

# Dirac Equation and Some Quasi-Exact Solvable Potentials in the Turbiner's Classification

S. Aghaei\* and A. Chenaghlou†

Department of Physics, Faculty of Sciences, Sahand University of Technology, P.O. Box 51335-1996, Tabriz, Iran

(Received February 6, 2013)

**Abstract** In this paper quasi-exact solvability (QES) of Dirac equation with some scalar potentials based on  $sl(2)$  Lie algebra is studied. According to the quasi-exact solvability theory, we construct the configuration of the classes II, IV, V, and X potentials in the Turbiner's classification such that the Dirac equation with scalar potential is quasi-exactly solved and the Bethe ansatz equations are derived in order to obtain the energy eigenvalues and eigenfunctions.

**PACS numbers:** 03.65.-w, 03.65.Pm, 03.65.Fd, 02.20.-a

**Key words:** Dirac equation, quasi-exact solvability, supersymmetric quantum mechanics

## 1 Introduction

In the relativistic quantum mechanics, exact solution of the Dirac equation has attracted considerable attention.<sup>[1–12]</sup> But, there are very few exactly solvable potentials. Quasi exact solvability theory provides a method for obtaining a part of the energy spectrum but not the whole spectrum.<sup>[13]</sup>

Lorentz scalar potential plays an important role in relativistic quantum mechanics and quantum field theory, and has various applications in these areas.<sup>[14–17]</sup> However, a few forms of Lorentz scalar potential lead to exactly solvable equations.

In Refs. [18–19], a general procedure for constructing quasi-exactly solvable systems based on  $sl(2)$  Lie algebra has been developed. Also, all the possible one-dimensional quasi-exactly solvable models that can be constructed were identified. This procedure is based on factorizability or equivalently supersymmetric structure of the Hamiltonian of the system. Moreover, this procedure suggests some configurations for the potential which provide determination of a part of the spectrum but not the whole spectrum. Meanwhile, possible forms of the Lorentz scalar potential which lead to the quasi-exactly solvable of the corresponding Dirac equation are determined.

Quasi-exactly solvable systems based on  $sl(2)$  Lie algebra have been classified by Turbiner in ten classes.<sup>[20]</sup> By factorizing the Dirac equation with the Lorentz scalar potential,<sup>[21]</sup> it is noticed that seven classes of quasi-exactly solvable potential can be constructed. These classes correspond to class I to class VI and class X in the Turbiner's classification. In Ref. [21], the construction of class I potential has been presented. In this paper by using the factorizability of the Dirac equation with the

Lorentz scalar potential, the construction of class II, class IV, class V, and class X are studied.

## 2 (2+1)-Dimensional Dirac Equation

In this section we briefly review the QES method of Ref. [21].

A (2+1)-dimensional Dirac Hamiltonian is given by

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta(m + V_s(x)), \quad (1)$$

where  $\boldsymbol{\alpha}$  and  $\beta$  are Dirac matrices and  $V_s(x)$  is the scalar potential. The representation of Dirac matrices has the following form:

$$\beta = \sigma_z, \quad \alpha_y = \sigma_y, \quad \alpha_x = \sigma_x, \quad (2)$$

in which  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$  are the Pauli matrices.

Regarding the form of potential, the wave function may be written as

$$\psi = e^{ik_y y} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix}, \quad (3)$$

where  $k_y$  is a real constant and  $f_{\pm}$  are some real functions of  $x$ . By using the relations (2) and (3) the Dirac equations takes the following form:

$$\begin{pmatrix} U(x) & p_x - ik_y \\ p_x + ik_y & -U(x) \end{pmatrix} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix} = E \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix}, \quad (4)$$

where  $U = m + V_s(x)$ . Equation (4) will take the supersymmetric form if the wave function is transformed by a unitary matrix  $T$  as:

$$\begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix} \longrightarrow \begin{pmatrix} i\psi_-(x) \\ i\psi_+(x) \end{pmatrix} \equiv T^+ \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix}, \quad (5)$$

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

So the Hamiltonian transform as

$$H \longrightarrow T^+ H T$$

\*E-mail: so\_ghaei@sut.ac.ir

†E-mail: a.chenaghlou@sut.ac.ir

$$= \begin{pmatrix} k_y & i\left(-\frac{d}{dx} + U(x)\right) \\ -i\left(-\frac{d}{dx} + U(x)\right) & -k_y \end{pmatrix}. \quad (6)$$

Now the Dirac equation reduces to

$$\begin{aligned} \left(-\frac{d}{dx} + U\right)\psi_+ &= (E - k_y)\psi_-, \\ \left(\frac{d}{dx} + U\right)\psi_- &= (E + k_y)\psi_+ \end{aligned} \quad (7)$$

or

$$\begin{aligned} \left(-\frac{d^2}{dx^2} + U^2 - U'\right)\psi_- &= \varepsilon\psi_-, \\ \left(-\frac{d^2}{dx^2} + U^2 - U'\right)\psi_+ &= \varepsilon\psi_+, \end{aligned} \quad (8)$$

where  $\varepsilon = E^2 - k_y^2$ . Moreover,  $U(r)$  plays the role of the superpotential. In order to solve Eq. (8), it is sufficient to obtain the upper component of the wave function i.e.  $\psi_-$ . By applying appropriate supercharge operator on the upper component, one may obtain the lower component.<sup>[22]</sup>

According to the theory of QES models,<sup>[13]</sup> it is necessary to perform an imaginary gauge transformation on the function  $\psi_-(x)$ , that is,

$$\psi(x) = \varphi e^{-g(x)}, \quad (9)$$

in which  $g(x)$  is the gauge function. The function  $\varphi(x)$  satisfies

$$-\frac{d^2\varphi}{dx^2} + 2g'\frac{d\varphi}{dx} + (V(x) + g'' - g'^2)\varphi = \varepsilon\varphi, \quad (10)$$

where  $V(x) = U^2 - U'$ .<sup>[22]</sup> For physical models, the phase factor  $e^{-g}$  is responsible for the asymptotic behaviors of the wave function in order to ensure the normalization condition. In fact, the form of  $V(x)$  guaranties the QES of the system which, in turn, leads to the fact that the gauge transformed Hamiltonian can be written as a quadratic combination of the generators of the  $sl(2)$  Lie algebra. Meanwhile, seven classes of QES potential can be constructed. These classes are class I to class VI and class X in Turbiner's classification.<sup>[20]</sup> The construction of class I has been investigated in Ref. [21]. Now we study class II, class IV, class V, and class X.

### 3 Class II

Considering the Turbiner's classification, the QES potential of class II is given by:<sup>[20]</sup>

$$\begin{aligned} V_N(x) &= d^2 e^{-4\alpha x} + 2ad e^{-3\alpha x} \\ &+ [a^2 + 2d(b - \alpha - N\alpha)] e^{-2\alpha x} \\ &+ (2ab - a\alpha + \lambda) e^{-\alpha x} + b^2, \end{aligned} \quad (11)$$

The gauge function  $g(x)$  corresponding to the potential has the form<sup>[20]</sup>

$$g(x) = \frac{d e^{-2\alpha x}}{2\alpha} + \frac{a e^{-\alpha x}}{\alpha} - bx. \quad (12)$$

It is evident that the parameters should be chosen such that the wave function becomes normalizable. For this

reason,  $a, d > 0$  and  $b < 0$ . The potential which leads to the ground state, with energy parameter  $\varepsilon = E^2 - k_y^2$ , is produced by

$$V_0 = g'^2 - g'' = U_0^2 - U_0'. \quad (13)$$

In order to notice that Eq. (10) is QES with the above potential, we perform the change of variable  $z = e^{-\alpha x}$ . So by substituting the class II potential and the corresponding gauge function, Eq. (10) is reduced to

$$\begin{aligned} \left[-\alpha^2 z^2 \frac{d^2}{dz^2} + (2d\alpha z^3 + 2a\alpha z^2 + 2b\alpha z - \alpha^2 z) \frac{d}{dz} \right. \\ \left. + (\lambda z - 2Nd\alpha z^2)\right] \varphi = \varepsilon\varphi. \end{aligned} \quad (14)$$

The above equation can be rewritten as

$$T_{II}\varphi = 0, \quad (15)$$

where

$$\begin{aligned} T_{II} &= -\alpha z \frac{d^2}{dz^2} + (2dz^2 + 2az + 2b - \alpha) \frac{d}{dz} \\ &+ \frac{\lambda}{\alpha} - 2Ndz - \frac{\varepsilon}{\alpha z}. \end{aligned} \quad (16)$$

Now the differential operator  $T_{II}$  may be written in terms of the  $sl(2)$  generators  $J^+$ ,  $J^0$  and  $J^-$  (Note that the  $sl(2)$  Lie algebra is the most general symmetry algebra for the one dimensional QES systems) as:<sup>[23]</sup>

$$\begin{aligned} T_{II} &= -\alpha J^0 J^- + 2dJ^+ + 2aJ^0 - \left[\alpha\left(\frac{N}{2} + 1\right) - 2b\right] J^- \\ &+ aN + \frac{\lambda}{\alpha} - \frac{\varepsilon}{\alpha z}, \end{aligned} \quad (17)$$

where

$$\left. \begin{aligned} J^+ &= z^2 \frac{d}{dz} - Nz, \\ J^0 &= z \frac{d}{dz} - \frac{N}{2}, \\ J^- &= \frac{d}{dz}, \end{aligned} \right\} N = 0, 1, 2, \dots \quad (18)$$

Therefore, the relation (8) with the class II potential is QES. The QES Hamiltonian has an algebraic sector with  $N + 1$  eigenvalues and eigenfunctions. The eigenfunctions has the following form

$$\psi = \varphi e^{-g(x)} = \prod_{i=1}^N (z - z_i) \exp\left[-\int U_0(x) dx\right], \quad (19)$$

where  $z_i$  ( $i = 1, 2, \dots, N$ ) are some parameters that can be obtained from the Bethe ansatz equations. Equation (19) can be rewritten as

$$\begin{aligned} \psi &= \exp\left[-\int U_N(x, \{z_i\}) dx\right], \\ U_N(x, \{z_i\}) &= U_0(x) - \sum_{i=1}^N \frac{z'(x)}{z(x) - z_i}. \end{aligned} \quad (20)$$

In fact,  $N + 1$  possible functions  $U_N$  exist for the  $N + 1$  sets of eigenfunctions  $\psi$ .

By assuming

$$\varphi = \prod_{i=1}^N (z - z_i), \quad (21)$$

one can obtain the Bethe ansatz equations for the relation (14). The operator of the relation (16) may be defined as  $T_{\text{II}} = a^+ a + \kappa$ , where the operator  $a$  is defined such that  $a\varphi = 0$  and  $\kappa$  is not a differential operator. It is easily seen that

$$a = \frac{d}{dz} - \sum_{i=1}^N \frac{1}{z - z_i}, \quad (22)$$

where  $z_i$ 's are the roots. On the other hand, the operator  $a^+$  has the following form:

$$a^+ = -\alpha z \frac{d}{dz} + (2dz^2 + 2az + 2b - \alpha) - \alpha z \sum_{i=1}^N \frac{1}{z - z_i}, \quad (23)$$

so we have

$$a^+ a = T_{\text{II}} - \left( \frac{\lambda}{\alpha} - 2Ndz - \frac{\varepsilon}{\alpha z} \right) + \alpha z \sum_{i \neq j} \frac{1}{(z - z_i)(z - z_j)} - (2dz^2 + 2az + 2b - \alpha) \sum_{i=1}^N \frac{1}{z - z_i}, \quad (24)$$

which, in turn, leads to

$$\left[ - \left( \frac{\lambda}{\alpha} - 2Ndz - \frac{\varepsilon}{\alpha z} \right) + \alpha z \sum_{i \neq j} \frac{1}{(z - z_i)(z - z_j)} - (2dz^2 + 2az + 2b - \alpha) \sum_{i=1}^N \frac{1}{z - z_i} \right] \varphi = 0. \quad (25)$$

Now, one may obtain the Bethe ansatz equations corresponding to the relation (14) which give the roots of function  $\varphi$  and the energy eigenvalues:

$$2dz_i^2 + 2az_i + 2b - \alpha - \alpha z_i \sum_{j \neq i} \frac{1}{z_i - z_j} = 0, \quad (26)$$

$$\lambda = \alpha(2b - \alpha) \sum_{i=1}^N \frac{1}{z_i}, \quad (27)$$

$$\varepsilon = 0. \quad (28)$$

In the class II and according to the relation (27),  $\lambda$  is obtained by the roots. So for a given  $N$ ,  $N + 1$  different potentials are generated with  $\varepsilon = 0$ . Firstly, we consider the case  $N = 0$ . In this case, one obtains  $\varepsilon = 0$  and  $\lambda = 0$ , and the form of the wave function is given by

$$\psi \propto e^{-g(x)}. \quad (29)$$

The potential which is responsible for the solvable ground state is

$$V(x) = V_0(\lambda = 0, x), \quad (30)$$

which is obtained by the following superpotential (according to the relation (13)):

$$U_0^{(0)} = g'(x) = -de^{-2\alpha x} - ae^{-\alpha x} - b.$$

So, the form of the scalar potential is such that the corresponding Dirac equation has solvable ground state, that is,

$$V_s(x) = -de^{-2\alpha x} - ae^{-\alpha x} - b - m. \quad (31)$$

For the case  $N = 1$ , the wave function is

$$\psi = (z - z_1) e^{-g(x)}. \quad (32)$$

The corresponding energy is  $\varepsilon = 0$ . The root  $z_1$  is obtained by using the relation (26)

$$dz_1^2 + az_1 + b - \frac{\alpha}{2} = 0. \quad (33)$$

The above equation has two roots as

$$z_1^\mp = \frac{-a \mp \sqrt{a^2 - 4d(b - \alpha/2)}}{2d}, \quad (34)$$

where  $z^-$  gives the ground state while  $z^+$  gives the first excited state. In other words, the potential  $V_1^{(0)} = V_1(x, \lambda^-)$  and  $V_1^{(1)} = V_1(x, \lambda^+)$ , are responsible for the solvability of the ground state and the first excited states, respectively. Note that

$$\lambda^\mp = \frac{\alpha(2b - \alpha)}{z_1^\mp}. \quad (35)$$

The above two potentials are obtained by the following superpotentials regarding to the relations (20):

$$U_1^{(0,1)} = U_0^{(0)} - \frac{z'}{z - z_1^\mp} = -de^{-2\alpha x} - ae^{-\alpha x} - b - \frac{-\alpha e^{-\alpha x}}{e^{-\alpha x} - z_1^\mp}. \quad (36)$$

So, the scalar potentials which lead to the solvability of the ground state and the first excited state are given by:

$$V_{s1}^{(0,1)} = -de^{-2\alpha x} - ae^{-\alpha x} - b - \frac{-\alpha e^{-\alpha x}}{e^{-\alpha x} - z_1^\mp} - m. \quad (37)$$

It must be mentioned that the ground state energy for the potential  $V_{s1}^{(0)}$  and the first excited energy for the potential  $V_{s1}^{(1)}$  are zero.

#### 4 Class IV

In the class IV, the potential has the following form:<sup>[20]</sup>

$$V_{\text{IV}} = -[a(a + \alpha) + \alpha k(\alpha k + \alpha + 2a)] \cosh^{-2} \alpha x - c(c + 2\alpha - 2a) \cosh^2 \alpha x + c^2 \cosh^4 \alpha x + a^2 + c\alpha - 2ca. \quad (38)$$

The corresponding gauge function is given by<sup>[20]</sup>

$$g(x) = c \cosh 2\alpha x / 4\alpha + (a/\alpha) \ln \cosh \alpha x. \quad (39)$$

The ground state potential is

$$V_0(x) = g'^2 - g'' = -a(a + \alpha) \cosh^{-2} \alpha x - c(c + 2\alpha - 2a) \cosh^2 \alpha x + c^2 \cosh^4 \alpha x + a^2 + c\alpha - 2ca = U_0^2 - U_0'. \quad (40)$$

Performing the change of variable  $z = \cosh^{-2} \alpha x$  in the relation (10), we have

$$\left\{ -4\alpha z^2(1 - z) \frac{d}{dz} + [(6\alpha + 4a)z^2 \right.$$

$$+ (4c - 4a - 4\alpha)z - 4c \left] \frac{d}{dz} - k(\alpha k + \alpha + 2a)z - \frac{\varepsilon}{\alpha} \right\} \varphi = 0. \quad (41)$$

The operator associated with the above equation can be rewritten in terms of the generators of the  $sl(2)$  Lie algebra as:<sup>[23]</sup>

$$\begin{aligned} T_{IV} \varphi &= 0, \\ T_{IV} &= 4\alpha J^+ J^0 - 4\alpha J^+ J^- + 2[\alpha(6j + 1) + 2a] J^+ \\ &\quad - 4[\alpha(2j + 1) + a - c] J^0 \\ &\quad - 4c J^- + 4(c - a - \alpha)j - 8\alpha j^2 - \frac{\varepsilon}{\alpha}, \end{aligned} \quad (42)$$

where  $j$  is the weight of the differential representation of  $sl(2)$  Lie algebra and  $k/2 = 2j$ . It is noticed that the corresponding operator of Eq. (41) can be written in terms of the Lie algebra operators. So this equation can be studied

by the QES method.

Now we write down the Bethe ansatz equations for the relation (41). The operator  $a$  and  $a^+$  are defined as:

$$\begin{aligned} a &= \frac{d}{dz} - \sum_{i=1}^N \frac{1}{z - z_i}, \\ a^+ &= (4\alpha z^3 - 4\alpha z^2) \frac{d}{dz} + (6\alpha + 4a)z^2 \\ &\quad + 4(c - a - \alpha)z - 4c \\ &\quad + (4\alpha z^3 - 4\alpha z^2) \sum_{i=1}^N \frac{1}{z - z_i}. \end{aligned} \quad (43)$$

It is easily seen that

$$a\varphi = a \prod_{i=1}^N (z - z_i) = 0 \Rightarrow a^\dagger a\varphi = 0,$$

On the other hand

$$\begin{aligned} a^+ a\varphi &= T_{IV} \varphi + \left\{ k(k\alpha + \alpha + 2a)z + \frac{\varepsilon}{\alpha} - (4\alpha z^3 - 4\alpha z^2) \sum_{i \neq j}^N \frac{1}{(z - z_i)(z - z_j)} \right. \\ &\quad \left. - [(6\alpha + 4a)z^2 + 4(c - a - \alpha)z - 4c] \sum_{i=1}^N \frac{1}{(z - z_i)} \right\} \varphi. \end{aligned} \quad (44)$$

So, we have

$$\begin{aligned} &\left\{ k(k\alpha + \alpha + 2a)z + \frac{\varepsilon}{\alpha} - (4\alpha z^3 - 4\alpha z^2) \sum_{i \neq j}^N \frac{1}{(z - z_i)(z - z_j)} \right. \\ &\quad \left. - [(6\alpha + 4a)z^2 + 4(c - a - \alpha)z - 4c] \sum_{i=1}^N \frac{1}{(z - z_i)} \right\} \varphi = 0. \end{aligned} \quad (45)$$

Now, we obtain the Bethe ansatz equations associated with the class IV potential that give the roots  $z_i$ 's and the energies of the higher degree  $N$ :

$$(6\alpha + 4a)z_i^2 + 4(c - a - \alpha)z_i - 4c + (4\alpha z_i^3 - 4\alpha z_i^2) \sum_{j \neq i}^N \frac{1}{(z_i - z_j)} = 0 \quad (46)$$

and

$$\varepsilon = -4\alpha c \sum_{i=1}^N \frac{1}{z_i}. \quad (47)$$

For the case  $N = 0$ ,  $\varepsilon = 0$  and the wave function has the form  $\psi \propto e^{-g(x)}$ . The potential of solvable ground state is:

$$V(x) = V_0(k = 0, x) = U_0^2 - U_0',$$

where

$$U_0 = g'(x) = c \cosh \alpha x \sinh \alpha x + a \tanh \alpha x,$$

or

$$V_s(x) = c \cosh \alpha x \sinh \alpha x + a \tanh \alpha x - m. \quad (48)$$

For the case  $N = 1$ , the wave function is given by

$$\psi = (z - z_1) e^{-g(x)}, \quad (49)$$

where the root  $z_1$  is obtained by the relation (46):

$$(3\alpha + 2a)z_1^2 + 2(c - a - \alpha)z_1 - 2c = 0, \quad (50)$$

which gives

$$z_1^\mp = \frac{-2(c - a - \alpha) \mp \sqrt{4(c - a - \alpha)^2 + 8c(3\alpha + 2a)}}{6\alpha + 4a}, \quad (51)$$

where  $z_1^-$  and  $z_1^+$  are related to the ground state and the first excited states, respectively. The energies of the ground state and the first excited states are given by the relation (47):

$$\varepsilon^\mp = -4\alpha c \sum_{i=1}^N \frac{1}{z_1^\mp}. \quad (52)$$

The scalar potential that is responsible for the solvability of the ground state and the first excited state is:

$$\begin{aligned} U_1 &= U_0^{(0)} - \frac{z'}{z - z_1^-} = c \cosh \alpha x \sinh \alpha x \\ &\quad + a \tanh \alpha x + \frac{2\alpha \coth^{-3} \alpha x \sinh \alpha x}{\cosh^{-2} \alpha x - z_1^-} \end{aligned} \quad (53)$$

or

$$V_{s1}(x) = c \cosh \alpha x \sinh \alpha x + a \tanh \alpha x + \frac{2\alpha \coth^{-3} \alpha x \sinh \alpha x}{\cosh^{-2} \alpha x - z_1^-} - m. \quad (54)$$

## 5 Class V

According to the Turbiner's classification, the QES potential in the class V has the form:

$$V_k = -b^2 \cosh^{-6} \alpha x + b[2a + 3b + \alpha(4N + 3)] \cosh^{-4} \alpha x - [(a + 3b)(a + b + \alpha) + 4Nb\alpha + \lambda] \cosh^{-2} \alpha x + (a + b)^2, \quad (55)$$

and, the corresponding gauge function is:

$$g(x) = -\frac{b}{2\alpha} \tanh^2 \alpha x + \frac{(a + b)}{\alpha} \ln \cosh \alpha x, \quad (56)$$

where  $a$ ,  $b$ , and  $\alpha$  are constants. This parameters should be chosen such that to ensure the normalizability of the wave function. The QES ground state potential, with energy parameter  $\varepsilon = E^2 - k_y^2$ , is produced by

$$V_0 = g'^2 - g'' = U_0^2 - U_0'. \quad (57)$$

For deriving the Bethe ansatz equations one may perform the change of variables  $z = \cosh^{-2} \alpha x$ . So, Eq. (10) is reduced to:

$$4\alpha(z^2 - z) \frac{d^2 \varphi}{dz^2} + [-4bz^2 + (6\alpha + 8b + 4a)z - 4(\alpha + a + b)] \frac{d\varphi}{dz} + \left(4Nbz - 4Nb - \frac{\lambda}{\alpha} - \frac{\varepsilon}{\alpha z}\right) \varphi = 0. \quad (58)$$

According to the QES theory, the wave function is considered as:  $\psi = \varphi e^{-g(x)}$ . Equation (58) can be written as

$$T_V \varphi = 0, \quad (59)$$

where

$$T_V = 4\alpha(z^2 - z) \frac{d^2}{dz^2} + [-4bz^2 + (6\alpha + 8b + 4a)z - 4(\alpha + a + b)] \frac{d}{dz} + \left(4Nbz - 4Nb - \frac{\lambda}{\alpha} - \frac{\varepsilon}{\alpha z}\right). \quad (60)$$

If we write the differential operator (60) in terms of a linear combination of a Lie algebra generators, we will use the QES method in order to obtain the potential  $U_N(x)$  which admits solvable states with higher weights.

The differential operator  $T_V$  may be written as follow:

$$T_V = 4\alpha J^+ J^- - 4\alpha J^0 J^- - 4b J^+ + (4\alpha N + 6\alpha + 8b + 4a) J^0 - (2\alpha N + 4\alpha + 4a + 4b) J^- + (6\alpha + 8b + 4a) \frac{N}{2} + 4\alpha N^2 - 4Nb - \frac{\lambda}{\alpha} - \frac{\varepsilon}{\alpha z}. \quad (61)$$

Now, we can obtain the Bethe ansatz equations for the relation (58). For this reason, we define operator  $a$  as:

$$a = \frac{d}{dz} - \sum_{i=1}^N \frac{1}{z - z_i},$$

where  $a\varphi = 0$  and  $z_i$ 's are the roots. On the other hand, the operator  $a^+$  is defined as follows:

$$a^+ = 4\alpha(z^2 - z) \frac{d}{dz} + [-4bz^2 + (6\alpha + 8b + 4a)z - 4(\alpha + a + b)] + 4\alpha(z^2 - z) \sum_{i=1}^N \frac{1}{z - z_i}.$$

So we have

$$a^+ a = T_V - \left(4bNz - 4Nb - \frac{\lambda}{\alpha} - \frac{\varepsilon}{\alpha z}\right) - 4\alpha(z^2 - z) \sum_{i \neq j} \frac{1}{(z - z_i)(z - z_j)} - [-4bz^2 + (6\alpha + 8b + 4a)z - 4(\alpha + a + b)] \sum_{i=1}^N \frac{1}{z - z_i}, \quad (62)$$

also, it is easily seen that  $a^+ a \varphi = T_V \varphi = 0$ . So

$$\left\{ - \left(4bNz - 4Nb - \frac{\lambda}{\alpha} - \frac{\varepsilon}{\alpha z}\right) - 4\alpha(z^2 - z) \sum_{i \neq j} \frac{1}{(z - z_i)(z - z_j)} - [-4bz^2 + (6\alpha + 8b + 4a)z - 4(\alpha + a + b)] \sum_{i=1}^N \frac{1}{z - z_i} \right\} \varphi = 0. \quad (63)$$

Therefore, the Bethe ansatz equations which give the roots of function  $\varphi$  and the energy eigenvalues are

$$4bz_i^2 - (6\alpha + 8b + 4)z_i + 4(\alpha + a + b) + 4z_i(z_i - 1) \sum_{j \neq i} \frac{1}{z_i - z_j} = 0, \quad (64)$$

$$\lambda = 4\alpha \left[ (\alpha + a + b) \sum_{i=1}^N \frac{1}{z_i} - Nb \right], \quad (65)$$

$$\varepsilon = 0. \quad (66)$$

Therefore, for a given  $N$ ,  $N + 1$  different potentials are

generated with  $\varepsilon = 0$ . Let us consider the case  $N = 1$  which admits two solutions. For this case the wave function is  $\psi = (z - z_1)e^{-g(x)}$ . The corresponding energy is  $\varepsilon = 0$ . According to Eq. (64), the root  $z_1$  satisfies

$$4bz_1^2 - (6\alpha + 8b + 4)z_1 - 4(\alpha + a + b) = 0, \quad (67)$$

which gives two solutions

$$z_1^\mp = \frac{(6\alpha + 8b + 4) \mp \sqrt{(6\alpha + 8b + 4)^2 + 64b(\alpha + a + b)}}{8b}, \quad (68)$$

as a result, we have

$$\lambda^\mp = \frac{4\alpha[(\alpha + a)]}{z_1^\mp}. \quad (69)$$

The potential  $V_1^{(0)} = V_1(x, \lambda^-)$  and  $V_1^{(1)} = V_1(x, \lambda^+)$  are responsible for the solvability of the ground state and the first excited states, respectively. This two potential are obtained by the following superpotentials regarding to the relations (20):

$$\begin{aligned} U_1^{(0,1)} &= U_0^{(0)} - \frac{z'}{z - z_1^\mp} \\ &= -b \cosh^{-2} \alpha x \tanh \alpha x + (a + b) \tanh \alpha x \\ &\quad - \frac{-2\alpha \sinh \alpha x \cosh^{-3} \alpha x}{\cosh^{-2} \alpha x - z_1^\mp}. \end{aligned} \quad (70)$$

According to the relation (66), the ground state energy for the potential  $V_1^{(0)}$  and the first excited energy for the potential  $V_1^{(1)}$  are zero. Similarly, QES potentials for higher degree  $N$  can be constructed.

## 6 Class X

The general form of the QES potential in class X has the form:

$$V_N = -a^2 \cos^2 \alpha x + a\alpha(2N + 1) \cos \alpha x + a^2. \quad (71)$$

The gauge function is

$$g(x) = \frac{a}{\alpha} \cos \alpha x. \quad (72)$$

where  $a$  and  $\alpha$  are the constants parameters which are responsible for the normalizability of the wave function. The QES potential that gives the ground state, with energy parameter  $\varepsilon = 0$ , is given by:

$$V_0 = U_0^2 - U_0', \quad (73)$$

with

$$U_0 = g'(x) = -\sin \alpha x. \quad (74)$$

In order to derive the Bethe ansatz equations for determining the potentials  $U_N(x)$  which admits solvable states with higher weights, let us perform the change of variable  $z = \cos \alpha x$ . Then, Eq. (10) becomes:

$$\left[ -\alpha^2(1 - z^2) \frac{d^2}{dz^2} + (\alpha^2 z + 2a\alpha(1 - z^2)) \frac{d}{dz} + 2aN\alpha z - \varepsilon \right] \varphi = 0 \quad (75)$$

or

$$T_X \varphi = 0, \quad (76)$$

where

$$T_X = -\alpha^2(1 - z^2) \frac{d^2}{dz^2} + (\alpha^2 z + 2a\alpha(1 - z^2)) \frac{d}{dz} + 2aN\alpha z - \varepsilon. \quad (77)$$

Now, we can write the differential operator  $T_X$  in terms of  $sl(2)$  Lie algebra generators as follow:

$$T_X = \alpha^2 J^+ J^- - \alpha^2 J^- J^+ + (N + 1)\alpha^2 J^0 + 2a\alpha J^- + \frac{\alpha^2}{2} N(N + 1) - \varepsilon. \quad (78)$$

Therefore, we have a QES problem.

In order to determine the Bethe ansatz equations, we define the operators  $a$  and  $a^+$  as:

$$\begin{aligned} a &= \frac{d}{dz} - \sum_{i=1}^N \frac{1}{z - z_i}, \\ a^+ &= -\alpha^2(1 - z^2) \frac{d}{dz} + \alpha^2 z + 2a\alpha(1 - z^2) - \alpha^2(1 - z^2) \sum_{i=1}^N \frac{1}{z - z_i}. \end{aligned} \quad (79)$$

As a result, we have:

$$\left[ -2aN\alpha z + \varepsilon + \alpha^2(1 - z^2) \sum_{i \neq j} \frac{1}{(z - z_i)(z - z_j)} - (\alpha^2 z + 2a\alpha(1 - z^2)) \sum_{i=1}^N \frac{1}{z - z_i} \right] \varphi = 0. \quad (80)$$

So, the Bethe ansatz equations that give the roots  $z_i$ 's and the energy parameters of the higher degree  $N$ , are:

$$-2a\alpha z_i^2 + \alpha^2 z_i + 2a\alpha - \alpha^2(1 - z_i^2) \sum_{j \neq i}^N \frac{1}{z_i - z_j} = 0, \quad (81)$$

$$\varepsilon = 2a\alpha \sum_{i=1}^N \frac{1}{z_i}. \quad (82)$$

For example, let us consider the case  $N = 1$ . In this case,

the wave function has the form:

$$\psi = (z - z_1) e^{g(x)},$$

where  $z_1$ , according to Eq. (81), satisfies the relation

$$-2a\alpha z_1^2 + \alpha^2 z_1 + 2a\alpha = 0.$$

Therefore, we have two roots

$$z_1^\mp = \frac{-\alpha^2 \mp \sqrt{\alpha^4 + 16a^2\alpha^2}}{-4a}$$

and the values of  $\varepsilon$  are

$$\varepsilon^\mp = 2a\alpha \frac{1}{z_1^\mp}.$$

The potential that give the ground state and first excited solvable state are

$$U_1 = U_0 - \frac{z'}{z - z_1},$$

$$U_1 = -a \cos \alpha x - \frac{-\alpha \sin \alpha x}{\cos \alpha x - z_1^-},$$

where the ground state energy is zero and the first excited energy is  $\varepsilon = \varepsilon^+ - \varepsilon^-$ .

## 7 Conclusion

In this paper by factorizing the Dirac equation in presence of the classes II, IV, V and X potentials in the Turbiner's classification, the corresponding operators were written in terms of the generators of the  $sl(2)$  Lie algebra. Then by deriving the Bethe ansatz equations, the wave functions and the corresponding energies of the ground state and the first excited states were calculated. It must be mentioned that there are some QES systems that are not factorizable or are not related to the  $sl(2)$  Lie algebra.

## References

- [1] Gao-Feng Wei and Shi-Hai Dong, Phys. Lett. B **686** (2010) 288.
- [2] Gao-Feng Wei, Xiao-Yong Duan, and Xu-Yang Liu, Int. J. Mod. Phys. A **25** (2010) 1649.
- [3] A.D. Alhaidari and A. Jellal, arXiv:1106.2236 [hep-th].
- [4] Min-Cang Zhang and Guo-Qing Huang-Fu, J. Math. Phys. **52** (2011) 053518.
- [5] Oktay Aydogdu and Ramazan Sever, Few Body Syst. **47** (2010) 193.
- [6] Y. Chargui, A. Trabelsi, and L. Chetouani, Phys. Lett. A **374** (2010) 531.
- [7] Y. Chargui, L. Chetouani, and A. Trabelsi, Commun. Theor. Phys. **53** (2010) 231.
- [8] Chun-Sheng Jia, Yong-Feng Diao, and Jian-Yi Liu, Int. J. Theor. Phys. **47** (2008) 664.
- [9] Chun-Sheng Jia, Xiao-Ping Li, and Lie-Hui Zhang, Few Body Syst. **52** (2012) 11.
- [10] Tomislav Prokopec, Michael G. Schmidt, and Jan Weenink, arXiv:1301.4132 [hep-th].
- [11] Alonso Contreras-Astorga, David J.C. Fernandez, and Javier Negro, Symmetry, Integrability and Geometry: Methods and Applications **8** (2012) 082.
- [12] Sameer M. Ikhdair, J. Mod. Phys. **3(2)** (2012) 170.
- [13] A.G. Ushveridze, Sov. Phys. -Lebedev Inst. Rep. **2** (1988) 50; *Quasi-Exactly Solvable Models in Quantum Mechanics*, IOP, Bristol (1994).
- [14] A. Chodos, R.L. Jaffe, K. Johnson, C.B. Thorn, and V. Weisskopf, Phys. Rev. D **9** (1974) 3471.
- [15] R.J. Bhaduri, *Models of Nucleons*, Addison-Wesley, Reading, MA (1988).
- [16] R.J. Furnstahl and Brian D. Serot, Nucl. Phys. A **673** (2000) 298.
- [17] E.S. Rodrigues, A.F. de Lima, and R. de Lima Rodrigues, arXiv:1301.6148 [math-ph].
- [18] C.L. Ho and P. Roy, J. Phys. A **39** (2003) 4617.
- [19] C.L. Ho and P. Roy, Ann. Phys. **312** (2004) 161.
- [20] A.V. Turbiner, Commun. Math. Phys. **118** (1988) 467.
- [21] C.L. Ho, Ann. Phys. **321** (2006) 2170.
- [22] F. Cooper, A. Khare, and U. Sukhatme, *Supersymmetric Quantum Mechanics*, World Scientific, Singapore (2001), and references there in.
- [23] A.V. Turbiner, J. Phys. A **25** (1992) L1087.