, we hope two systems or networks achieve synchronization in a finite time. To achieve finite-time synchronization, the effective method is using finite-time control techniques.^[26] The idea of finite-time control techniques has been introduced to the finite-time synchronization for chaotic systems^[27] and complex networks.^[28] In Ref. [28], Yang and Cao investigated the finite-time inner synchronization of complex networks. In Ref. [29], Shen and Cao investigated finite-time synchronization of an array of coupled neural networks. However, to the best of our knowledge, there have not been any general results for the finite-time generalized outer synchronization of coupled complex networks with different node dynamics.

Motivated by the above discussion, in this paper, we consider the finite-time generalized outer synchronization between two complex networks with different dynamical behaviors. Based on the finite-time stability theory of differential equations, sufficient conditions guaranteeing the successful finite-time generalized outer synchronization are developed. Numerical simulations will be discussed to illustrate the validity of the theoretical results. The effect of the control parameter on the synchronization time are numerically demonstrated. In our study, each network can be undirected or directed, connected or disconnected, and two networks have different dynamics and topologies. The work in this paper is closely related to Ref. [28]. It is worth pointing out that different from Ref. [28], which studied finite-time inner synchronization of complex networks, we consider finite-time generalized outer synchronization between two completely different complex networks.

The rest of this paper is organized as follows. In Sec. 2, network models and preliminaries are given. Sufficient conditions for the finite-time generalized outer synchronization are derived in Sec. 3. Numerical examples are shown in Sec. 4. Finally, some conclusions are drawn in Sec. 5.

Notations: Throughout this paper unless specified we let $\|\cdot\|$ be Euclidean norm, I be an identity matrix of suit-

able dimensions. If A is a vector or matrix, its transpose is denoted by A^T . If A is a symmetric matrix, $\lambda_{\max}(A)$ denotes its largest eigenvalue.

2 Network Modeling and Preliminaries

Consider a general complex network consisting of N dynamical nodes with linear couplings, which is described by

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{j=1}^N a_{ij} P x_j, \quad i = 1, 2, \dots, N, \quad (1)$$

where $x_i(t) = (x_{i1}, \dots, x_{in})^T \in R^n$ is the state vector of the i -th node, $f: R^n \rightarrow R^n$ is a continuously differentiable nonlinear vector function governing the evolution of the i -th isolated node $x_i(t)$. $P \in R^{n \times n}$ is the inner connection matrix between two connected nodes; $A = (a_{ij})_{N \times N}$ represents the coupling configurations of the network, whose entries a_{ij} are defined as follows: if there is a link from node j to node i ($i \neq j$) then set $a_{ij} > 0$, otherwise $a_{ij} = 0$ ($i \neq j$). The diagonal elements of matrix A are defined as

$$a_{ii} = - \sum_{j=1, j \neq i}^N a_{ij}, \quad i = 1, 2, \dots, N.$$

To realize the finite-time generalized outer synchronization between two coupled complex networks, we refer to model (1) as the drive network, and the response network is given by the following equation:

$$\dot{y}_i(t) = g(y_i(t)) + \sum_{j=1}^N b_{ij} Q y_j + u_i(t), \quad i = 1, 2, \dots, N, \quad (2)$$

where $y_i(t) = (y_{i1}, \dots, y_{im})^T \in R^m$ is the state vector of node i , $g: R^m \rightarrow R^m$ is a continuously differentiable nonlinear vector function governing the evolution of the i -th isolated node $y_i(t)$. $Q \in R^{m \times m}$ is the inner connection matrices, $B = (b_{ij})_{N \times N}$ is the coupling configuration matrix, which has the same meaning as that of matrix A . u_i ($i = 1, 2, \dots, N$) are the controllers to be designed.

Remark 1 In this paper, the configuration matrices A and B of networks (1) and (2) are not assumed to be symmetric or irreducible, which means that networks (1) and (2) can be undirected or directed networks, and they may also contain isolated nodes and clusters.

The definition of finite-time generalized synchronization is given as follows.

Definition 1 Let $\phi_i: R^n \rightarrow R^m$ be continuously differentiable vector maps. We say networks (1) and (2) are finite-time generalized outer synchronization, if for any initial states $x_i(0)$, $y_i(0)$, there exists a finite time function T_0 such that

$$\lim_{t \rightarrow T_0} \|y_i(t, y_i(0)) - \phi_i(x_i(t, x_i(0)))\| = 0, \\ i = 1, 2, \dots, N,$$

where T_0 is called the settling-time function.

Throughout this paper we make the following assumption:

Assumption 1 For function $g(x)$ there exists a positive constant l such that

$$\begin{aligned} & [x(t) - y(t)]^T [g(x(t)) - g(y(t))] \\ & \leq [x(t) - y(t)]^T l [x(t) - y(t)], \quad \forall x, y \in R^n. \end{aligned} \quad (3)$$

Remark 2 Condition (3) is usually called global Lipschitz condition, and l is called Lipschitz constant. It is easy to check that some well-known chaotic systems, such as Chuan's circuit^[30] and Rössler-like system,^[31] satisfy Assumption 1.

In order to get our main results in the next section, we state here some needed lemmas.

Lemma 1^[26] Consider a continuous system in the form of

$$\dot{x}(t) = F(x(t)), \quad (4)$$

where $F : \mathcal{N} \rightarrow R^n$ is continuous on open neighborhood $\mathcal{N} \subseteq R^n$ of the origin and $F(0) = 0$. Suppose there exists a continuously differentiable function $V : \mathcal{N} \rightarrow R$ satisfying that $V(x)$ is positive definite and

$$\dot{V}(t)|_{(4)} \leq -KV^\theta(t),$$

where $K > 0$, $0 < \theta < 1$ are two real numbers. Then the origin of Eq. (4) is finite-time stable, and there exists a settling-time function $T(x_0)$ satisfying $T(x_0) \leq V(x_0)^{1-\theta}/K(1-\theta)$, where x_0 is the initial state. If in addition, $N = R^n$, V is proper, and \dot{V} takes negative values on $R^n \setminus \{0\}$, then the origin of Eq. (4) is globally finite-time stable.

Lemma 2^[32] Let $a_1, a_2, \dots, a_n > 0$ and $0 < r < p$. Then

$$\left(\sum_{i=1}^n a_i^p \right)^{1/p} \leq \left(\sum_{i=1}^n a_i^r \right)^{1/r}.$$

3 Sufficient Conditions for Finite-Time Generalized Outer Synchronization

In this section, we will investigate the finite-time generalized outer synchronization between networks (1) and (2) by employing the finite-time control theory, and the main results are given in the following theorem and corollaries.

Theorem 1 Suppose that Assumption 1 holds. There exists a sufficiently large positive constant k such that $k > l + \lambda_{\max}(\mathcal{B}^s)$, where $\mathcal{B} = B \otimes Q$, $\mathcal{B}^s = (\mathcal{B} + \mathcal{B}^T)/2$. Then drive network (1) and response network (2) can achieve generalized outer synchronization in a finite time under the following controllers:

$$\begin{aligned} u_i &= \mathcal{D}\phi_i(x_i(t))\dot{x}_i(t) - g(\phi_i(x_i)) - \sum_{j=1}^N b_{ij}Q\phi_j(x_j(t)) \\ & \quad - ke_i(t) - \eta \text{sign}(e_i(t))|e_i(t)|^\theta, \end{aligned} \quad (5)$$

where $\mathcal{D}(\phi_i(x_i))$ is the Jacobian matrix of the map $\phi_i(x_i)$, $e_i = y_i - \phi_i(x_i)$ is the generalized synchronization error, $|e_i(t)|^\theta = (|e_{i1}(t)|^\theta, \dots, |e_{in}(t)|^\theta)^T$, $\text{sign}(e_i(t)) =$

$\text{diag}(\text{sign}(e_{i1}(t)), \dots, \text{sign}(e_{in}(t)))$, $0 < \theta < 1$, k , and η are positive constants to be determined.

Proof From Eqs. (1) and (2), we can get the following error system:

$$\begin{aligned} \dot{e}_i(t) &= g(y_i) - g(\phi_i(x_i)) + \sum_{j=1}^N b_{ij}Qe_j(t) \\ & \quad - ke_i(t) - \eta \text{sign}(e_i)|e_i(t)|^\theta. \end{aligned} \quad (6)$$

Therefore, under Assumption 1, it is from the theory of differential equation that the error system (6) possesses a unique global solution on $t \geq 0$, denoted by $e_i(t, e_i(0))$ for any initial data $e_i(0) = y_i(0) - \phi_i(x_i(0))$. And $e_i(t, 0) \equiv 0$ is a trivial solution of the error dynamics (6). From Definition 1, we know that if this trivial solution is globally finite-time stable, then generalized outer synchronization between networks (1) and (2) could be realized in a finite time.

Let $e(t) = (e_1^T(t), e_2^T(t), \dots, e_N^T(t))^T$, and consider the following Lyapunov function:

$$V = \frac{1}{2}e^T(t)e(t) = \frac{1}{2} \sum_{i=1}^N e_i^T(t)e_i(t).$$

The derivative of V along the trajectory of Eq. (6) is

$$\begin{aligned} \dot{V}(t)|_{(6)} &= \sum_{i=1}^N e_i^T(t)[g(y_i) - g(\phi_i(x_i)) + \sum_{j=1}^N b_{ij}Qe_j(t) \\ & \quad - ke_i(t) - \eta \text{sign}(e_i)|e_i(t)|^\theta]. \end{aligned} \quad (7)$$

From Assumption 1, we have

$$e_i^T(t)[g(y_i) - g(\phi_i(x_i))] \leq le_i^T(t)e_i(t). \quad (8)$$

Substituting Eq. (8) into Eq. (7) yields

$$\begin{aligned} \dot{V}(t)|_{(6)} &\leq (l-k) \sum_{i=1}^N e_i^T(t)e_i(t) + \sum_{i=1}^N e_i^T(t) \sum_{j=1}^N b_{ij}Qe_j(t) \\ & \quad - \eta \sum_{i=1}^N e_i^T(t) \text{sign}(e_i)|e_i(t)|^\theta. \end{aligned} \quad (9)$$

Note that

$$\begin{aligned} \sum_{i=1}^N e_i^T(t) \sum_{j=1}^N b_{ij}Qe_j(t) &= e^T(t)B \otimes Qe(t) \\ &\leq \lambda_{\max}(\mathcal{B}^s)e^T(t)e(t). \end{aligned} \quad (10)$$

Moreover, we notice that

$$\begin{aligned} \sum_{i=1}^N e_i^T(t) \text{sign}(e_i)|e_i(t)|^\theta &= \sum_{i=1}^N (|e_i(t)|^\theta)^T \text{sign}(e_i)e_i(t) \\ &= \sum_{i=1}^N (|e_i(t)|^\theta)^T |e_i(t)| = \sum_{i=1}^N \sum_{j=1}^N |e_{ij}|^{\theta+1}. \end{aligned} \quad (11)$$

By Lemma 2, we have

$$\left(\sum_{i=1}^N \sum_{j=1}^N |e_{ij}|^{\theta+1} \right)^{1/(\theta+1)} \geq \left(\sum_{i=1}^N \sum_{j=1}^N |e_{ij}|^2 \right)^{1/2}.$$

Hence,

$$\begin{aligned} \sum_{i=1}^N \sum_{j=1}^N |e_{ij}|^{\theta+1} &\geq \left(\sum_{i=1}^N \sum_{j=1}^N |e_{ij}|^2 \right)^{(\theta+1)/2} \\ &= \left(\sum_{i=1}^N e_i^T(t) e_i(t) \right)^{(\theta+1)/2}. \end{aligned} \quad (12)$$

Then by Eqs. (9)–(12), we can obtain that

$$\begin{aligned} \dot{V}(t)|_{(6)} &\leq - (k - l - \lambda_{\max}(\mathcal{B}^s)) e^T(t) e(t) \\ &\quad - \eta (2V(t))^{(\theta+1)/2}. \end{aligned} \quad (13)$$

If

$$k > l + \lambda_{\max}(\mathcal{B}^s), \quad (14)$$

then we have

$$\dot{V}(t)|_{(6)} \leq -\eta (2V(t))^{(\theta+1)/2}.$$

Therefore, from Lemma 1, the trivial solution of Eq. (6) is globally finite-time stability. This means that outer synchronization between networks (1) and (2) could be achieved in a finite time for any initial data, and the synchronization time is estimated by

$$T_0 \leq T_1 = \frac{(2V(0))^{(1-\theta)/2}}{\eta(1-\theta)}, \quad (15)$$

where

$$V(0) = \frac{1}{2} \sum_{i=1}^n e_i^2(0).$$

The proof is completed. \square

Remark 3 The inequality (13) indicates that the convergence rate of the proposed algorithm is closely related to the topology and control parameters k, η and θ . Especially, in the bidirectional interaction case, i.e., when $B^T = B$ and $Q = I$, we can show that larger $\lambda_{\max}(B)$, will lead to lower convergence rate and the synchronization rate increase when the parameters k and η increase.

Remark 4 The convergence time of the proposed algorithm is closely related to the protocol parameters η and θ . From Eq. (15), one can see that for fixed parameter θ and $V(0)$, the synchronization time decreases as η increases. In addition, by some straightforward arguments, it can be shown that smaller θ can lead to a shorter convergence time when initial states of two networks differ a little from each other, and there exists a θ^* which stands for a minimum convergence time when initial states of two networks differ a lot from each other.

From Definition 1, we know that the finite-time complete outer synchronization is a kind of of special case of the finite-time generalized outer synchronization if $\phi_i(x(t)) = x(t)$, $i = 1, 2, \dots, N$. Based on Theorem 1, we can easily derive the following corollaries:

Corollary 1 Let Assumption 1 hold. If networks (1) and (2) have identical dynamics, namely, $f = g$ and

$k > l + \lambda_{\max}(\mathcal{B}^s)$, then the two networks can achieve finite-time complete outer synchronization under the following control scheme:

$$\begin{aligned} u_i &= -ke_i(t) - \eta \text{sign}(e_i(t)) |e_i(t)|^\theta \\ &\quad + \sum_{j=1}^N (a_{ij}P - b_{ij}Q)x_j(t), \quad i = 1, 2, \dots, N, \end{aligned}$$

where \mathcal{B}^s is defined as that in Theorem 1.

Corollary 2 Let Assumption 1 hold. If networks (1) and (2) have the same topological structures and uniform inner-coupling matrices, i.e., $A = B$, $P = Q$ and also have identical node dynamics, namely, $f = g$, then the two networks can achieve finite-time complete outer synchronization under the following control scheme:

$$u_i = -ke_i(t) - \eta \text{sign}(e_i(t)) |e_i(t)|^\theta, \quad i = 1, 2, \dots, N,$$

where $k > l + \lambda_{\max}(\mathcal{B}^s)$, and \mathcal{B}^s is defined as that in Theorem 1.

4 Simulation Results

In this section, some numerical simulations are performed to verify the feasibility and effectiveness of the proposed method for finite-time generalized outer synchronization between two different networks. For this purpose, we use the Lorenz system to describe the node dynamics of the driving network and take the Rössler-like system as the node dynamics of the response network.

The Lorenz system can be described as^[33]

$$\dot{x} = f(x) = \begin{pmatrix} -a & a & 0 \\ c & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{pmatrix}, \quad (16)$$

where $x = (x_1, x_2, x_3)^T \in R^3$ is the state vector. As shown in Fig. 1, system (16) has a double-scrolling chaotic attractor when $a = 10$, $b = 8/3$, $c = 28$.

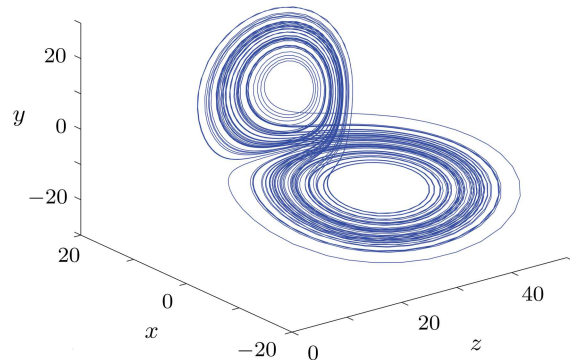


Fig. 1 Chaotic attractor generated by the system (16) when $a = 10$, $b = 8/3$, $c = 28$.

The Rössler-like system can be described as^[31]

$$\dot{y} = g(y) = \alpha \begin{pmatrix} -\Gamma & -\beta & -\lambda \\ 1 & \gamma & 0 \\ 0 & 0 & -\mu \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ \alpha\mu\psi(y_1) \end{pmatrix} \triangleq \Lambda y + \Psi(y), \quad (17)$$

where $y = (y_1, y_2, y_3)^T \in R^3$ is the state vector,

$$\psi(s) = \begin{cases} 0, & s < 2.56, \\ \xi(s - 2.56), & s \geq 2.56. \end{cases}$$

As shown in Fig. 2, the Rössler-like system has a chaotic attractor when $\alpha = 0.03$, $\beta = 1.5$, $\gamma = 0.2$, $\mu = 1.5$, $\lambda = 0.75$, $\xi = 21.43$, and $\Gamma = 0.075$. And it is easy to compute that $\lambda_{\max}(\Lambda + \Lambda^T) = 0.0208$ and

$$\begin{aligned} & (x - y)^T (g(x) - g(y)) \\ &= (x - y)^T \Lambda (x - y) + \alpha\mu(x_3 - y_3)(\psi(x_1) - \psi(y_1)) \\ &\leq \left[\lambda_{\max} \left(\frac{\Lambda + \Lambda^T}{2} \right) + \frac{\alpha\mu\xi}{2} \right] (x - y)^T (x - y). \end{aligned}$$

Therefore Assumption 1 is satisfied with $l = 0.4926$.

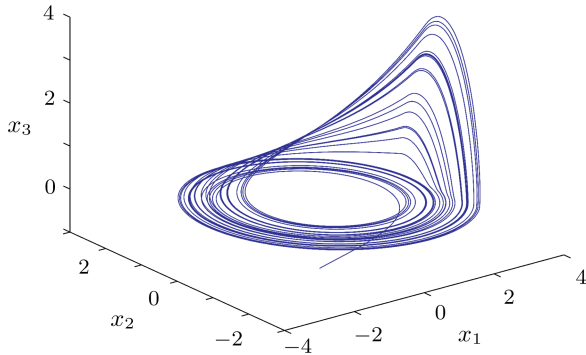


Fig. 2 Chaotic attractor generated by the system (17) when $\alpha = 0.03$, $\beta = 1.5$, $\gamma = 0.2$, $\mu = 1.5$, $\lambda = 0.75$, $\xi = 21.43$, and $\Gamma = 0.075$.

The map ϕ_i is defined as

$$\phi_i(x_i) = \left(x_{i1} - x_{i2}, x_{i2}^2 + 1, \frac{1}{2}x_{i3}^2 - x_{i1} \right)^T, \\ i = 1, 2, \dots, N.$$

Then

$$\mathcal{D}\phi_i(x_i) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2x_{i2} & 0 \\ -1 & 0 & x_{i3} \end{pmatrix}.$$

In the numerical simulations, for brevity, we always set $P = Q = I$. The configuration matrix for the drive

network is given as follows:

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & -5 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & -4 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \end{pmatrix}.$$

The configuration matrix for the response network is given as follows:

$$B = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & -3 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & -4 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -3 \end{pmatrix}.$$

It is easy to compute that $\lambda_{\max}(\mathcal{B}^s) = 0.1424$. Take $k = 1$, $\eta = 3$, $\theta = 0.6$, we simulate the evolution of the networks according to the controllers defined in Eq. (5). The initial state of the i -th node of the drive network is $x_i(0) = [2 - \sin i, 3 - e^{-2i}, 4 + \sin i]$, while the initial state of the i -th node of the response network is $y_i(0) = [2 + \cos i, 3 - \sin i, 4 + e^{-i}]$. According to Theorem 1, networks (1) and (2) can reach generalized outer synchronization in a finite time. By computing Eq. (15), we get $T_1 = 1.5129$. Let $E(t) = \sum_{i=1}^N \|y_i(t) - \phi_i(x_i(t))\|$ denote the total synchronization error between the two networks. The evolution of synchronization errors $e_{ij}(t)$ ($i = 1, 2, \dots, 10; j = 1, 2, 3$) and $E(t)$ are shown in Figs. 3, respectively. From Fig. 3, one can find that the generalized outer synchronization is realized at about $t = 0.66$, and the simulations match the theoretical results perfectly.

To study the effect of the control parameter θ on the settling time, we simulate the evolution of two networks with the controllers defined in Eq. (5) through taking different values of θ . Figures 4 give the evolutions of the total error function $E(t)$ with different values of θ . As shown in Fig. 4(a), for little initial states differences of two networks, networks with small θ converge faster than those with large θ . As shown in Fig. 4(b), larger θ can lead to a shorter convergence time when initial states of two networks different a lot from each other, which is consistent with the analysis of Remark 4.

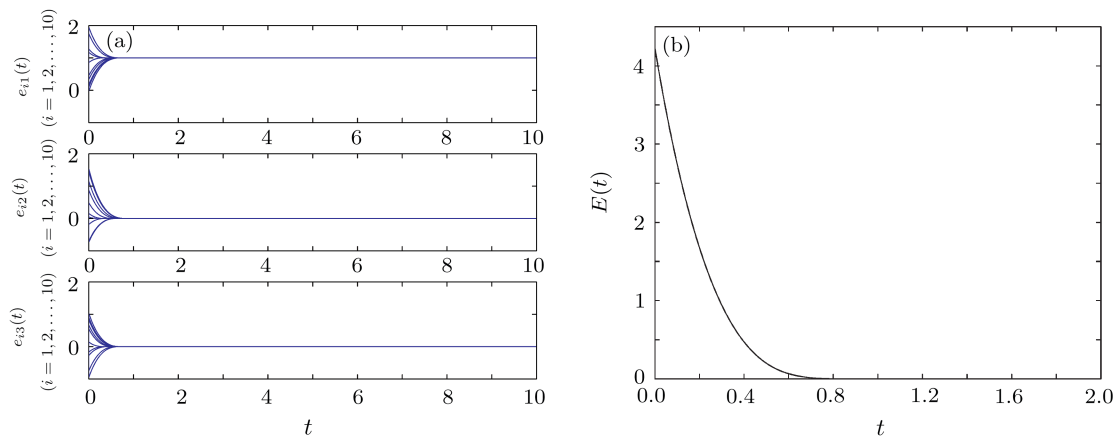


Fig. 3 Trajectories of the synchronization error (a) and the total synchronization error (b) between networks (1) and (2) with $k = 1, \eta = 3, \theta = 0.6$.

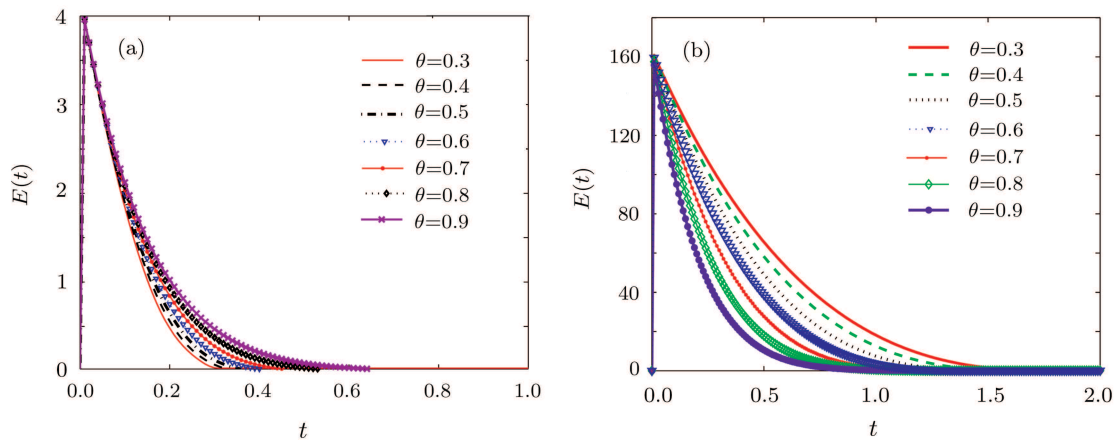


Fig. 4 The variations of total synchronization error $E(t)$ between networks (1) and (2) with $k = 1.5, \eta = 5$ and $\theta = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. The initial conditions of two networks in Fig. 4(a) are randomly taken from the interval $[-0.1, 0.1]$. In Fig. 4(b), the initial conditions of the drive network are randomly taken from the interval $[-0.2, 0.2]$, the the initial conditions of the response network are randomly taken from the interval $[-20, 20]$.

5 Conclusions

Synchronization of complex networks has been extensively studied in the past decade. But few studies concern the finite-time synchronization of complex networks. Particularly, to the best of our knowledge, the finite-time outer synchronization between two different complex dynamical networks has not been reported in the literature to date. In this paper, we have investigated the finite-time generalized outer synchronization between two coupled complex networks with different dynamical behaviors. The two networks can be undirected or directed,

and they may also contain isolated nodes and clusters. By using the finite-time stability theory and adding suitable controllers, sufficient conditions for finite-time generalized outer synchronization are obtained. Numerical simulations are given to illustrate the effectiveness of the analytical results. The effect of the control parameter on the synchronization time is also numerically demonstrated.

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