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Conformists and Contrarians in a Kuramoto Model with Uniformly Distributed Natural Frequencies*

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Abstract *A generalization of the Kuramoto model in which oscillators are coupled to the mean field with random signs is investigated in this work. We focus on a situation in which the natural frequencies of oscillators follow a uniform probability density. By numerically simulating the model, we find that the model supports a modulated travelling wave state except for already reported π state and travelling wave state in the one with natural frequencies following Lorenztian probability density or a delta function. The dependence of the observed dynamics on the parameters of the model is explored and we find that the onset of synchronization in the model displays a non-monotonic dependence on both positive and negative coupling strength.*

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Key words: Kuramoto model, modulated travelling wave state, conformist, contrarian

1 Introduction

The Kuramoto model[1] of coupled oscillators has been used to model diverse systems in physics, chemistry and biology,[2] particularly those involving synchronization transitions. Examples include groups of fireflies flashing in unison,[3–4] Josephson junction arrays,[5] charge density waves,[6] collective atomic recoil lasers,[7] electrochemical oscillators,[8] and human crowd behavior.[9]

The original Kuramoto model consists of N phase oscillators globally coupled together. Each oscillator has its own natural frequency ω drawn from a given probability density $g(\omega)$ and interacts with the mean field with a global coupling strength K. K is assumed to be positive to account for an attractive interaction. A natural generalization of the model is to allow K to be negative for a repulsive interaction, inspired by models of spin glasses.[10] Actually, both attractive and repulsive interaction can be found in neural networks with excitatory and inhibitory coupling.[11] In neural networks, excitatory neurons response to stimuli with $K > 0$ while and inhibitory neurons react to stimuli with $K < 0$. Along this line, some authors treated the coupling as local ones describing the interaction between two oscillators and they found the evidences of glassy behavior when both positive and negative coupling strength are allowed at the same time.[12] Tsimring *et al.* considered the case in which the interaction between oscillators and the mean field is a global and repulsive one and found that synchronization fails for an array of nonidentical phase oscillators provided that the number of oscillators is sufficiently large.[13] Recently, Hong and Strogatz studied the situation in which the coupling strength is treated as an oscillator's ability reacting to the mean field individually.[14] In their work, both positive and negative coupling strength are present in the population. Oscillators with positive K behave like conformists by tending to fall in line with prevailing rhythm in the population, whereas those with negative K are repelled by the rhythm and act like contrarians. The authors found some expected stationary states such as incoherence states and partially synchronized states in which conformists and contrarians locked in antiphase. They also found a surprising time-dependent state, a travelling wave state in which the mean field oscillates at a frequency different from the population's mean natural frequency and the phase difference between conformists and contrarians are locked at an angle away from π. By using the remarkable ansatz discovered by Ott and Antonsen (OA),[15] they found that the model is exactly solvable when the natural frequencies of oscillators are drawn from Lorentzian probability density and the numerical results are demonstrated analytically. Furthermore, when the phase oscillators in the population are identical, Hong and Strogatz found one more stationary state, the blurred π state,[16] with the aid of a theoretical method proposed by Watanabe and Strogatz (WS).[17]

Now it is interesting to ask whether the observations by Hong and Strogatz are robust and whether we can find other dynamical behaviors in the model with conformists

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and contrarians if other types of probability density for the natural frequencies of oscillators are adopted. Since the coupled phase oscillators with a Gaussian probability density (or some other unimodal probability densities) of natural frequency have been numerically demonstrated to behave similarly to those with a Lorentzian distribution, here we focus on the model with a uniform probability density of natural frequency, which can not be solved analytically by either the OA ansatz or the WS method. We find another long-term output: a modulated travelling wave state. We also find that the onset of synchronization in the population depends non-monotonically on the coupling strength between conformists (or contrarians) and the mean field, which is different from those in previous literatures.

2 Model

The model is described as

$$\dot{\phi}_i = \omega_i + \frac{K_i}{N}\sum_{j=1}^{N}\sin(\phi_j - \phi_i)\,, \quad i = 1, 2, \ldots, N\,. \tag{1}$$

$\phi_i(t)$ is the phase of the i-th oscillator at time t. ω_i is the natural frequency of the i-th oscillator and is chosen at random from $[-\gamma, \gamma]$ where γ is the width of natural frequency distribution. K_i is the coupling strength of the i-th oscillator to the mean field and is chosen from a double-δ probability density $\Gamma(K) = (1-p)\delta(K-K_1) + p\delta(K-K_2)$ where $K_1 < 0$ and $K_2 > 0$ represent the couplings for the contrarians and conformists, respectively, and p is the probability that a random oscillator is a conformist.

The key idea behind the analysis of the Kuramoto model is the introduction of a mean-field-like quantity, namely, a complex order parameter $R\,\mathrm{e}^{\Phi}$ which is defined as

$$Z = R\,\mathrm{e}^{\mathrm{i}\Phi} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\phi_j}\,, \tag{2}$$

where the amplitude $0 \le R \le 1$ measures the phase coherence in the population and Φ gives the average phase. In terms of R and Φ, Eq. (1) can be rewritten as

$$\dot{\phi}_i = \omega_i + K_i R\sin(\Phi - \phi_i)\,, \quad i = 1, 2, \ldots, N\,. \tag{3}$$

When $|\omega_i| < |K_i R|$ the oscillator i gets trapped by the mean field, otherwise it runs on its own natural frequency. Throughout this work, we set $N = 10\,000$.

3 Results and Analysis

The system of Eq. (1) is numerically simulated by a 4th-order Runge–Kutta algorithm with a time step $\delta t = 0.01$ and the quantities of interest are measured after a sufficient long transient is discarded.

There are several parameters in charge of the dynamics of the model. We first fix $\gamma = 0.1$, $K_1 = -0.5$ and $K_2 = 1$ and then investigate how the dynamics of the Eq. (1) depends on the fraction of conformists p. For this aim, we consider the variation of the order parameter against p. Besides the order parameter R defined in Eq. (2), we also consider the order parameter in contrarians $R_1 = [1/N(1-p)]|\sum \mathrm{e}^{\mathrm{i}\phi_j}|$ and the order parameter in conformists $R_2 = (1/Np)|\sum \mathrm{e}^{\mathrm{i}\phi_j}|$. The results are presented in Fig. 1. Clearly, all of these three order parameters increase with the increase of p and we always have $R_2 > R_1 > R$ since the synchronization of contrarians is assisted by synchronized conformists. Figure 1 shows that there exist several regimes for different dynamical states. The dynamical states observed in Refs. [14, 16] are found here either. The incoherent state is stable in the range of $p < 0.35$ where each oscillator oscillates at its own natural frequency. The π state is stable in the range of $p > 0.7$ in which conformists and contrarians form two partially (or globally) synchronized clusters and the peaks of their phase distributions are separated from each other by an angle of π. In the range of $0.5 < p < 0.7$, the travelling wave state is realized in which the phase distributions of conformists and contrarians spontaneously travel at a constant speed along the phase axis and the peaks of the phase distributions of conformists and contrarians always maintain a constant separation but not at the angle of π. However, there exists a new state in which the travelling wave is modulated (we call it a modulated travelling wave state).

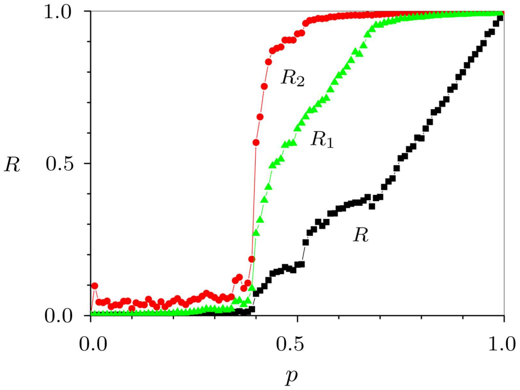

Fig. 1 (Color online) The order parameters are plotted against the fraction of conformists in the population p. Black solid squares denotes the system's order parameter R; Red solid circles line denotes the order parameter in conformists R_2; Blue solid triangles denote the order parameter in contrarians R_1. $\gamma = 0.1$, $K_1 = -0.5$, and $K_2 = 1.0$.

Now we explore the dynamics of the modulated travelling wave state. We present the time sequences of the order parameters after transient in Fig. 2(a) where $p = 0.45$. Different from the travelling wave state where the order parameters stay at constants, here three order parameters oscillate periodically and seemingly in the manner of a period-2 oscillation. The temporal behavior of the order parameters can be explained by the evolutions of the phase distributions for both conformists and contrarians. As shown by Figs. 2(b) and 2(c), the phase distributions of the two subpopulations are not stationary and the time evolutions of them consist of two components: one is a slow one which travels along the phase axis at a constant speed and the other is a fast one which periodically modulates the phase distributions of conformists and contrarians. Under the modulation of the fast component, the peak values of the two phase distributions vary periodically. Interestingly, Figs. 2(b) and 2(c) show that the phase distributions are always not symmetrical about the peak, which is different from the travelling wave state as shown in Refs. [14, 16]. In one half period of the fast component, the phase distributions have long tails at high phase whereas, in the other half period, the long tails of the phase distributions appear at low phase. In each modulation period of the fast component, the asymmetry in the phase distributions alternates twice which accounts for the seemingly period-2 behavior of the order parameters. The careful comparison between Figs. 2(b) and 2(c) reveals that the separation between the peaks of the phase distributions of conformists and contrarians is not a constant and always away from π which is a remnant of the travelling wave state.

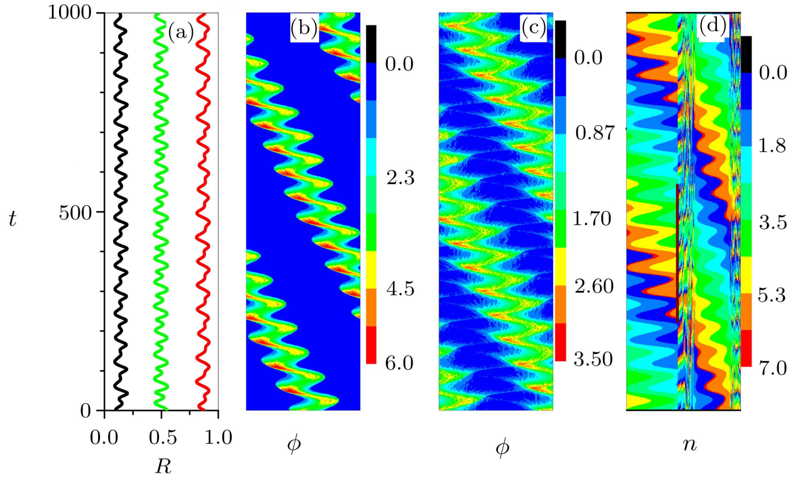

Fig. 2 (Color online) (a) The time sequences of the order parameters R (black line), R_1 (green line) and R_2 (red line). (b) The time evolution of the phase distribution in the subpopulation of conformists. (c) The time evolution of the phase distribution in the subpopulation of contrarians. (d) The time evolution of oscillators' phases where the oscillators are numbered according to their coupling strength and natural frequencies. For the convenience of plotting, the phase distributions are multiplied by a constant. $p = 0.45$, $\gamma = 0.1$, $K_1 = -0.5$, and $K_2 = 1$.

Furthermore, we investigate how conformists and contrarians organize themselves in a modulated travelling wave state. To do it, we number oscillators according to their coupling strength and their natural frequencies. If an oscillator i has its coupling strength K_1 and an oscillator j has the coupling strength K_2, then we have $n_i > n_j$ regardless of their natural frequencies. On the other hand, we have $n_i < n_j$ if the two oscillators have the same coupling strength but $\omega_i < \omega_j$. As shown by Fig. 2(d), adjacent oscillators stay closely in phase and the phase of each oscillator swings back and forth along its averaged trajectory in the period of the fast component. Another feature from the phase evolution of oscillators is that unsynchronized oscillators are not symmetrically distributed around the mean natural frequency. Actually, the extent of asymmetry depends on the travelling speed of the slow component. Consider that, for simplicity in a travelling wave state, the average phase of the order parameter follows $\Phi = \Omega t + \Phi'$ where $\Omega = (1/N) \times \sum_{j=1}^{N} \langle \dot{\phi_j} \rangle_T$ is the speed of the travelling wave. By introducing a new phase variable for each oscillator $\phi'_i = \phi_i - \Omega$, the time evolution of Eq. (3) can be rewritten as $\dot{\phi'_i} = \omega_i - \Omega + K_i R \sin(\Phi' - \phi'_i)$. Consequently, an oscillator will be synchronized if its frequency satisfies $|\omega_i - \Omega| < |K_i R|$ and more larger Ω, more asymmetrical the synchronized oscillators around the mean natural frequency.

It is important to mention that the modulated travelling wave state does not directly bifurcate from the incoherent state. Actually, it emerges from a travelling wave state. We present the dynamical state in Fig. 3 for $p = 0.4$ which lies between the incoherent state and the modulated travelling wave state. Though the order parameters in Fig. 3(a) display violent fluctuation due to the finite size N, the evolutions of the phase distributions in Figs. 3(b) and 3(c) show that the system is in a partially synchronized state and the synchronized oscillators do not oscillate at the mean natural frequency. The evolution of the oscillators' phases in Fig. 3(d) shows clearly a travelling wave state both in the middle of conformists and contrarians with a much slow speed.

Then we explore how the parameters of γ and p influence the travelling wave state and the modulated travelling wave state. To do it, we consider two quantities. One is the speed of a travelling wave Ω and the other is the standard deviation of instantaneous wave speed σ. When $\Omega \neq 0$ and $\sigma = 0$, the system of coupled phase oscillators displays a travelling wave behavior. When $\Omega \neq 0$ and $\sigma \neq 0$, the travelling wave state is modulated. Figure 4 plots Ω in (a) and σ in (b) as functions of p and γ. Notice that the travelling wave state occurs in a wider window as γ tends to 0. On the other hand, there are two regimes for the modulated travelling wave state. One locates at the lower p boundary of the travelling wave which occurs at sufficiently large γ. The other locates at the upper p boundary which occurs at sufficiently low γ. The modulated travelling wave at sufficiently low γ is a little different from the one observed in Fig. 2 in which the order parameters do not behave as period-2 solutions (the results are not presented here). In both regimes, the wave speed Ω is extremely low.

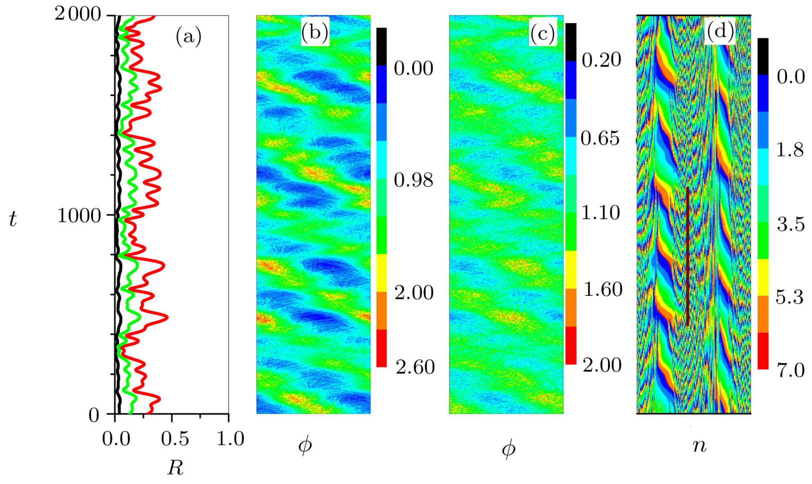

Fig. 3 (Color online) (a) The time sequences of the order parameters R (black line), R_1 (green line) and R_2 (red line). (b) The time evolution of the phase distribution in the subpopulation of conformists. (c) The time evolution of the phase distribution in the subpopulation of contrarians. (d) The time evolution of oscillators' phases where the oscillators are numbered according to their coupling strength and natural frequencies. $p = 0.4$, $\gamma = 0.1$, $K_1 = -0.5$ and $K_2 = 1$.

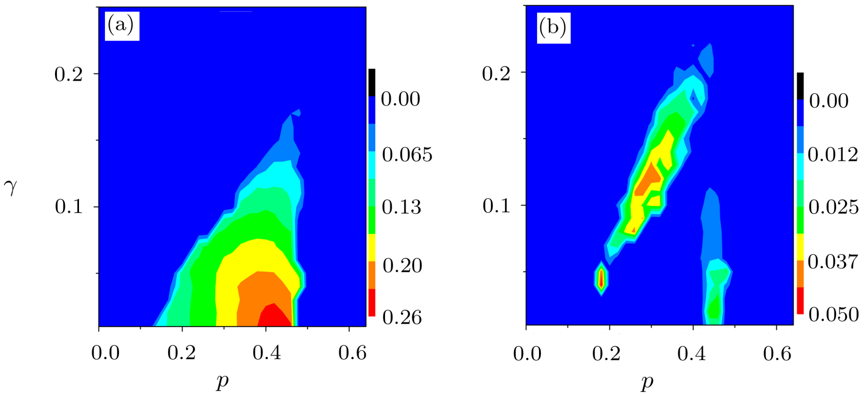

Fig. 4 (Color online) The wave speed Ω in (a) and the standard variation of instantaneous wave speed σ in (b) as a function of γ and p. $K_1 = -0.5$ and $K_2 = 1$.

The above results are obtained at $K_1 = -0.5$ and $K_2 = 1$. For other combinations of K_1 and K_2, the dependence of the travelling wave state and the modulated travelling wave state on γ and p is similar to those in Fig. 4. In the studies of the Kuramoto model and its different variants, the stability of the incoherent state which signatures the onset of synchronization is an important topic. Here, we illustrate how the stability of the incoherent state depends on K_1 and K_2. We set $\gamma = 0.05$ and focus on the critical p_c at which the incoherent state loses its stability. The result of p_c is presented in Fig. 5 as a function of K_1 and K_2.

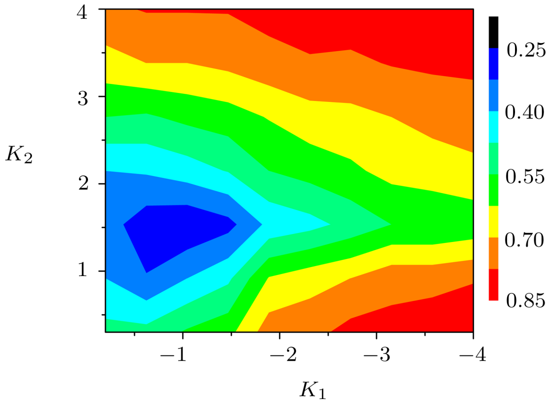

Fig. 5 (Color online) The critical fraction of conformists p_c for the onset of synchronization in the population is plotted as a function of K_1 and K_2. $\gamma = 0.05$.

Interestingly, we find that p_c does not monotonically depend on K_1 and K_2 and there exists a combination of K_1 and K_2 at which the critical number of conformists reaches a minimum for the onset of synchronization. Remember that $p_c = (2\gamma - K_1)/(K_2 - K_1)$ in Ref. [14] where the natural frequencies of oscillators are drawn from a Lorenztian distribution and $p_c = -K_1/(K_2 - K_1)$ in Ref. [16] where oscillators are identical. The finding in Fig. 5 is quite different from those in Refs. [14, 16] in which p_c varies with K_2 (or K_1) at a given k_1 (or at a given k_2) monotonically, which also reflects the important role of probability density of natural frequency on the dynamics of a model.

4 Conclusions

In this work, we considered a generalization of the Kuramoto model which consists of conformists which response to the mean field positively and contrarians which react to the mean field negatively. We focused on a special situation where the natural frequencies of oscillators uniformly distribute in the range of $[-\gamma, \gamma]$. We have investigated the dynamics of the model by numerical simulations. We found that there exists a new dynamical state, a modulated travelling wave state, except for already reported states such as the π state and the travelling wave state. The characteristics of the modulated travelling wave state has been investigated. We also explored the dependence of the observed dynamics on the parameters of the model. We found that the onset of synchronization in the population displays a non-monotonically dependence on both the positive and the negative coupling strength, which is different from those in previous literatures.

References

[1] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence*, Springer, New York (1984).

[2] A.T. Winfree, *The Geometry of Biological Time*, Springer, New York (1980); J.A. Acebron, *et al.*, Rev. Mod. Phys. **77** (2005) 137.

[3] J. Buck and E. Buck, Sci. Am. **234** (1976) 74.

[4] J. Buck, Quart. Rev. Biol. **63** (1988) 265.

[5] J.W. Swift, S.H. Strogatz, and K. Wiesenfeld, Physica D **55** (1992) 239; K. Wiesenfeld, P. Colet, and S.H. Strogatz, Phys. Rev. Lett. **76** (1996) 404; Phys. Rev. E **57** (1998) 1563; K. Park and M.Y. Choi, Phys. Rev. B **56** (1997) 387; G. Filatrella, N.F. Pedersen, and K. Wiesenfeld, Phys. Rev. E **61** (2000) 2513; B.C. Daniels, S.T.M. Dissanayake, and B.R. Trees, ibid. **67** (2003) 026216.

[6] G. Gruner and A. Zettl, Phys. Rep. **119** (1985) 117; G. Gruner, Rev. Mod. Phys. **60** (1988) 1129; S.H. Strogatz and R.E. Mirollo, J. Phys. A **21** (1988) L699; Physica D **31** (1989) 143; C.M. Marcus, S.H. Strogatz, and R.M. Westervelt, Phys. Rev. B **40** (1989) 5588.

[7] C. von Cube, S. Slama, *et al.*, Phys. Rev. Lett. **93** (2004) 083601; J. Javaloyes, M. Perrin, and Politi, Phys. Rev. E **78** (2008) 011108.

[8] N. Mazouz, K. Krischer, *et al.*, J. Phys. Chem. B **101** (1997) 2403; I.Z. Kiss, W. Wang, and J.L. Hudson, Chaos **12** (2002) 252; M. Wickramasinghe and I.Z. Kiss, Phys. Rev. E **83** (2011) 016210.

[9] Z. Neda, E. Ravasz, *et al.*, Nature (London) **403** (2000) 849; Z. Neda, E. Ravasz, *et al.*, Phys. Rev. E **61** (2000) 6987.

[10] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett. **35** (1975) 1792.

[11] C. Börgers and N. Kopell, Neural Comput. **15** (2003) 509.

[12] H. Daido, Prog. Theor. Phys. **77** (1987) 622; Phys. Rev. Lett. **68** (1992) 1073; L.L. Bonilla, *et al.*, J. Stat. Phys. **70** (1993) 921; D.H. Zanette, Europhys. Lett. **72** (2005) 190.

[13] L.S. Tsimring. N.F. Rulkov, M.L. Larsen, and M. Gabbay, Phys. Rev. Lett. **95** (2005) 014101.

[14] H. Hong and S.H. Strogatz, Phys. Rev. Lett. **106** (2011) 054102.

[15] E. Ott and T.M. Antonsen, Chaos **18** (2008) 037113.

[16] H. Hong and S.H. Strogatz, Phys. Rev. E **84** (2011) 046202.

[17] S. Watanabe and S.H. Strogatz, Physica D **74** (1994) 197.