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# Wronskian Form Solutions for a Variable Coefficient Kadomtsev–Petviashvili Equation*

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***Abstract*** *Starting from a simple transformation, and with the aid of symbolic computation, we establish the relationship between the solution of a generalized variable coefficient Kadomtsev–Petviashvili (vKP) equation and the solution of a system of linear partial differential equations. According to this relation, we obtain Wronskian form solutions of the vKP equation, and further present $N$-soliton-like solutions for some degenerated forms of the vKP equation. Moreover, we also discuss the influences of arbitrary constants on the soliton and $N$-soliton solutions of the KPII equation.*

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**Key words:** variable coefficient KP equation, wronsian form solution, multi-soliton-like solution, symbolic computation

## 1 Introduction

As a generalization of the well known Kadomtsev–Petviashvili (KP) equation[1]

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0\,, \tag{1}$$

the variable coefficient Kadomtsev–Petviashvili (vKP) equation[2]

$$(u_t + f(t)uu_x + g(t)u_{xxx})_x + h(t)u_x + p(t)u_{yy} = 0\,, \tag{2}$$

in which $f(t)$, $g(t)$, $h(t)$, and $p(t)$ are differentiable functions, provides a description of surface waves in a more realistic situation. For example, it can better model the propagation of (small-amplitude) surface waves in straits or large channels of (slowly) varying width and depth with nonvanishing vorticity.[3] The KP equation (1) is known to be integrable, however, the vKP equation (2) can pass the Painlevé tests for complete integrability only when certain coefficient restrictions are satisfied.[4]

It is a difficult task for us to solve the nonlinear evolution equations (NLEEs) explicitly, and presently a general scheme for NLEEs solving is still unavailable. Even though, the investigation of exact solutions to NLEEs has attracted much research attention, since the exact solutions of NLEEs often model natural phenomena and facilitate the testing of numerical solvers. One of the most effective constructive algorithms for solving the NLEEs is the Wronskian determinant method,[5] which constructs the solutions of the NLEEs through Wronskian determinants. The algorithm has successfully been applied to different continuous and discrete NLEEs.[6−10] In this paper, we would like to construct Wronskian form solutions of Eq. (2). The solutions which we obtain establish a relationship between the vKP equation (2) and a system of linear partial differential equations. The special structure of Wronskian not only makes it easy to verify the solutions through direct substitution, but also facilitates the investigation of soliton interactions.

## 2 Wronskian Form Solutions for the vKP Equation

To explicitly solve the vKP equation (2), we make a simple transformation

$$u = F + \frac{12g(\ln\omega)_{xx}}{f}\,, \tag{3}$$

where $F = F(x,y,t)$ and $\omega = \omega(x,y,t)$ are functions to be determined. This transformation can be obtained through the method presented in Ref. [11], and can also be obtained from Darboux transformation.[12] Substituting Eq. (3) into Eq. (2), collecting and eliminating the coefficients of different powers of $1/\omega$, yields a set of partial differential equations (We denote them Eqs and list them in the appendix). The transformation (3) gives a solution of the vKP equation (2) provided $F$, $\omega$, and the coefficients $f$, $g$, $h$, $p$ satisfy the system Eqs.

The Wronskian form $N$-soliton solution of the KPII equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0\,, \tag{4}$$

was firstly given by Freeman and Nimmo.[5] Based on the bilinear approach,[13] Yao *et al.*[14] and Wu *et al.*[15] considered the vKP equation (2) with $h(t) = 0$, and performed different $N$-th order Wronskian form solution of the equation under the constraint $f(t) = Cg(t)$. Lü *et al.*[16] constructed new Wronskian solutions of the KP equation (1).

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In this Letter, we will construct Wronskian form solutions of the vKP equation (2) through the transformation (3).

For convenience, here we list two equations of the system Eqs as follows:

$$(F_t + f(t)FF_x + g(t)F_{xxx})_x + h(t)F_x + p(t)F_{yy} = 0, \quad (5)$$

$$4g\omega_x\omega_{xxx} + \omega_x\omega_t - 3g\omega_{xx}^2 + p\omega_y^2 + Ff\omega_x^2 = 0\,. \quad (6)$$

Equation (5) implies that $F$ should be a solution of the vKP equation (2). If setting $F = F(t)$, i.e. $F$ does not depend on $x$ and $y$, then Eq. (5) is satisfied identically. Further assuming that

$$\omega_y = \sqrt{3g/p}(\omega_{xx} + \lambda\omega_x)\,, \quad (7)$$

with $\lambda$ be arbitrary constant, then Eq. (6) can be reduced to

$$\omega_t = -g(3\lambda^2\omega_x + 6\lambda\omega_{xx} + 4\omega_{xxx}) - Ff\omega_x\,. \quad (8)$$

Note that Eqs. (7) and (8) are linear partial differential equations in the unknown function $\omega$.

It is interesting that Eqs. (7) and (8) together with $F = F(t)$ reduce the system Eqs to a single equation

$$g'f - f'g + fgh = 0\,. \quad (9)$$

This leads to the following proposition.

***Proposition 1*** If the coefficients constraint

$$h = \frac{f'g - g'f}{fg}\,, \quad (10)$$

is satisfied, then the formula (3) gives a solution of the vKP equation (2), where $\omega$ satisfies Eqs. (7) and (8).

In fact, we can further get the following result.

***Proposition 2*** Under the constraint (10), the vKP equation (2) possesses the following Wronskian form solution

$$u = F + \frac{12g(\ln W)_{xx}}{f}\,, \quad (11)$$

where $F$ is an arbitrary analytical function of $t$, and

$$W = \mathrm{Wr}(\omega_1, \omega_2, \ldots, \omega_N) = \begin{vmatrix} \omega_1 & \omega_1^{(1)} & \omega_1^{(2)} & \cdots & \omega_1^{(N-1)} \\ \omega_2 & \omega_2^{(1)} & \omega_2^{(2)} & \cdots & \omega_2^{(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_N & \omega_N^{(1)} & \omega_N^{(2)} & \cdots & \omega_N^{(N-1)} \end{vmatrix}, \quad (12)$$

with $(\omega_1, \omega_2, \ldots, \omega_N)$ be linearly independent solutions of both Eqs. (7) and (8), and $\omega_i^{(n)} := \partial_x^n \omega_i$ for $1 \le i \le N$.

To prove the Proposition 2, we introduce an identity called the Plücker relation[17] satisfied by the maximal minors of an $N \times M$ matrix

$$\Pi = \begin{pmatrix} \omega_1 & \omega_1^{(1)} & \omega_1^{(2)} & \cdots & \omega_1^{(M-1)} \\ \omega_2 & \omega_2^{(1)} & \omega_2^{(2)} & \cdots & \omega_2^{(M-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_N & \omega_N^{(1)} & \omega_N^{(2)} & \cdots & \omega_N^{(M-1)} \end{pmatrix}, \quad (13)$$

with $N < M$. A maximal minor $\pi(l_1, l_2, \ldots, l_N)$ is the determinant of the submatrix formed by $N$ columns $\Pi_i = (\omega_1^{(i-1)}, \omega_2^{(i-1)}, \ldots, \omega_N^{(i-1)})^{\mathrm{T}}$ of $\Pi$, where $i \in \{l_1, l_2, \ldots, l_N\}$, i.e.,

$$\pi(l_1, l_2, \ldots, l_N) = \det[\Pi_{l_1}, \Pi_{l_2}, \ldots, \Pi_{l_N}]\,. \quad (14)$$

If set

$$\pi[i, j] := \pi(1, 2, \ldots, N-2, N-2+i, N-2+j)\,,$$

then we have the following Plücker relation

$$\pi[1,2]\pi[3,4] - \pi[1,3]\pi[2,4] + \pi[1,4]\pi[2,3] = 0\,. \quad (15)$$

We now prove the Proposition 2:

***Proof*** Substituting Eqs. (10) and (11) into the vKP equation (2), integrating twice with respect to $x$ and setting the integration constants to zero yields the following equation in $W$:

$$WW_{xt} - W_xW_t + 3gW_{xx}^2 + gWW_{xxxx} - 4gW_xW_{xxx} + pWW_{yy} - pW_y^2 - FfW_x^2 + FfWW_{xx} = 0\,. \quad (16)$$

Thus if $W$ is a solution of Eq. (16), then Eq. (11) is a solution of the vKP equation (2). If $W$ satisfies (12), namely, $W = \pi[1,2]$, then we have

$$\begin{aligned} \pi[1,3] &= W_x\,, \\ \pi[1,4] &= \frac{1}{2}(W_{xx} + W_y/\sqrt{3g/p} - \lambda W_x)\,, \\ \pi[2,3] &= \frac{1}{2}(W_{xx} - W_y/\sqrt{3g/p} + \lambda W_x)\,, \\ \pi[3,4] &= \frac{1}{12g}(gW_{xxxx} + pW_{yy} + W_{xt} + FfW_{xx})\,, \\ \pi[2,4] &= \frac{1}{12g}[W_t + 3\lambda^2 gW_x + 4gW_{xxx} + FfW_x \\ &\quad + 6\lambda g(W_y/\sqrt{3g/p} - \lambda W_x)]\,. \end{aligned} \quad (17)$$

Substituting Eq. (17) into the Plücker relation (15), we have

$$\begin{aligned} 0 &= \pi[1,2]\pi[3,4] - \pi[1,3]\pi[2,4] + \pi[1,4]\pi[2,3] \\ &= \frac{1}{12g}(WW_{xt} - W_xW_t + 3gW_{xx}^2 \\ &\quad + gWW_{xxxx} - 4gW_xW_{xxx} + pWW_{yy} \\ &\quad - pW_y^2 - FfW_x^2 + FfWW_{xx})\,, \end{aligned} \quad (18)$$

which is equivalent to Eq. (16). This completes the proof of the Proposition.

When $h = 0$, the coefficient constraint (10) is just $f = Cg$ as mentioned in Ref. [14]. But in this case, if $\lambda \ne 0$ or $F \ne 0$, we can obtain new Wronskian form solutions that have not been obtained in Ref. [14].

Proposition 2 indicates that any linearly independent set of solutions of the linear system (7) and (8) will give rise to a solution of the vKP equation (2). Thus a large class of solutions for the equation can be generated in this way. Particularly, when $g(t)/p(t)$ is a positive constant, multi-soliton-like solutions of the vKP equation (2) can be obtained from the choice

$$\omega_n = \sum_{m=1}^{M} a_{nm}\, \mathrm{e}^{\theta_m}\,, \quad n = 1, 2, \ldots, N\,, \tag{19}$$

where

$$\theta_m = c_m x + \sqrt{3g/p}(c_m^2 + \lambda c_m) y - \int [g(4c_m^3 + 6\lambda c_m^2 + 3\lambda^2 c_m) + c_m f F] \mathrm{d}t\,, \tag{20}$$

with distinct nonzero parameters: $c_1 < c_2 < \cdots < c_M$. Since the functions $\{\omega_n\}_{n=1}^N$ are linearly independent, the coefficients therefore define an $N \times M$ constant matrix $A := (a_{nm})$ of rank $N$. The Wronskian (12) can then be expressed as

$$W = \sum_{1\le m_1 \le \cdots \le m_N \le M} \Big[A(m_1, \ldots, m_N) \exp[\theta(m_1, \ldots, m_N)] \prod_{1\le s<r\le N} (c_{m_r} - c_{m_s})\Big], \tag{21}$$

by expanding the determinant using the Binet-Cauchy formula.[18] In Eq. (21), $\theta(m_1, \ldots, m_N) := \theta_{m_1} + \theta_{m_2} + \cdots + \theta_{m_n}$, and $A(m_1, \ldots, m_N)$ is the maximal minor, i.e. the determinant of the $N \times N$ sub-matrix of $A$ obtained from columns $1 \le m_1 < \cdots < m_N \le M$. Regularity of the solutions in the entire $(x, y)$-plane for all values of time $t$ can be guaranteed, if all the $N \times N$ maximal minors of $A$ are non-negative.

When $f(t) = -3/2$, $g(t) = -1/4$, $h(t) = 0$, and $p(t) = -3/4$, the vKP equation (2) becomes the following KPII equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0\,. \tag{22}$$

Wronskian-induced soliton solutions of this equation have been classified in Ref. [17]. The $N$-soliton solution presented in Ref. [17] can be obtained from Eq. (11) by choosing $\omega_n$ as Eq. (19) with $F = \lambda = 0$.

Now we consider the influences of $\lambda$ and $F$ on the soliton and $N$-soliton solutions of the KPII equation (22). If $M = 2$, $N = 1$ and $F$ be constant, Eq. (19) leads to the following single-soliton solution of the KPII equation (22)

$$u = F - \frac{2\left(\begin{vmatrix}\omega_1 & \omega_1^{(1)}\\ \omega_2 & \omega_2^{(1)}\end{vmatrix}\begin{vmatrix}\omega_1 & \omega_1^{(1)}\\ \omega_2 & \omega_2^{(1)}\end{vmatrix}_{xx} - \begin{vmatrix}\omega_1 & \omega_1^{(1)}\\ \omega_2 & \omega_2^{(1)}\end{vmatrix}_x^2\right)}{\begin{vmatrix}\omega_1 & \omega_1^{(1)}\\ \omega_2 & \omega_2^{(1)}\end{vmatrix}^2}$$

$$= F + a\mathrm{sech}^2(\boldsymbol{c}\cdot\boldsymbol{r} + \tau t)\,, \tag{23}$$

where $\boldsymbol{r} := (x, y)$, wave vector $\boldsymbol{c} := (c_x, c_y) = ((c_2 - c_2)/2, (c_2 - c_1)(c_1 + c_2 + \lambda)/2)$, the amplitude

$$a = (c_2 - c_1)^2/2\,, \tag{24}$$

and the frequency

$$\tau = (c_2^3 - c_1^3)/2 + 3\lambda(c_2^2 - c_1^2)/4 + (3\lambda^2 + 6F)(c_2 - c_1)/8\,, \tag{25}$$

with $c_1 < c_2$. This soliton is localized in the $(x, y)$-plane along a line which makes an angle $\psi$ measured counter-clockwise from the $y$-axis, where

$$\tan\psi = c_y/c_x = c_1 + c_2\,, \quad -\pi/2 < \psi < \pi/2\,. \tag{26}$$

We can verify that for given amplitude $a$ and angle $\psi$, the frequency $\tau$ can be expressed in the form

$$\tau = \sqrt{2a}(2a + 3\tan^2\psi + 6F)/8\,. \tag{27}$$

This implies that when $F = 0$, the conditions $\lambda = 0$ and $\lambda \neq 0$ give rise to equivalent single-soliton solutions of the KPII equation. According to the analysis in Ref. [17], we can conclude that different values of $\lambda$ also lead to the equivalent $N$-soliton solutions of the KPII equation.

On the other hand, according to Eq. (27), the amplitude $a$, the angle $\psi$, together with $F$ affects the frequency of the single-soliton. Thus for given $a$ and $\psi$, we can control the velocity of the soliton through $F$. Furthermore, if $F \neq 0$, Eqs. (11) and (19) give new $N$-soliton solution of the KPII equation (22) that has not been reported in Ref. [17] or elsewhere.

It is obvious that different values of $\lambda$ can lead to nonequivalent non-soliton solutions (for example, rational solutions) of the KPII equation.

When $f(t) = 1/(2 - \sin t)$, $g(t) = 1$, $h(t) = f'/f$, and $p(t) = 3$, then Eq. (19) gives rise to exact solution of the following equation[19]

$$\Big(u_t + \frac{1}{2 - \sin t} uu_x + u_{xxx}\Big)_x + \frac{\cos t}{2 - \sin t} u_x + 3u_{yy} = 0. \tag{28}$$

In particular, if $F = 0$, we get multi-soliton-like breather solution.

Figure 1 shows a bi-soliton-like breather of Eq. (28) by choosing $N = 2$, $M = 4$, and the coefficient matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{29}$$

in Eq. (19). As we can see, the amplitude of the solution wave varies periodically with time evolution. For the cylindrical KP equation[20]

$$(u_t + 6uu_x + u_{xxx})_x + \frac{1}{2t} u_x + \frac{3\sigma^2}{t^2} u_{yy} = 0\,, \tag{30}$$

its coefficient functions $f(t) = 6$, $g(t) = 1$, $h(t) = 1/(2t)$, and $p(t) = 3\sigma^2/t^2$ do not satisfy the restriction (10), and its solution can not be expressed in the Wronskian form (11). However, it is pointed out in Refs. [12] and [20] that if $u(x, y, t)$ is a solution of the KP equation (1), then $u(x - y^2 t/(12\sigma^2), yt, t)$ is a solution of the cylindrical KP equation (30). In this sense, the solution of the cylindrical KP equation can also be induced by Wronskian.

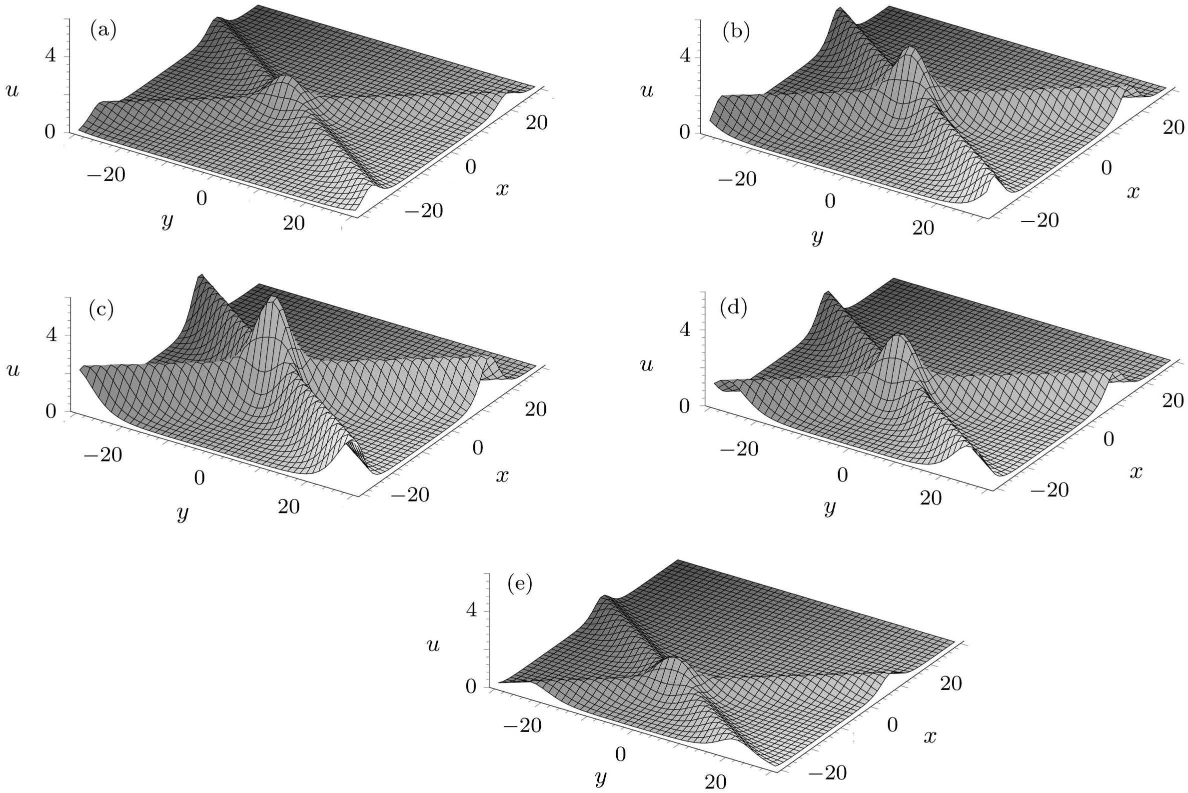


**Fig. 1** Evolution of the bi-soliton-like breather for Eq. (28) obtained from Eq. (19), with the integration constant be zero, $(c_1, c_2, c_3, c_4, \lambda) = (-1/5, -3/4, 1/2, 0, 1/4)$, and $F = 0$ at times (a) $t = \pi/2$, (b) $t = \pi$, (c) $t = 3\pi/2$, (d) $t = 2\pi$, (e) $t = 5\pi/2$.

## 3 Conclusion

The Wronskian form solutions of the KPI and KPII equations have been reported in previous literatures.[5,16−17] Compared with those known solutions, our solution has less restrictions on the entries of the Wronskian determinant, which enables us to control the amplitude, position, and velocity of the corresponding soliton solution waves. The solution waves of the variable coefficient NLEEs can be much different from those of the constant coefficient ones. The bi-soliton-like breather shown in this paper may have potential technological applications.

## Appendix

The partial differential equation system Eqs relating the functions $f$, $g$, $h$, $p$, $F$, and $\omega$.

$$(F_t + fFF_x + gF_{xxx})_x + hF_x + pF_{yy} = 0\,,$$

$$4g\omega_x\omega_{xxx} + \omega_x\omega_t - 3g\omega_{xx}^2 + p\omega_y^2 + Ff\omega_x^2 = 0\,,$$

$$f^2g\omega_{xx}F_{xx} + fg^2\omega_{xxxxxx} + 2f^2gF_x\omega_{xxx} + fg_t\omega_{xxx} + fg\omega_{xxxt} - f_tg\omega_{xxx} + f^2gF\omega_{xxxx} + fgp\omega_{xxyy} + fgh\omega_{xxx} = 0\,,$$

$$9fg^2\omega_x^2\omega_{xxxx} + 2f^2g\omega_x^3F_x - 3fg^2\omega_{xx}^3 + fg_t\omega_x^3 - f_tg\omega_x^3 + 3fg\omega_{xt}\omega_x^2 + 6f^2gF\omega_x^2\omega_{xx} + fgh\omega_x^3 + 3fg\omega_t\omega_x\omega_{xx} + fgp\omega_y^2\omega_{xx} + 4fgp\omega_y\omega_x\omega_{xy} + fgp\omega_{yy}\omega_x^2 = 0\,,$$

$$3f^2gF\omega_{xx}^2 + f^2g\omega_x^2F_{xx} - 2fg^2\omega_{xxx}^2 + 2fg\omega_{xy}^2 - 3f_tg\omega_x\omega_{xx} + 6f^2F_xg\omega_x\omega_{xx} + 4f^2gF\omega_x\omega_{xxx} + 3fg_t\omega_x\omega_{xx} + 3fg\omega_{xx}\omega_{xt} + 3fg\omega_x\omega_{xxt} + fg\omega_t\omega_{xxx} + 3fg^2\omega_{xx}\omega_{xxxx} + 6fg^2\omega_x\omega_{xxxxx} + 3fgh\omega_x\omega_{xx} + 2fgp\omega_y\omega_{xxy} + fgp\omega_{yy}\omega_{xx} + 2fgp\omega_x\omega_{xyy} = 0.$$

# References

[1] B.B. Kadomtsev and V.I. Petviashvili, Sov. Phys. Dokl. **15** (1970) 539.

[2] Z.N. Zhu, Phys. Lett. A **182** (1993) 277.

[3] W.L. Chan, K.S. Li, and Y.S. Li, J. Math. Phys. **33** (1992) 3759.

[4] G.M. Wei, Y.T. Gao, T. Xu, X.H. Meng, and C.Y. Zhang,

Chin. Phys. Lett. **25** (2008) 1599.

[5] N.C. Freeman and J.J.C. Nimmo, Phys. Lett. A **95** (1983) 1.

[6] W.X. Ma and Y. You, Trans. Amer. Math. Soc. **357** (2005) 1753.

[7] W.X. Ma and K. Maruno, Phys. A **343** (2004) 219.

[8] Y.Y. Sun and D.J. Zhang, Commun. Theor. Phys. **57** (2012) 923.

[9] F.C. You and J. Zhang, Commun. Theor. Phys. **56** (2011) 11.

[10] Y. Zheng, F.F. Cheng, D.J. Ding, and X.L. Dang, Commun. Theor. Phys. **55** (2011) 20.

[11] Z.S. Lü and H.Q. Zhang, Chaos, Solitons and Fractals **19** (2004) 527.

[12] V.B. Matveev and M.A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin (1991).

[13] R. Hirota, *The Direct Method in Soliton Theory*, Cambridge University, Cambridge (2004).

[14] Z.Z. Yao, C.Y. Zhang, H.W. Zhu, X.H. Meng, X. Lü, W.R. Shan, and B. Tian, Commun. Theor. Phys. **49** (2008) 1125.

[15] J.P. Wu and X.G. Geng, Chin. Phys. Lett. **30** (2013) 060502.

[16] D.Z. Lü, Y.Y. Cui, C.H. Liu, and M. Zhang, Commun. Theor. Phys. **54** (2010) 484.

[17] S. Chakravarty and Y. Kodama, Stud. Appl. Math. **123** (2009) 83.

[18] F.R. Gantmacher, *The Theory of Matrices*, Chelsea Publishing Company, New York (1959).

[19] Z.S. Lü and F.D. Xie, Math. Comput. Model. **52** (2010) 1423.

[20] C. Klein, V.B. Matveev, and A.O. Smirnov, Theor. Math. Phys. **152** (2007) 1132.