

An Interacting Gauge Field Theoretic Model for Hodge Theory: Basic Canonical Brackets

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Abstract We derive the basic canonical brackets amongst the creation and annihilation operators for a two (1 + 1)-dimensional (2D) gauge field theoretic model of an interacting Hodge theory where a $U(1)$ gauge field (A_μ) is coupled with the fermionic Dirac fields (ψ and $\bar{\psi}$). In this derivation, we exploit the spin-statistics theorem, normal ordering and the strength of the underlying six infinitesimal continuous symmetries (and the concept of their generators) that are present in the theory. We do not use the definition of the canonical conjugate momenta (corresponding to the basic fields of the theory) anywhere in our whole discussion. Thus, we conjecture that our present approach provides an alternative to the canonical method of quantization for a class of gauge field theories that are physical examples of Hodge theory where the continuous symmetries (and corresponding generators) provide the physical realizations of the de Rham cohomological operators of differential geometry at the algebraic level.

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1 Introduction

One of the earliest methods of quantization scheme (for a given physical system) is the canonical method of quantization where a set of three basic ideas is primarily exploited together. First of all, by using the spin-statistics theorem, we differentiate between the commuting (bosonic) and anticommuting (fermionic) variables of a given Lagrangian. Thereafter, we define the canonical conjugate momenta corresponding to the basic dynamical variables of the theory. We obtain the basic (graded) Poisson brackets amongst the dynamical variables and corresponding conjugate momenta at the classical level. These are, finally, promoted to the basic (anti)commutators at the quantum level and the quantization of the system ensues. If the physical system supports the existence of creation and annihilation operators, then, the normal ordering is required for the physical quantities to make some useful sense.

In the field theoretic models, we define a Lorentz scalar Lagrangian density for a given physical system. We follow the above sequence for the procedure of quantization. That is to say, we distinguish between the bosonic and fermionic fields (by spin-statistics theorem) and define the canonical conjugate momenta corresponding to the basic

dynamical fields. The basic (graded) Poisson brackets of the theory at classical level are promoted to the canonical (anti)commutators at the quantum level. However, in the field theoretic models, the fields (and corresponding momenta) are operators which are expressed, normally, in terms of the creation and annihilation operators. The above cited canonical quantum (anti)commutators are, at this stage, usually expressed in terms of the creation and annihilation operators. In other works, these basic canonical brackets are written in the language of the above creation and annihilation operators. Physical quantities of interest (e.g. Hamiltonian, conserved charges, etc.) are expressed in terms of the creation and annihilation operators, too. However, to avoid the unwanted infinities, the operators (present in the above physical quantities) are normal ordered so that they could make some physical sense.^[1–2]

In our present investigation, we shall utilize the virtues of normal ordering and spin-statistic theorem. However, we shall not take the help of mathematical definition of canonical conjugate momenta for the basic dynamical fields of our theory (i.e. 2D interacting $U(1)$ gauge theory of photon and Dirac fields). Rather, in the place of the latter, we shall utilize the beauty and strength of the

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physical symmetry principles (and the concept of a generator for a given infinitesimal continuous symmetry). Our present method of quantization, eventhough algebraically more involved, is physically more appealing because it is the symmetry properties of our theory that play a key role in our computations of the basic canonical brackets. In contrast, it is the mathematical definition of the canonical momentum that plays a decisive role in the determination of the basic brackets in the case of canonical method of quantization.

Recently we have exploited the central theme of our approach to quantize the 2D free Abelian 1-form and 4D free Abelian 2-form gauge theories where the mathematical definition of canonical conjugate momenta has not been used anywhere.^[3] It is the symmetry properties of the above field theoretic models for the Hodge theory^[4–7] (and their corresponding generators) that have played a decisive role in the derivation of the basic canonical brackets amongst the creation and annihilation operators of the above theories which contain bosonic as well as fermionic operators. Our present model of 2D quantum electrodynamics (QED) is a field theoretic example of an interacting Hodge theory (see, e.g. [8–9]) because the symmetries provide the physical realizations of de Rham cohomological operators.^[10–12] It is a challenging endeavor to check the sanctity of our method of quantization^[3] in the context of our present interacting theory as well. Of course, the interacting theory is more general than its free counterpart because the latter is a limiting case of the former.

Our present endeavour is essential on the following grounds. First and foremost, it is very urgent problem for us to extend our method of quantization (that is valid for the 2D Abelian 1-form and 4D Abelian 2-form free theories) to an interacting model for the Hodge theory where there is an interaction between the matter fields and gauge field. Second, it is always important and interesting to provide an alternative to the mathematical definition in the language of some basic physical properties. For instance, we provide an alternative to the mathematical definition of the canonical conjugate momenta in terms of the continuous symmetries and concept for the generator of a continuous symmetry transformation. Finally, our present endeavor adds a new dimension to the utility of symmetry principles (for a class of gauge field theories that turn out to be the field theoretic models for the Hodge theory) because it encompasses in its ever-widening folds the basic canonical brackets, too.

The contents of our present investigation are organized as follows. First of all, we discuss a set of six continuous symmetries of a 2D QED which is dynamically closed system of a U(1) gauge field and Dirac fields in Sec. 2. Our forthcoming Sec. 3 contains the derivation of canonical basic brackets from the (anti-)BRST symmetries and corresponding generators. We derive the same basic brackets

by exploiting the (anti-)co-BRST symmetries in Sec. 4. Our Sec. 5 deals with a concise derivation of the above brackets by exploiting the unique bosonic symmetry of the theory. For the sake of comparison, we discuss, in a concise manner, the canonical method of quantization for our present theory in Sec. 6. Finally, we make some concluding remarks and point out a few future directions for further investigations in our Sec. 7.

In our Appendix A, we derive the basic brackets amongst the creation and annihilation operators that appear in the normal mode expansions of the Dirac fields in their full generality (where we do not take the help of the condition $t = 0$). We also establish that the canonical brackets at the field level are equivalent to the canonical brackets at the level of the creation and annihilation operators.

2 Preliminaries: Lagrangian Formalism

In this section, we discuss various continuous symmetries of the 2D QED where there is an interaction between the U(1) gauge field (A_μ) and the Dirac fields (ψ and $\bar{\psi}$). We begin with the following locally gauge invariant Lagrangian density (see, e.g. [1–2])

$$\mathcal{L}_0 = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi, \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is derived from the 2-form $F^{(2)} = dA^{(1)}$. In the above, $d = dx^\mu \partial_\mu$ (with $d^2 = 0$) is the exterior derivative and the 1-form $A^{(1)} = dx^\mu A_\mu$ defines the vector potential (A_μ). We have taken $D_\mu \psi = \partial_\mu \psi + ie A_\mu \psi$ as the covariant derivative on the Dirac field ψ . The above Lagrangian density is a closed system of a massless vector gauge boson and Dirac fields where the interaction term is $(-e \bar{\psi} \gamma^\mu A_\mu \psi)$. This derivation is true in any arbitrary dimension of spacetime. Here the quantity e is the electric charge of the Dirac fields.

In the specific two (1+1) dimensions of spacetime, $F_{\mu\nu}$ has only one non-vanishing component which is nothing but the electric field $E = -\varepsilon^{\mu\nu} \partial_\mu A_\nu = \partial_0 A_1 - \partial_1 A_0$ where we have taken the Levi-Civita tensor $\varepsilon_{\mu\nu}$ with the conventions $\varepsilon_{01} = +1 = -\varepsilon^{01}$, $\varepsilon_{\mu\nu} \varepsilon^{\nu\lambda} = \delta_\mu^\lambda$, etc., and the Greek indices $\mu, \nu, \lambda, \dots = 0, 1$. One of most elegant approaches to quantize the above gauge theory is the Becchi–Rouet–Stora–Tyutin (BRST) formalism where the gauge-fixing and Faddeev–Popov ghost terms are incorporated in the Lagrangian density. Such an (anti-)BRST invariant 2D Lagrangian density, in the Feynman gauge, is as follows (see, e.g. [8–9] for details)

$$\begin{aligned} \mathcal{L}_b &= \mathcal{L}_0 - \frac{1}{2} (\partial \cdot A)^2 - i \partial_\mu \bar{C} \partial^\mu C \\ &\equiv \frac{1}{2} E^2 + \bar{\psi} (i\gamma^\mu D_\mu - m) \psi \\ &\quad - \frac{1}{2} (\partial \cdot A)^2 - i \partial_\mu \bar{C} \partial^\mu C, \end{aligned} \quad (2)$$

where $(\bar{C})C$ are the fermionic ($\bar{C}^2 = C^2 = 0$, $C\bar{C} + \bar{C}C = 0$) (anti-)ghost fields, the Dirac fields $(\psi, \bar{\psi})$ obey the anticommutativity property ($\psi\bar{\psi} + \bar{\psi}\psi = 0$) at the *classical* level and $\bar{\psi} = \psi^\dagger\gamma^0$. Here $\gamma^\mu = (\gamma^0, \gamma^1)$ are the 2×2 Dirac gamma matrices in 2D and one can choose them in terms of the Pauli σ -matrices as $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$ so that $\gamma_5 = \gamma^0\gamma^1 = \sigma_3$. It can be readily checked that $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ and $\gamma_\mu\gamma_5 = \varepsilon_{\mu\nu}\gamma^\nu$ where $\eta_{\mu\nu} = \text{diag}(+1, -1)$ is the flat 2D Minkowski metric. We adopt here the convention of left derivative w.r.t. the fermionic fields $(\psi, \bar{\psi}, C, \bar{C})$ so that we obtain the conjugate momenta as: $\Pi_\psi = -i\psi^\dagger$, $\Pi_C = i\dot{\bar{C}}$, $\Pi_{\bar{C}} = -i\dot{C}$.

The Lagrangian density (2) respects the following on-shell ($\square C = \square \bar{C} = 0$) nilpotent ($s_{(a)b}^2 = 0$) (anti-)BRST symmetry transformations ($s_{(a)b}$) (see, e.g. Refs. [8–9])

$$\begin{aligned} s_{ab}A_\mu &= \partial_\mu\bar{C}, & s_{ab}\bar{C} &= 0, \\ s_{ab}C &= +i(\partial \cdot A), & s_{ab}E &= 0, \\ s_{ab}\psi &= -ie\bar{C}\psi, & s_{ab}\bar{\psi} &= ie\bar{C}\bar{\psi} \equiv -ie\bar{\psi}\bar{C}, \\ s_bA_\mu &= \partial_\mu C, & s_bC &= 0, \\ s_b\bar{C} &= -i(\partial \cdot A), & s_bE &= 0, \\ s_b\psi &= -ieC\psi, & s_b\bar{\psi} &= ieC\bar{\psi} \equiv -ie\bar{\psi}C, \end{aligned} \quad (3)$$

where we have used, at a couple of places, the anticommutativity property ($C\psi + \psi C = 0$, $\bar{C}\bar{\psi} + \bar{\psi}\bar{C} = 0$, etc.) of the fermionic ($\psi^2 = \bar{\psi}^2 = C^2 = \bar{C}^2 = 0$) fields $(\psi, \bar{\psi}, C, \bar{C})$. It is straightforward to check that the above on-shell nilpotent (anti-) BRST symmetry transformations are absolutely anticommuting ($s_b s_{sb} + s_{ab} s_b = 0$) on the on-shell where $\square C = \square \bar{C} = 0$. The corresponding on-shell nilpotent and conserved ($\dot{Q}_{(a)b} = 0$) (anti-)BRST charges, that generate the symmetry transformations (3), are

$$\begin{aligned} Q_{ab} &= \int dx [E\partial_1\bar{C} - e\bar{\psi}\gamma_0\bar{C}\psi - (\partial \cdot A)\dot{\bar{C}}], \\ Q_b &= \int dx [E\partial_1 C - e\bar{\psi}\gamma_0 C\psi - (\partial \cdot A)\dot{C}], \end{aligned} \quad (4)$$

which can be re-expressed in a simpler form if we use the equation of motion $\partial_\mu F^{\mu\nu} + \partial^\nu(\partial \cdot A) = e\bar{\psi}\gamma^\nu\psi$. In other words, we also have the simpler forms:

$$\begin{aligned} Q_{ab} &= \int dx [\partial_0(\partial \cdot A)\bar{C} - (\partial \cdot A)\dot{\bar{C}}], \\ Q_b &= \int dx [\partial_0(\partial \cdot A)C - (\partial \cdot A)\dot{C}]. \end{aligned}$$

We note that under the following on-shell ($\square C = \square \bar{C} = 0$) nilpotent ($s_{(a)d}^2 = 0$) (anti-)co-BRST symmetry transformations ($s_{(a)d}$) (see, e.g. Refs. [8–9])

$$\begin{aligned} s_{ad}A_\mu &= -\varepsilon_{\mu\nu}\partial^\nu C, & s_{ad}C &= 0, & s_{ad}\bar{C} &= iE, \\ s_{ad}(\partial \cdot A) &= 0, & s_{ad}\psi &= -ieC\gamma_5\psi, & s_{ad}\bar{\psi} &= ie\bar{\psi}C\gamma_5, \\ s_dA_\mu &= -\varepsilon_{\mu\nu}\partial^\nu\bar{C}, & s_d\bar{C} &= 0, & s_dC &= -iE, \\ s_d(\partial \cdot A) &= 0, & s_d\psi &= -ie\bar{C}\gamma_5\psi, \\ s_d\bar{\psi} &= ie\bar{\psi}\bar{C}\gamma_5, \end{aligned} \quad (5)$$

the Lagrangian density (2) transforms to a total spacetime derivative thereby showing the symmetry property of the action integral ($S = \int d^2x \mathcal{L}_b$) due to the Gauss's divergence theorem. It should be noted that the transformations (5) are symmetry transformations for the Lagrangian density (2) only in the massless limit ($m = 0$). In other words, the above symmetry (5) is true for the chiral fermions only. According to Noether's theorem, we have conserved charges $Q_{(a)d}$ corresponding to the above symmetry transformations (5) as:

$$\begin{aligned} Q_{ad} &= \int dx [E\dot{C} + e\bar{\psi}\gamma_1 C\psi + \partial_1(\partial \cdot A)C], \\ Q_d &= \int dx [E\dot{\bar{C}} + e\bar{\psi}\gamma_1\bar{C}\psi + \partial_1(\partial \cdot A)\bar{C}], \end{aligned} \quad (6)$$

which turn out to be the generators of the transformations (5). These charges can be written in a simpler form if we use the equation of motion $\partial_\mu F^{\mu\nu} + \partial^\nu(\partial \cdot A) = e\bar{\psi}\gamma^\nu\psi$ to re-express (6) as $Q_{ad} = \int dx [E\dot{C} - \dot{E}C]$ and $Q_d = \int dx [E\dot{\bar{C}} - \dot{E}\bar{C}]$.

There is a unique (i.e. $\{s_b, s_d\} = s_\omega = -\{s_{ab}, s_{ad}\}$) bosonic symmetry (s_ω) in our theory which is obtained from the anticommutators of $s_{(a)b}$ and $s_{(a)d}$.^[4,8–9] Under this symmetry transformation, the relevant fields of the theory transform as:

$$\begin{aligned} s_\omega A_\mu &= \partial_\mu E - \varepsilon_{\mu\nu}\partial^\nu(\partial \cdot A), \\ s_\omega\psi &= -ie[\gamma_5(\partial \cdot A) + E]\psi, \\ s_\omega C &= 0, & s_\omega\bar{C} &= 0, \\ s_\omega(\partial \cdot A) &= \square E, & s_\omega E &= \square(\partial \cdot A), \\ s_\omega\bar{\psi} &= +ie[\gamma_5(\partial \cdot A) + E]\bar{\psi} \equiv ie\bar{\psi}[E - \gamma_5(\partial \cdot A)]. \end{aligned} \quad (7)$$

It can be checked that the Lagrangian density (2) transforms to a total spacetime derivative under (7). As a consequence, the action integral remains invariant under the infinitesimal and continuous symmetry transformations (7). The conserved charge corresponding to the above transformations (7) is

$$\begin{aligned} Q_\omega &= \int dx [(\partial \cdot A)\partial_1(\partial \cdot A) - E\partial_1 E - e(\partial \cdot A)\bar{\psi}\gamma_1\psi \\ &\quad + eE\bar{\psi}\gamma_0\psi], \end{aligned} \quad (8)$$

which turns out to be the generator of transformations (7).

Finally, we have a global ghost-scale symmetry in the theory where $C \rightarrow e^{+\Lambda} C$, $\bar{C} \rightarrow e^{-\Lambda} \bar{C}$. Here Λ is an infinitesimal spacetime independent scale parameter. The infinitesimal version of this transformation (s_g) is

$$s_g C = +C, \quad s_g \bar{C} = -\bar{C}, \quad s_g \Phi = 0, \quad \Phi = A_\mu, \psi, \bar{\psi}, \quad (9)$$

where, for the sake of brevity, we have set $\Lambda = 1$. The corresponding Noether's conserved charge (Q_g) is as follows

$$Q_g = i \int dx [C\dot{\bar{C}} + \bar{C}\dot{C}]. \quad (10)$$

The above charge Q_g is the generator of the transformations (9). Thus, as claimed earlier, we have *six* infinitesimal and continuous symmetries in the theory.

The decisive features of the on-shell nilpotent (anti-)BRST and (anti-)co-BRST symmetries are the invariances of the kinetic and gauge-fixing terms of the Lagrangian density (2), respectively. The ghost term, on the other hand, remains invariant under the bosonic symmetry transformations. The key feature of the ghost-scale symmetry is the observation that only the (anti-)ghost fields transform globally and rest of the fields of the theory remain unchanged under it. These symmetries, at the algebraic level, provide the physical realizations of the de Rham cohomological operators of differential geometry (see, e.g. [4, 8–9] for details). As a consequence, our present 2D interacting theory (i.e. 2D QED) is a physical model for the Hodge theory.

3 (Anti-)BRST Symmetries: Basic Brackets

In our earlier work on the 2D Abelian 1-form gauge theory,^[3] we have exploited the ideas of symmetry principles (along with spin-statistics theorem and normal ordering) to derive the basic non-vanishing canonical brackets amongst the creation and annihilation operators of the gauge field A_μ and (anti-)ghost fields (\bar{C}) C as:

$$\begin{aligned} [a_\mu(k), a_\nu^\dagger(k')] &= \eta_{\mu\nu} \delta(k - k'), \\ \{c(k), \bar{c}^\dagger(k')\} &= +i \delta(k - k'), \\ \{c^\dagger(k), \bar{c}(k')\} &= -i \delta(k - k'), \end{aligned} \quad (11)$$

where the above operators are present in the normal mode expansions of the basic fields of the Lagrangian density (2) (in the limit $\psi = 0, \bar{\psi} = 0$) as (see, e.g. [1,3]):

$$\begin{aligned} A_\mu(x, t) &= \int \frac{dk}{\sqrt{2\pi} 2k_0} [a_\mu(k) e^{+ik \cdot x} + a_\mu^\dagger(k) e^{-ik \cdot x}], \\ C(x, t) &= \int \frac{dk}{\sqrt{2\pi} 2k_0} [c(k) e^{+ik \cdot x} + c^\dagger(k) e^{-ik \cdot x}], \\ \bar{C}(x, t) &= \int \frac{dk}{\sqrt{2\pi} 2k_0} [\bar{c}(k) e^{+ik \cdot x} + \bar{c}^\dagger(k) e^{-ik \cdot x}]. \end{aligned} \quad (12)$$

Here the two-vector $k_\mu = (k_0, k_1 = k)$ is the momentum vector and the dagger operators $a_\mu^\dagger(k)$, $c^\dagger(k)$, and $\bar{c}^\dagger(k)$ are the creation operators for a photon, a ghost and an anti-ghost quanta, respectively. The non-dagger operators $a_\mu(k)$, $c(k)$, and $\bar{c}(k)$ stand for the corresponding annihilation operators for a single quantum. It is also clear that the operators ($a_\mu^\dagger(k), a_\mu(k)$) are bosonic in nature as against the operators ($c(k), \bar{c}(k), c^\dagger(k), \bar{c}^\dagger(k)$) that are fermionic. The (anti-)BRST symmetries (being supersymmetric type in nature), our present interacting theory is endowed with bosonic as well as fermionic creation and annihilation operators.

In the above derivation, we have mainly utilized the definition of the generator of a continuous symmetry transformation. According to the common folklore in

quantum field theory, the conserved charges (that are derived due to the presence of the continuous symmetries in the theory) generate the infinitesimal and continuous symmetry transformations as

$$s_r \Phi = \pm i [\Phi, Q_r]_\pm, \quad r = b, ab, d, ad, \omega, g, \quad (13)$$

where Φ is the generic field of the theory and Q_r are the conserved charges. The (\pm) signs, as the subscripts on the square bracket, correspond to the (anti)commutator for the generic field Φ being (fermionic) bosonic in nature. The (\pm) signs, in front of the expression on the r.h.s. (i.e. $\pm i [\Phi, Q_r]_\pm$), need explanation. The pertinent points, regarding the choice of a specific sign for a specific purpose, are as follows:

- (i) for $s_r = s_b, s_{ab}, s_d, s_{ad}$ only the negative sign is to be taken into account (i.e. $s_b A_\mu = -i [A_\mu, Q_b]$, $s_b \bar{C} = -i [\bar{C}, Q_b]$, etc.), and
- (ii) for $s_r = s_g, s_\omega$ the negative sign is to be taken into account for the bosonic field and the positive sign is to be chosen for the fermionic field (e.g. $s_g A_\mu = -i [A_\mu, Q_g]$, $s_g C = +i [C, Q_g]$, $s_g \bar{C} = +i [\bar{C}, Q_g]$, etc.).

In the derivation of the non-vanishing basic brackets (11), we have utilized the above rules in the computation of the basic quantum (anti)commutators. It is obvious that the rest of brackets are zero for the free theory (as far as the non-vanishing brackets (11) and related brackets are concerned).

In our present endeavor, we shall focus on the derivation of the basic brackets for the fields ψ and $\bar{\psi}$. Following (3) and (13), it is clear that the Dirac fields transform, under the on-shell nilpotent (anti-)BRST symmetries, as

$$\begin{aligned} s_b \psi &= -i \{\psi, Q_b\} = -ie C \psi, \\ s_{ab} \psi &= -i \{\psi, Q_{ab}\} = -ie \bar{C} \psi, \\ s_b \bar{\psi} &= -i \{\bar{\psi}, Q_b\} = +ie C \bar{\psi}, \\ s_{ab} \bar{\psi} &= -i \{\bar{\psi}, Q_{ab}\} = +ie \bar{C} \bar{\psi}, \end{aligned} \quad (14)$$

where we have used only the basic concepts of the continuous symmetries (and their generators) as well as the spin-statistics theorem. We can, at this stage, take the following mode expansions for the Dirac fields in the momentum phase space^[13]

$$\begin{aligned} \psi(x, t) &= \int \frac{dk}{\sqrt{(2\pi) 2k_0}} \sum_\alpha (b^\alpha(k) u^\alpha(k) e^{+ik \cdot x} \\ &\quad + (d^\alpha)^\dagger(k) v^\alpha(k) e^{-ik \cdot x}), \end{aligned} \quad (15)$$

$$\begin{aligned} \psi^\dagger(x, t) &= \int \frac{dk}{\sqrt{(2\pi) 2k_0}} \sum_\alpha ((b^\alpha)^\dagger(k) (u^\alpha)^\dagger(k) e^{-ik \cdot x} \\ &\quad + d^\alpha(k) (v^\alpha)^\dagger(k) e^{+ik \cdot x}), \end{aligned} \quad (16)$$

where $(b^\alpha)^\dagger(k)$, $(d^\alpha)^\dagger(k)$ are the creation operators and the corresponding operators without dagger (i.e. $b^\alpha(k)$, $d^\alpha(k)$) are the annihilation operators. All these operators are fermionic in nature. The bosonic variables $u^\alpha(k)$ and

$v^\alpha(k)$ are the plane wave solutions of the Dirac equation for the positive and negative energies, respectively. We shall be exploiting some of the key properties associated with these operators. A few of these (that are useful to our current endeavor) are listed below (see, e.g. [13])

$$\begin{aligned} \sum_{\alpha} u^{\alpha}(k)\bar{u}^{\alpha}(k) &= (\gamma^{\mu}k_{\mu} + m), \\ \sum_{\alpha} v^{\alpha}(k)\bar{v}^{\alpha}(k) &= (\gamma^{\mu}k_{\mu} - m), \\ (u^{\beta})^{\dagger}(k)u^{\alpha}(k) &= 2k_0\delta^{\alpha\beta}, \\ (v^{\beta})^{\dagger}(k)v^{\alpha}(k) &= 2k_0\delta^{\alpha\beta}, \end{aligned} \quad (17)$$

where $\bar{u}^{\alpha}(k) = (u^{\alpha})^{\dagger}(k)\gamma^0$, $\bar{v}^{\alpha}(k) = (v^{\alpha})^{\dagger}(k)\gamma^0$.

It will be worthwhile to mention that, in the case of 2D free U(1) gauge theory (i.e. $\psi = \bar{\psi} = 0$), it was quite easy to express all the conserved charges in terms of the creation and annihilation operators in a compact form without any exponentials because, in these expressions, the field variables turned out to be quadratic (bilinear) only (see, e.g. [3] for details). This is, however, not the case as far as the interacting 2D QED is concerned. It can be seen that, in the expressions for Q_b

and Q_{ab} [cf. (4)], we have terms (e.g. $\int dx eC\psi^{\dagger}\psi$ etc.), that are not quadratic. Thus, to obtain the canonical brackets between the fermionic annihilation and creation operators (e.g. $b^{\alpha}(k)$, $(b^{\alpha})^{\dagger}(k)$, etc.), we have to adopt a different technique because non-quadratic terms can not be expressed in a compact form in terms of the creation and annihilation operators without any exponentials. To elucidate this point, we do the following explicit exercise which conveys the main ideology of our approach.

Let us try to obtain the canonical anticommutators from the following principle of the continuous symmetry transformation [cf. (13)]:

$$s_b\psi = -ieC\psi = -i\{\psi, Q_b\} \implies eC\psi = \{\psi, Q_b\}, \quad (18)$$

where the BRST charge Q_b is the generator of the transformation s_b . Since the nilpotent (anti-)BRST charges are conserved quantities, it is simpler to perform all the computations with $t = 0$ (see, e.g. [13] for details). The results, thus obtained, would be same as the ones obtained in all their generality (i.e. without taking the limit $t = 0$ (see, e.g., Appendix A below)). Exploiting the mode expansions of the basic fields given in (12) and (15), in this limit (i.e. $t = 0$), the l.h.s. is

$$\begin{aligned} e \int \frac{dkdk'}{(2\pi)\sqrt{(2k_0 2k'_0)}} \sum_{\alpha} (c(k)b^{\alpha}(k')u^{\alpha}(k') \exp[-i(k+k')x] + c(k)(d^{\alpha})^{\dagger}(k')v^{\alpha}(k') \exp[-i(k-k')x] \\ + c^{\dagger}(k)b^{\alpha}(k')u^{\alpha}(k') \exp[+i(k-k')x] + c^{\dagger}(k)(d^{\alpha})^{\dagger}(k')v^{\alpha}(k') \exp[+i(k+k')x]), \end{aligned} \quad (19)$$

where, in the exponentials, we have only the space part of the 2D dot products and k and k' correspond to the space part of the momenta for the fields $C(x)$ and $\psi(x)$ in the phase space. The powers of exponential play a very important role when we compare the l.h.s. with r.h.s. (while exploiting the basic equation (18) related to the key tenets of continuous symmetries and their generators).

As far as the computation of the r.h.s. is concerned, it should be noted that the following portions of the BRST and anti-BRST charges, namely;

$$Q_b^{(R)} = e \int dx C(x)\psi^{\dagger}(x)\psi(x), \quad Q_{ab}^{(R)} = e \int dx \bar{C}(x)\psi^{\dagger}(x)\psi(x), \quad (20)$$

contribute in the computations of $s_{(a)b}\psi$ and $s_{(a)b}\bar{\psi}$. Here the superscript (R) on the charges (i.e. $Q_{(a)b}^{(R)}$) denotes the relevant portion of the conserved and nilpotent charges $Q_{(a)b}$ [cf. (4)]. Furthermore, it should be emphasized that the rest of the quadratic parts of the (anti-)BRST charges ($Q_{(a)b}$) can be expressed in terms of the creation and annihilation operators and they appear in a compact form as is the case in Ref. [3]. The explicit form of the contributing factor (in the case of $s_b\psi$), from the r.h.s., is

$$e \int dy \{\psi(x), C(y)\psi^{\dagger}(y)\psi(y)\}. \quad (21)$$

The above expression can be written in terms of the mode expansions of the fields $C(y)$, $\psi(y)$, $\psi^{\dagger}(y)$, and $\psi(x)$ by exploiting (12), (15), and (16). It is evident that the comparison of the l.h.s. with the r.h.s. leads to the cancellation of the factor “ e ” present on both the sides [cf. (19), (21)].

The relevant Q_b part (i.e. $\int dy C(y)\psi^{\dagger}(y)\psi(y)$) for our computation, in terms of the mode expansion, can be written (for $t = 0$) as

$$\begin{aligned} \sum_{\gamma} \sum_{\sigma} \int dy \int \frac{dk_2 dk_3 dk_4}{(2\pi)^{3/2} \sqrt{(2k_2^0 2k_3^0 2k_4^0)}} (a_1 \exp[-i(k_2 - k_3 + k_4)y] + a_2 \exp[-i(k_2 - k_3 - k_4)y] \\ + a_3 \exp[-i(k_2 + k_3 + k_4)y] + a_4 \exp[-i(k_2 + k_3 - k_4)y] + a_5 \exp[+i(k_2 + k_3 - k_4)y] \\ + a_6 \exp[+i(k_2 + k_3 + k_4)y] + a_7 \exp[+i(k_2 - k_3 - k_4)y] + a_8 \exp[+i(k_2 - k_3 + k_4)y]), \end{aligned} \quad (22)$$

where k_2 , k_3 , and k_4 are the momenta associated with the mode expansions of $C(y)$, $\psi^\dagger(y)$, and $\psi(y)$, respectively. The operators a_i ($i = 1, 2, 3, \dots, 8$) are

$$\begin{aligned} a_1 &= c(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & a_2 &= c(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4), \\ a_3 &= c(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & a_4 &= c(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4), \\ a_5 &= c^\dagger(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & a_6 &= c^\dagger(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4), \\ a_7 &= c^\dagger(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & a_8 &= c^\dagger(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4). \end{aligned} \quad (23)$$

It will be noted that we have not written operators a_i in the normal ordered form and maintained the order as they have appeared (because the relevant part of Q_b is still not in the quadratic form). Ultimately, we have to compute the anticommutator between the eight terms of the relevant part of Q_b and the two terms of the mode expansion of $\psi(x, t)$ [cf. (15)] that are written in terms of operators $b^\alpha(k)$ and $(d^\alpha)^\dagger(k)$.

It is clear, from the expansion of $\psi(x, t)$ and the above eight terms, that there would, in totality, be sixteen anticommutators in the computation of $\{\psi, Q_b\}$. At this stage, the fermionic (i.e. $\psi^2(x, t) = 0$, $(\psi^\dagger)^2(x, t) = 0$) nature of the fields $\psi(x, t)$ and $\psi^\dagger(x, t)$ helps us immensely. For instance, it can be seen that the following relationships amongst the creation and annihilation operators ensue from the first condition $\psi^2(x, t) = 0$, namely

$$\psi^2(x, t) = \frac{1}{2}\{\psi(x, t), \psi(x, t)\} = 0 \implies \{b^\alpha(k), b^\beta(k')\} = 0, \quad \{(d^\alpha)^\dagger(k), (d^\beta)^\dagger(k')\} = 0, \quad \{b^\alpha(k), (d^\beta)^\dagger(k')\} = 0. \quad (24)$$

Similarly $(\psi^\dagger)^2(x, t) = 0$ implies the following

$$\{(b^\alpha)^\dagger(k), (b^\beta)^\dagger(k')\} = 0, \quad \{(b^\alpha)^\dagger(k), d^\beta(k')\} = 0, \quad \{d^\alpha(k), d^\beta(k')\} = 0. \quad (25)$$

Note that the relations (24) and (25) are also valid for $k = k'$ (because of the limiting case). The above canonical brackets help us in evaluating sixteen anticommutators in a simple manner because many of them vanish.

It is straightforward to check that the first term of the mode expansion of $\psi(x, t)$ [cf. (15)] and the first term of the expansion (22) lead to the following anticommutator (for $t = 0$), namely;

$$\sum_{\beta} \sum_{\gamma} \sum_{\sigma} \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} \{b^\beta(k_1)u^\beta(k_1) e^{-ik_1 x}, a_1 e^{-i(k_2 - k_3 + k_4)y}\}, \quad (26)$$

where k_1, k_2, k_3, k_4 are the momenta corresponding to the fields $\psi(x)$, $C(y)$, $\psi^\dagger(y)$ and $\psi(y)$, respectively, in the phase space. It is clear from the expression for “ a_1 ” [cf. (23)] and the relevant anticommutator of (24) that we have

$$\begin{aligned} & - \sum_{\beta} \sum_{\gamma} \sum_{\sigma} \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} c(k_2) \{b^\beta(k_1), (b^\gamma)^\dagger(k_3)\} u^\beta(k_1) (u^\gamma)^\dagger(k_3) b^\sigma(k_4) u^\sigma(k_4) \\ & \times e^{-ik_1 x} e^{-i(k_2 - k_3 + k_4)y}. \end{aligned} \quad (27)$$

Here a couple of points are to be noted. First, since we are exploiting here the equal-time anticommutators, all the field expansions have been written for $t = 0$. Second, since the (anti-)ghost fields are decoupled from the rest of the theory, the annihilation and creation operators $c(k_2)$ and $c^\dagger(k_2)$ anticommute with the rest of the fermionic creation and annihilation operators. This can also be verified by the fact that, under the ghost transformations, the Dirac field ψ does *not* transform ($s_g \psi = i[\psi, Q_g] = 0$). Taking the help of Q_g from (10) and $\psi(x, t)$ from (15), it is clear that $\{b^\beta(k_1), c(k_2)\} = 0$, $\{b^\beta(k_1), c^\dagger(k_2)\} = 0$, $\{(d^\beta)^\dagger(k_1), c(k_2)\} = 0$, $\{(d^\beta)^\dagger(k_1), c^\dagger(k_2)\} = 0$, etc., where we have used the expression for Q_g when it is expressed in terms of the creation and annihilation operators as: $Q_g = - \int dk [\bar{c}^\dagger(k) c(k) + c^\dagger(k) \bar{c}(k)]$ (see, e.g. [3]).

Comparing the above exponential with the exponential of the first term [cf. (19)] of the l.h.s. (i.e. $C(x)\psi(x)$), it is straightforward that if we choose

$$\{b^\beta(k_1), (b^\gamma)^\dagger(k_3)\} = -\delta^{\beta\gamma} \delta(k_1 - k_3), \quad (28)$$

we can match the exponential of the first term of the l.h.s. [cf. (19)] if $k_2 \rightarrow k$ and $k_4 \rightarrow k'$ because we have the following definition of the Dirac δ -function in the space part of the 2D spacetime, manifold, namely;

$$\int \frac{dk_1}{(2\pi)} e^{-i(k_1 x - k_3 y)} \Big|_{k_1=k_3} = \delta(x - y). \quad (29)$$

With input from (28), we obtain explicitly the following expression

$$\begin{aligned} & \sum_{\beta} \sum_{\gamma} \sum_{\sigma} \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} c(k_2) \delta^{\beta\gamma} \delta(k_1 - k_3) u^\beta(k_1) (u^\gamma)^\dagger(k_3) b^\sigma(k_4) u^\sigma(k_4) e^{-ik_1 x} e^{-i(k_2 - k_3 + k_4)y} \\ & = \sum_{\sigma} \int dy \frac{dk_1 dk_2 dk_4}{(2\pi)^2 2k_1^0 \sqrt{(2k_2^0 2k_4^0)}} e^{-ik_1(x-y)} e^{-i(k_2 + k_4)y} c(k_2) \left(\sum_{\beta} u^\beta(k_1) \bar{u}^\beta(k_1) \right) \gamma^0 b^\sigma(k_4) u^\sigma(k_4), \end{aligned} \quad (30)$$

where we have inserted $(\gamma^0)^2 = I$ appropriately at a suitable place.

The above type of expression also emerges from the anticommutator of the second term of $\psi(x, t)$ [cf. (15)] with the third term of (22), namely

$$\begin{aligned} & \sum_{\beta} \sum_{\gamma} \sum_{\sigma} \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} \{ (d^\beta)^\dagger(k_1) v^\beta(k_1) e^{+ik_1 x}, a_3 e^{-i(k_2+k_3+k_4)y} \} \\ &= - \sum_{\beta} \sum_{\gamma} \sum_{\sigma} \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} c(k_2) \{ (d^\beta)^\dagger(k_1), d^\gamma(k_3) \} \\ & \quad \times v^\beta(k_1) (v^\gamma)^\dagger(k_3) b^\sigma(k_4) u^\sigma(k_4) e^{ik_1 x} e^{-i(k_2+k_3+k_4)y}, \end{aligned} \quad (31)$$

due to the fact that $\{ (d^\beta)^\dagger(k_1), b^\sigma(k_4) \} = 0$ [cf. (24)] and $c(k_2)$ does anticommute with the operator $(d^\beta)^\dagger(k_1)$.

Finally, we obtain the analogue of equation (30) as

$$\sum_{\sigma} \int dy \frac{dk_1 dk_2 dk_4}{(2\pi)^2 2k_1^0 \sqrt{(2k_2^0 2k_4^0)}} e^{+ik_1(x-y)} e^{-i(k_2+k_4)y} c(k_2) \left(\sum_{\beta} v^\beta(k_1) \bar{v}^\beta(k_1) \right) \gamma^0 b^\sigma(k_4) u^\sigma(k_4), \quad (32)$$

if we exploit the anticommutator $\{ (d^\beta)^\dagger(k_1), d^\gamma(k_3) \} = -\delta^{\beta\gamma} \delta(k_1 - k_3)$. Exploiting the trick of our Appendix A and using Eq. (11), it can be checked that the sum of (30) and (32) leads to

$$\sum_{\sigma} \int \frac{dk_2 dk_4}{(2\pi) \sqrt{(2k_2^0 2k_4^0)}} e^{-i(k_2+k_4)x} c(k_2) b^\sigma(k_4) u^\sigma(k_4), \quad (33)$$

which is the first term (modulo “e”) of the l.h.s. of (19) in the limit $k_2 \rightarrow k, k_4 \rightarrow k'$ and $\sigma \rightarrow \alpha$. In exactly similar fashion, if we use (62) (see below), it can be checked that the sum of the anticommutators between

(i) The first term of $\psi(x, t)$ and the second term of (22) and that of the second term of $\psi(x, t)$ with the fourth term of (22) yields the second term of the l.h.s. [cf. Eq. (19)],

(ii) The first term of $\psi(x, t)$ and the fifth term of (22) and that of the second term of $\psi(x, t)$ with the seventh term of (22) produces the third term of the l.h.s. [cf. Eq. (19)], and

(iii) The first term of $\psi(x, t)$ and the sixth term of (22) and that of the second term of $\psi(x, t)$ with the eighth term of (22) leads to the fourth term of the l.h.s. [cf. Eq. (19)].

We lay emphasis on the fact that it is the comparison of the exponentials from the l.h.s. and r.h.s. of (18) that dictates the non-vanishing brackets to be (62) (see, Sec. 6 below). Rest of the anticommutators amongst the fermionic creation and annihilation operators turn out to be zero. Exactly similar kind of computations, with the anti-BRST charge Q_{ab} , produces the basic canonical brackets to be (62) (see below). We conclude that symmetry transformations (and their generators) of a Hodge theory can replace the mathematical definition of the canonical conjugate momenta.

4 (Anti)-co-BRST Symmetries: Basic Brackets

The Lagrangian density (2) also respects the nilpotent (anti)-co-BRST symmetry transformations $s_{(a)d}$ in the massless ($m = 0$) limit of the Dirac fields. From the (anti)-co-BRST symmetry transformations (5) and the corresponding charges (6), the canonical brackets amongst the creation and annihilation operators can be computed.

The non-vanishing brackets for the free theory (without matter field) has already been derived in our previous work.^[3] We shall focus, therefore, only on the derivation of the canonical brackets for the matter fields of 2D QED. Once again, as has been done with (anti)-BRST charges, the decisive role is played by the definition of the generator ($Q_{(a)d}$) of the continuous symmetries ($s_{(a)d}$) (i.e. $s_{(a)d}\psi = -i\{\psi, Q_{(a)d}\}$). Let us take an example for the sake of clarification, namely;

$$s_d\psi = -i\{\psi, Q_d\} = -ie\bar{C}\gamma_5\psi \implies \{\psi, Q_d\} = e\bar{C}\gamma_5\psi. \quad (34)$$

The term that would contribute from Q_d [cf. (6)] is

$$Q_d^{(R)} = e \int dx \psi^\dagger \gamma_0 \gamma_1 \bar{C} \psi = -e \int dx \psi^\dagger \gamma_5 \bar{C} \psi, \quad (35)$$

for the determination of the continuous symmetry transformation connected with s_d . Here the superscript (R) on the conserved charge Q_d (i.e. $Q_d^{(R)}$) denotes the relevant part of the exact expression for Q_d , which contributes in our computation. Similarly, one can write all the rest of the fermionic transformations $s_{(a)d}$ in terms of the corresponding generators. Following the tricks developed in the case of the nilpotent (anti)-BRST symmetry transformations, it can be checked that exactly the same non-vanishing [cf. (62)] anticommutators (amongst the creation and annihilation operators) emerge from this exercise, too.

To corroborate the above statement, we provide some of the key steps that are needed in the determination of the anticommutators (62) from the continuous symmetry transformations $s_{(a)d}$ generated by the (anti)-co-BRST

charges $Q_{(a)d}$. It is evident from Eqs. (34) and (35) that we have, for the specific case of co-BRST transformation ($s_d\psi = -ie\gamma_5\bar{C}\psi = -i\{\psi, Q_d\}$), the following expression:

$$e\bar{C}\gamma_5\psi = +e\gamma_5\int dy\{\psi(x), \bar{C}(y)\psi^\dagger(y)\psi(y)\}, \quad (36)$$

where we have used $\bar{C}\psi^\dagger + \psi^\dagger\bar{C} = 0$. The above expression, finally, reduces to the following straightforward equality:

$$\bar{C}(x)\psi(x) = \int dy\{\psi(x), \bar{C}(y)\psi^\dagger(y)\psi(y)\}, \quad (37)$$

which is similar in appearance as the corresponding nilpotent BRST transformations [cf. (18), (21)] with the replacement $C \rightarrow \bar{C}$. In fact, the expression in (37) is exactly same as the one connected with the anti-BRST symmetry transformations and their corresponding generator (i.e. anti-BRST charge).

The l.h.s. of (37) (for $t = 0$) can be written, analogous to (19), as

$$\int \frac{dkdk'}{(2\pi)\sqrt{(2k_0 2k'_0)}} \sum_\alpha (\bar{c}(k)b^\alpha(k')u^\alpha(k') \exp[-i(k+k')x] + \bar{c}(k)(d^\alpha)^\dagger(k')v^\alpha(k') \exp[-i(k-k')x] + \bar{c}^\dagger(k)b^\alpha(k')u^\alpha(k') \times \exp[+i(k-k')x] + \bar{c}^\dagger(k)(d^\alpha)^\dagger(k')v^\alpha(k') \exp[+i(k+k')x]). \quad (38)$$

Similarly, the r.h.s. of (37) can be written, analogous to (22), as

$$\sum_\gamma \sum_\sigma \int dy \int \frac{dk_2 dk_3 dk_4}{(2\pi)^{3/2} \sqrt{(2k_2^0 2k_3^0 2k_4^0)}} (b_1 \exp[-i(k_2 - k_3 + k_4)y] + b_2 \exp[-i(k_2 - k_3 - k_4)y] + b_3 \exp[-i(k_2 + k_3 + k_4)y] + b_4 \exp[-i(k_2 + k_3 - k_4)y] + b_5 \exp[+i(k_2 + k_3 - k_4)y] + b_6 \exp[+i(k_2 + k_3 + k_4)y] + b_7 \exp[+i(k_2 - k_3 - k_4)y] + b_8 \exp[+i(k_2 - k_3 + k_4)y]), \quad (39)$$

where we have taken $t = 0$ for the relevant portion of the Q_d because it is a conserved quantity. Furthermore, we are using here the equal-time anticommutators, which enforce us to take the mode expansion of $\psi(x, t)$ at $t = 0$, too. The momenta k_2, k_3 , and k_4 , in the above, are associated with the mode expansions of $\bar{C}(y), \psi^\dagger(y)$, and $\psi(y)$, respectively. The explicit form of the operators b_i ($i = 1, 2, 3, \dots, 8$), in the above, are

$$\begin{aligned} b_1 &= \bar{c}(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & b_2 &= \bar{c}(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4), \\ b_3 &= \bar{c}(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & b_4 &= \bar{c}(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4), \\ b_5 &= \bar{c}^\dagger(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & b_6 &= \bar{c}^\dagger(k_2)(b^\gamma)^\dagger(k_3)(u^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4), \\ b_7 &= \bar{c}^\dagger(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)b^\sigma(k_4)u^\sigma(k_4), & b_8 &= \bar{c}^\dagger(k_2)d^\gamma(k_3)(v^\gamma)^\dagger(k_3)(d^\sigma)^\dagger(k_4)v^\sigma(k_4). \end{aligned} \quad (40)$$

It should be noted that we have not yet written the operators b_i in the normal ordered form and we have maintained the order as they appear in the expression for a portion of Q_d (where the local fields $\bar{C}(y), \psi^\dagger(y)$, and $\psi(y)$ are present).

Exploiting the inputs from (24) and the fact that the ghost transformation for the field $\psi(x)$ is zero (i.e. $s_g\psi = +i\{\psi, Q_g\} = 0$), it is clear, from the expression $Q_g = i\int dx[C\dot{\bar{C}} + \bar{C}\dot{C}] \equiv -\int dk[\bar{c}^\dagger(k)c(k) + c^\dagger(k)\bar{c}(k)]$ [cf. (10)] and expansion in (15) (for the field $\psi(x)$), that the following anticommutators are true, namely;

$$\begin{aligned} \{b^\beta(k_1), c(k_2)\} &= 0, & \{b^\beta(k_1), c^\dagger(k_2)\} &= 0, & \{b^\beta(k_1), \bar{c}(k_2)\} &= 0, & \{b^\beta(k_1), \bar{c}^\dagger(k_2)\} &= 0, \\ \{(d^\beta)^\dagger(k_1), c(k_2)\} &= 0, & \{(d^\beta)^\dagger(k_1), c^\dagger(k_2)\} &= 0, & \{(d^\beta)^\dagger(k_1), \bar{c}(k_2)\} &= 0, & \{(d^\beta)^\dagger(k_1), \bar{c}^\dagger(k_2)\} &= 0. \end{aligned} \quad (41)$$

We note that all the arguments of the BRST transformations (i.e. $s_b\psi = -i\{\psi, Q_b\} = -ieC\psi$), discussed in the main body of our previous section, would be applicable in our present discussion (connected with the continuous symmetry transformation $s_d\psi = -i\{\psi, Q_d\} = -ie\gamma_5\bar{C}\psi$) as well. It is obvious now that we shall obtain, ultimately, the non-vanishing brackets as (62) and the rest of the brackets would turn out to be zero. Similar kind of computations can be performed with Q_{ad} which will, once again, lead to the derivation of the non-vanishing canonical brackets (62). We would like to lay emphasis on the fact that the normal ordering in (40) would not affect the main results of our analysis. In other words, the non-vanishing brackets (62) would remain unaffected by normal ordering in Eq. (40).

5 Unique Bosonic Symmetry: Basic Brackets

We have a unique bosonic symmetry (s_ω) in our theory. The generator (Q_ω) of the transformations has been quoted in Sec. 2 [cf. (8)]. This bosonic charge Q_ω generates the bosonic transformations s_ω on the fermionic fields as^[8-9]

$$s_\omega\psi = -ie[\gamma_5(\partial \cdot A) + E]\psi, \quad s_\omega\bar{\psi} = +ie\bar{\psi}[E - \gamma_5(\partial \cdot A)], \quad (42)$$

which constitute the symmetry invariance of the Lagrangian density (2). The key equation [consistent with (13)] that leads to the determination of the canonical brackets amongst the creation and annihilation operators is the following

relationship:

$$s_\omega \psi = +i[\psi, Q_\omega] = -ie[\gamma_5(\partial \cdot A) + E]\psi, \quad s_\omega \bar{\psi} = +i[\bar{\psi}, Q_\omega] = +ie\bar{\psi}[E - \gamma_5(\partial \cdot A)]. \quad (43)$$

It is evident that, finally, we have to compute $[\psi, Q_\omega] = -e[\gamma_5(\partial \cdot A) + E]\psi$ and $[\bar{\psi}, Q_\omega] = +e\bar{\psi}[E - \gamma_5(\partial \cdot A)]$ for the determination of the canonical anticommutators amongst the creation and annihilation operators. For this purpose, the portion of the bosonic charge (Q_ω) that would contribute is: $Q_\omega^{(R)} = e \int dx [(\partial \cdot A)\psi^\dagger \gamma_5 \psi + E\psi^\dagger \psi]$ (where $(\gamma_0)^2 = I$, $\gamma_5 = \gamma^0 \gamma^1$ are used). Here the superscript (R) denotes the relevant part of Q_ω that contribute in the computation of $s_\omega \psi$ and $s_\omega \bar{\psi}$. It is to be emphasized that, for the free theory (without matter fields), the basic canonical brackets have been determined in our previous work.^[3] This is why, we have concentrated only on the determination of brackets for the matter fields from their symmetry transformations.

Once again, applying the rules of the commutators and anticommutators, we end up with the non-vanishing anticommutators as (62) (see below) that have been derived by canonical method. Some of the key steps of our present exercise are illustrated here for the readers' convenience. It is evident from (43) that

$$-e[\gamma_5(\partial \cdot A) + E]\psi = [\psi, Q_\omega], \quad (44)$$

where the term that contributes from Q_ω is: $e \int dy [(\partial \cdot A)\gamma_5 \psi^\dagger \psi + E\psi^\dagger \psi]$. Thus, the factor "e" cancels from l.h.s. and r.h.s. of Eq. (44). Plugging in the expansions from (12) (with $\partial \cdot A = \partial_\mu A^\mu$, $E = -\varepsilon^{\mu\nu} \partial_\mu A_\nu$) and (15), we obtain the l.h.s. of (44) (for $t = 0$) as follows:

$$i \int \frac{dk dk'}{(2\pi)\sqrt{(2k_0 2k'_0)}} (\varepsilon^{\mu\nu} k_\nu + \gamma_5 k^\mu) \sum_\alpha [a_\mu^\dagger(k) u^\alpha(k') b^\alpha(k') e^{+i(k-k')x} + a_\mu^\dagger(k) (d^\alpha)^\dagger(k') v^\alpha(k') e^{+i(k+k')x} - a_\mu(k) u^\alpha(k') b^\alpha(k') e^{-i(k+k')x} - a_\mu(k) (d^\alpha)^\dagger(k') v^\alpha(k') e^{-i(k-k')x}], \quad (45)$$

where k and k' are the momenta in the phase space corresponding to the field expansions in (12) (for $A_\mu(x)$ field) and (15) (for $\psi(x)$ field).

The relevant expression for Q_ω on the r.h.s. is: $\int dy [(\partial \cdot A)(y)\gamma_5 + E(y)]\psi^\dagger(y)\psi(y)$. Exploiting the expansions of (12), (15), and (16) at $t = 0$, we obtain the following expression for the relevant Q_ω operator, namely;

$$-i \sum_\gamma \sum_\sigma \int dy \frac{dk_2 dk_3 dk_4}{(2\pi)^{3/2} \sqrt{(2k_2^0 2k_3^0 2k_4^0)}} (\varepsilon^{\mu\nu} k_{2\nu} + \gamma_5 k_2^\mu) [c_\mu^{(1)} \exp(+i(k_2 + k_3 - k_4)y) + c_\mu^{(2)} \exp(+i(k_2 + k_3 + k_4)y) + c_\mu^{(3)} \exp(+i(k_2 - k_3 - k_4)y) + c_\mu^{(4)} \exp(+i(k_2 - k_3 + k_4)y) + c_\mu^{(5)} \exp(-i(k_2 - k_3 + k_4)y) + c_\mu^{(6)} \exp(-i(k_2 - k_3 - k_4)y) + c_\mu^{(7)} \exp(-i(k_2 + k_3 + k_4)y) + c_\mu^{(8)} \exp(-i(k_2 + k_3 - k_4)y)], \quad (46)$$

where k_2 , k_3 , and k_4 are the momenta corresponding to the fields $A_\mu(y)$, $\psi^\dagger(y)$, and $\psi(y)$, respectively. The operators $c_\mu^{(i)}$ ($i = 1, 2, 3, \dots, 8$) are

$$\begin{aligned} c_\mu^{(1)} &= a_\mu^\dagger(k_2) (b^\gamma)^\dagger(k_3) (u^\gamma)^\dagger(k_3) b^\sigma(k_4) u^\sigma(k_4), & c_\mu^{(2)} &= a_\mu^\dagger(k_2) (b^\gamma)^\dagger(k_3) (u^\gamma)^\dagger(k_3) (d^\sigma)^\dagger(k_4) v^\sigma(k_4), \\ c_\mu^{(3)} &= a_\mu^\dagger(k_2) d^\gamma(k_3) (v^\gamma)^\dagger(k_3) b^\sigma(k_4) u^\sigma(k_4), & c_\mu^{(4)} &= a_\mu^\dagger(k_2) d^\gamma(k_3) (v^\gamma)^\dagger(k_3) (d^\sigma)^\dagger(k_4) v^\sigma(k_4), \\ c_\mu^{(5)} &= -a_\mu(k_2) (b^\gamma)^\dagger(k_3) (u^\gamma)^\dagger(k_3) b^\sigma(k_4) u^\sigma(k_4), & c_\mu^{(6)} &= -a_\mu(k_2) (b^\gamma)^\dagger(k_3) (u^\gamma)^\dagger(k_3) (d^\sigma)^\dagger(k_4) v^\sigma(k_4), \\ c_\mu^{(7)} &= -a_\mu(k_2) d^\gamma(k_3) (v^\gamma)^\dagger(k_3) b^\sigma(k_4) u^\sigma(k_4), & c_\mu^{(8)} &= -a_\mu(k_2) d^\gamma(k_3) (v^\gamma)^\dagger(k_3) (d^\sigma)^\dagger(k_4) v^\sigma(k_4), \end{aligned} \quad (47)$$

where the order of the creation and annihilation operators has been maintained (without any kind of implementation of the normal ordering).

Now the stage is set for the explicit computation of the bracket $[\psi, Q_\omega]$. From expansion of $\psi(x)$ in (15), it is evident that there would be sixteen commutators but many of them would vanish due to our earlier arguments. It can be checked that commutator of the first term of expansion of $\psi(x)$ (in (15)) and the first term of the relevant part of Q_ω in (46) (for $t = 0$) yields:

$$-i \sum_\beta \sum_\gamma \sum_\sigma \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} (\varepsilon^{\mu\nu} k_{2\nu} + \gamma_5 k_2^\mu) [b^\beta(k_1), a_\mu^\dagger(k_2) (b^\gamma)^\dagger(k_3) b^\sigma(k_4)] u^\beta(k_1) (u^\gamma)^\dagger(k_3) u^\sigma(k_4) \times \exp(-ik_1 x) \exp(+i(k_2 + k_3 - k_4)y). \quad (48)$$

Exploiting the appropriate rules of the commutators we obtain, ultimately, the following existing bracket, namely;

$$-i \sum_\beta \sum_\gamma \sum_\sigma \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} (\varepsilon^{\mu\nu} k_{2\nu} + \gamma_5 k_2^\mu) a_\mu^\dagger(k_2) \{b^\beta(k_1), (b^\gamma)^\dagger(k_3)\} \times b^\sigma(k_4) u^\beta(k_1) (u^\gamma)^\dagger(k_3) u^\sigma(k_4) \exp[-ik_1 x + i(k_2 + k_3 - k_4)y], \quad (49)$$

where we have already exploited (24) and the commutator $[b^\beta(k_1), a_\mu^\dagger(k_2)] = 0$ due to the fact that there is no explicit mixing between the fields $A_\mu(x)$ and $\psi(x)$ as far as our basic continuous symmetry transformations $s_{(a)b}$ and $s_{(a)d}$ are concerned. Furthermore, under the ghost transformations, fermionic fields ψ and ψ^\dagger do not transform at all. Hence, there is no mixing here as well.

The exponentials of the first term of the l.h.s. of (44) (which is explicitly expressed as (45)) and the above commutator would match if we exploit the relevant canonical bracket of (62) i.e. $\{b^\beta(k_1), (b^\gamma)^\dagger(k_3)\} = -\delta^{\beta\gamma}\delta(k_1 - k_3)$. As a consequence, we obtain the following explicit expression:

$$i \sum_\sigma \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} (\varepsilon^{\mu\nu} k_{2\nu} + \gamma_5 k_2^\mu) \delta(k_1 - k_3) \\ \times a_\mu^\dagger(k_2) \sum_\beta (u^\beta(k_1) \bar{u}^\beta(k_1)) \gamma^0 b^\sigma(k_4) u^\sigma(k_4) e^{-ik_1(x-y)} e^{+i(k_2-k_4)y}. \quad (50)$$

Similar kind of terms would be generated from the commutator of the second term of ψ [cf. (15)] and the third term of the relevant portion of Q_ω [cf. (46)]. This can be expressed as

$$i \sum_\sigma \int dy \frac{dk_1 dk_2 dk_3 dk_4}{(2\pi)^2 \sqrt{(2k_1^0 2k_2^0 2k_3^0 2k_4^0)}} (\varepsilon^{\mu\nu} k_{2\nu} + \gamma_5 k_2^\mu) \delta(k_1 - k_3) \\ \times a_\mu^\dagger(k_2) \sum_\beta (v^\beta(k_1) \bar{v}^\beta(k_1)) \gamma^0 b^\sigma(k_4) u^\sigma(k_4) e^{+ik_1(x-y)} e^{+i(k_2-k_4)y}. \quad (51)$$

The sum of (50) and (51) (with the help of (17), definition and properties of the Dirac δ -function) produces the first term of the l.h.s. [cf. (45)], where one has to make the replacements: $k_2 \rightarrow k$, $k_4 \rightarrow k'$, and $\sigma \rightarrow \alpha$.

It is now straightforward to check that the rest of the terms of the l.h.s. [cf. (45)] can also be produced with various combinations of commutators from the first and second terms of ψ in (15) and some appropriate terms of (46). It is crucial to note that, in all these computations, the canonical brackets (62) play very decisive roles as they are the root cause behind the emergence of the correct powers of the exponentials on the r.h.s. In other words, we conclude that an accurate comparison of the exponentials from the l.h.s. and r.h.s. of (44) leads to the derivation of the canonical brackets amongst the creation and annihilations operators of (62). We would like to lay emphasis on the fact that the form of (62) would remain unaffected even if we perform the normal ordering in (47).

We also very briefly comment, in this section, on the (anti)commutators generated by the continuous ghost symmetry transformations. We note that, under the ghost continuous symmetry transformations [cf. (9)], all the physical fields $(A_\mu, \psi, \bar{\psi})$ remain unchanged (i.e. $s_g A_\mu = s_g \psi = s_g \bar{\psi} = 0$). Using $Q_g = -\int dk [\bar{c}^\dagger(k) c(k) + c^\dagger(k) \bar{c}(k)]$ and (13), it is elementary to check that the (anti-)ghost creation and annihilation operators $(c(k), c^\dagger(k), \bar{c}(k), \bar{c}^\dagger(k))$ anticommute with such kind of fermionic operators that appear in the mode expansions of ψ and ψ^\dagger (i.e. $b^\alpha(k), (b^\alpha)^\dagger(k), d^\alpha(k), (d^\alpha)^\dagger(k)$) and commute with the bosonic operators (i.e. $a_\mu(k), a_\mu^\dagger(k)$) that appear in the normal mode expansion of the field A_μ . As evident from our earlier discussions, we have taken into account the anticommutativity property of $c(k), c^\dagger(k), \bar{c}(k), \bar{c}^\dagger(k)$ with the creation and annihilation operators of ψ and ψ^\dagger and commutativity property with the operators

$a_\mu(k)$ and $a_\mu^\dagger(k)$ of the bosonic gauge field A_μ . The above properties are consistent with our argument.

6 Canonical Quantization Scheme: Lagrangian Formalism

For the sake of completeness of our present work, we derive here the canonical brackets for all the creation and annihilation operators of the interacting 2D model of Hodge theory (i.e. 2D QED). It is evident that the canonical conjugate momenta from the Lagrangian density (2), for the basic fields of the free (i.e. $\psi = \bar{\psi} = 0$) theory, are (see, e.g. [3] for details)

$$\Pi^\mu = \frac{\partial \mathcal{L}^{(b)}}{\partial (\partial_0 A_\mu)} = -F^{0\mu} - \eta^{0\mu} (\partial \cdot A), \\ \Pi_C = \frac{\partial \mathcal{L}^{(b)}}{\partial (\partial_0 C)} = +i \dot{C}, \\ \Pi_{\bar{C}} = \frac{\partial \mathcal{L}^{(b)}}{\partial (\partial_0 \bar{C})} = -i \dot{\bar{C}}. \quad (52)$$

As a consequence, we have the following canonical commutator and anti-commutators for the theory in 2D, namely;

$$[A_\mu(x, t), \Pi_\nu(y, t)] = i\eta_{\mu\nu} \delta(x - y), \\ \{\bar{C}(x, t), \Pi_{\bar{C}}(y, t)\} = i\delta(x - y) \Rightarrow \{\bar{C}(x, t), \dot{\bar{C}}(y, t)\} \\ = -\delta(x - y), \\ \{C(x, t), \Pi_C(y, t)\} = i\delta(x - y) \Rightarrow \{C(x, t), \dot{C}(y, t)\} \\ = \delta(x - y). \quad (53)$$

All the rest of the brackets are zero. It is clear that here two of the main ingredients of the canonical quantization scheme have been exploited. These are the elevation of the (graded) Poisson brackets to the canonical (anti)commutators and the spin-statistics theorem. The top entry, in the above, implies the following commutators in terms of the components of the 2D gauge field A_μ and

the corresponding conjugate momenta:

$$\begin{aligned} [A_0(x, t), (\partial \cdot A)(y, t)] &= -i\delta(x - y), \\ [A_i(x, t), E_j(y, t)] &= i\delta_{ij}\delta(x - y). \end{aligned} \quad (54)$$

The above form of the commutators would turn out to be useful later.

To simplify the rest of our computations, we re-express the normal mode expansions of the basic fields [cf. (12)] as^[2]

$$\begin{aligned} A_\mu(x, t) &= \int dk [a_\mu(k)f^*(k, x) + a_\mu^\dagger(k)f(k, x)], \\ C(x, t) &= \int dk [c(k)f^*(k, x) + c^\dagger(k)f(k, x)], \\ \bar{C}(x, t) &= \int dk [\bar{c}(k)f^*(k, x) + \bar{c}^\dagger(k)f(k, x)], \end{aligned} \quad (55)$$

where the new functions

$$f(k, x) = \frac{e^{-ik \cdot x}}{\sqrt{(2\pi 2k_0)}}, \quad f^*(k, x) = \frac{e^{ik \cdot x}}{\sqrt{(2\pi 2k_0)}}, \quad (56)$$

form an orthonormal set because they satisfy the following conditions

$$\begin{aligned} \int dx f^*(k, x) i_{\partial_0}^{\leftarrow} f(k', x) &= \delta(k - k'), \\ \int dx f^*(k, x) i_{\partial_0}^{\leftarrow} f^*(k', x) &= 0, \\ \int dx f(k, x) i_{\partial_0}^{\rightarrow} f(k', x) &= 0. \end{aligned} \quad (57)$$

We have taken into account, in the above, the following standard definition

$$A_{\partial_0}^{\leftarrow} B = A(\partial_0 B) - (\partial_0 A)B, \quad (58)$$

for the operator $i_{\partial_0}^{\leftarrow}$ between two arbitrary non-zero variables A and B . Using the above relations, it is straightforward to check that

$$\begin{aligned} a_\mu(k) &= \int dx A_\mu(x, t) i_{\partial_0}^{\leftarrow} f(k, x), \\ a_\mu^\dagger(k) &= \int dx f^*(k, x) i_{\partial_0}^{\leftarrow} A_\mu(x, t), \\ c(k) &= \int dx C(x, t) i_{\partial_0}^{\leftarrow} f(k, x), \\ \bar{c}(k) &= \int dx \bar{C}(x, t) i_{\partial_0}^{\leftarrow} f(k, x), \\ c^\dagger(k) &= \int dx f^*(k, x) i_{\partial_0}^{\leftarrow} C(x, t), \\ \bar{c}^\dagger(k) &= \int dx f^*(k, x) i_{\partial_0}^{\leftarrow} \bar{C}(x, t). \end{aligned} \quad (59)$$

Thus, we have expressed the creation and annihilation operators in terms of the basic fields and the orthonormal functions $f(k, x)$ and $f^*(k, x)$.

At this stage, a few important comments are in order. First and foremost, it is straightforward to check that only the canonical brackets (11) survive in the explicit computations. Second, there exist six anticommutators from the four fermionic operators $c(k)$, $c^\dagger(k)$, $\bar{c}(k)$, $\bar{c}^\dagger(k)$. Out of

which, four turn out to be zero because of the orthonormality relations (59) and due to the fact that $C^2 = \bar{C}^2 = 0$, $\{C(x, t), \dot{C}(y, t)\} = 0$, $\{\bar{C}(x, t), \dot{\bar{C}}(y, t)\} = 0$. Finally, there exist three basic commutators from the operators $a_\mu(k)$ and $a_\mu^\dagger(k)$. Out of which, two turn out to be zero (i.e. $[a_\mu(k), a_\nu(k')] = [a_\mu^\dagger(k), a_\nu^\dagger(k')] = 0$). The proof for it is simple because the commutation relations in (54) can be recast in the form $[A_\mu(x, t), \dot{A}_\nu(y, t)] = -i\eta_{\mu\nu}\delta(x - y)$ due to the fact that (i) $\dot{A}_0 = (\partial \cdot A) + \partial_i A_i$ and $A_i = E_i + \partial_i A_0$, and (ii) the spatial derivative of the gauge field A_μ commutes with itself (i.e. $[A_\mu(x, t), \partial_i A_\nu(y, t)] = 0$).

It is straightforward to check that the canonical brackets of (53) and (54) lead to the derivation of exactly the same brackets as are listed in (11). This can be checked directly by exploiting the explicit expressions for the creation and annihilation operators quoted in (59) and using the canonical brackets listed in (53) and (54). In this computation, the concept of normal ordering has not yet been exploited because we have not dealt with any non-trivial physical quantity (e.g. Hamiltonian, conserved charges, etc.) for our analysis and computation.

The stage is now set to discuss the canonical quantization in terms of the fermionic creation and annihilation operators of the Dirac fields present in the Lagrangian density (2) for the 2D QED. As pointed out earlier, we have only the conjugate momentum corresponding to the ψ field (i.e. $\Pi_\psi = -i\psi^\dagger(x)$). Thus, the analogue of (53) (for the case of Dirac fields) is

$$\begin{aligned} \{\psi(x, t), \Pi_\psi(y, t)\} &= i\delta(x - y) \Rightarrow \{\psi(x, t), \psi^\dagger(y, t)\} \\ &= -\delta(x - y), \end{aligned} \quad (60)$$

and rest of the relevant (anti)commutators are zero amongst the fermionic variables (ψ, ψ^\dagger) . Furthermore, the Dirac fermionic fields ψ and ψ^\dagger have the zero equal-time (anti)commutators with the rest of the basic fields (i.e. A_μ, C, \bar{C}) and their corresponding conjugate momenta that are derived from (2). It can be checked that the creation and annihilation operators in the expansions of ψ and ψ^\dagger [cf. (15)–(16)], can be explicitly expressed in terms of these fields itself (by exploiting the relationships enumerated in (17)). In their gory details, these operators can be written as

$$\begin{aligned} b^\alpha(k) &= \int \frac{dk e^{-ik \cdot x}}{\sqrt{(2\pi 2k_0)}} (u^\alpha)^\dagger(k) \psi(x, t), \\ (b^\alpha)^\dagger(k) &= \int \frac{dk e^{+ik \cdot x}}{\sqrt{(2\pi 2k_0)}} \psi^\dagger(x, t) u^\alpha(k), \\ d^\alpha(k) &= \int \frac{dk e^{-ik \cdot x}}{\sqrt{(2\pi 2k_0)}} \psi^\dagger(x, t) v^\alpha(k), \\ (d^\alpha)^\dagger(k) &= \int \frac{dk e^{+ik \cdot x}}{\sqrt{(2\pi 2k_0)}} (v^\alpha)^\dagger(k) \psi(x, t), \end{aligned} \quad (61)$$

where we have made use of the useful relations (17) (i.e. $(u^\alpha)^\dagger(k)u^\beta(k) = 2k_0\delta^{\alpha\beta}$, $(v^\alpha)^\dagger(k)v^\beta(k) = 2k_0\delta^{\alpha\beta}$). It is

now straightforward to check that

$$\begin{aligned} \{b^\alpha(k), (b^\beta)^\dagger(k')\} &= -\delta^{\alpha\beta}\delta(k-k'), \\ \{d^\alpha(k), (d^\beta)^\dagger(k')\} &= -\delta^{\alpha\beta}\delta(k-k'), \end{aligned} \quad (62)$$

and rest of the (anti)commutators amongst $b^\alpha(k)$, $(b^\beta)^\dagger(k)$, $d^\alpha(k)$, and $(d^\beta)^\dagger(k)$ are found to be zero where, in these proofs, we have to make use of $\{\psi(x, t), \psi(y, t)\} = 0$, $\{\psi^\dagger(x, t), \psi^\dagger(y, t)\} = 0$ and various relations that exist amongst $u^\alpha(k)$, $(u^\beta)^\dagger(k)$, $v^\alpha(k)$, and $(v^\beta)^\dagger(k)$ (see, e.g. [13] for details). Thus, we obtain the same canonical anticommutators [cf. (28)] as derived earlier by exploiting the continuous and nilpotent BRST symmetry transformations [cf. (18)].

Finally, we conclude that the basic canonical brackets, amongst the creation and annihilation operators of the bosonic and fermionic fields of the 2D QED with Dirac fields, can be derived in a straightforward manner

- (i) from the continuous symmetry considerations, and
- (ii) by exploiting the definition of momenta from the Lagrangian density.

From both the above methods, the basic brackets [cf. (11), (62)] turn out to be exactly the same. Thus, the basic brackets (11) and (62) are hidden, in a subtle way, in the continuous symmetry transformations of our present interacting Hodge theory itself.

7 Conclusions

The central result of our present investigation is the derivation of the basic canonical brackets by exploiting the continuous symmetry transformations that are present in the interacting 2D Abelian 1-form gauge theory with Dirac fields. These brackets exist amongst the creation and annihilation operators that appear in the normal mode expansions of the basic dynamical fields of the interacting theory. In our present endeavor, the key ideas that have been exploited are the spin-statistics theorem, normal ordering (in the expressions for the charges) and continuous symmetry transformations. The last of the above ideas is a novel one and it differs from the standard method of canonical quantization scheme where the classical (graded) Poisson-brackets (with the mathematical definition of the canonical momenta) are promoted to the quantum (anti)commutators in addition to the spin-statistics theorem and normal ordering.

It should be noted that, in some of the standard text books (see, e.g. [14]), the canonical brackets amongst the creation and annihilation operators have been obtained by exploiting the Poincaré operators (like momenta and angular momenta) which are generators of the global spacetime transformations (i.e. translations plus Lorentz rotations). In our case, however, we have exploited only

the continuous internal symmetry transformations connected with the BRST formalism. These continuous symmetry transformations are needed to prove that the 2D QED with Dirac fields is a field-theoretic model for the Hodge theory^[8–9] where the above symmetry transformations (and corresponding charges) provide the physical realizations of the de Rham cohomological operators of differential geometry (see, e.g. [4, 8–9]).

One of the most beautiful observations in our present investigation is the emergence of one and the same non-vanishing basic canonical brackets [cf. (11), (62)] from all the continuous symmetry transformations that are present in the theory. In fact, even though the continuous transformations [cf. (3), (5), (7), (9)] (and their corresponding generators) look drastically different, the hidden basic brackets [cf. (11), (62)], that emerge from the application of (13), are exactly the same. This key observation ensures that the symmetry principles encode in their folds the canonical brackets, too. To the best of our knowledge, our method of the derivation of (11) and (62) is a novel result in the realm of quantization of the gauge field theories.

We have purposely discussed, separately and independently, the free 2D U(1) gauge theory^[3] and the interacting 2D QED with Dirac fields. This is due to the fact that all the conserved charges in the case of the former turn out to be quadratic (bilinear) in fields.^[3] As a consequence, they can be expressed in terms of the creation and annihilation operators in a neat and compact form without any exponentials. This is not the situation with the 2D QED with Dirac fields. It can be seen that the continuous (anti-)BRST and (anti-)co-BRST transformations of the Dirac fields ($\psi(x, t)$ and $\psi^\dagger(x, t)$) are generated by the charges that contain trilinear terms which can not be expressed in terms of the creation and annihilation operators in a compact and neat fashion. Thus, the derivation of the basic canonical brackets, from the above trilinear terms, becomes quite involved. However, we have been able to get rid of this problem and accomplished our goal in a clear fashion. To demonstrate that our method of quantization is general in nature, we have applied it to the BRST quantization of 4D free Abelian 2-form gauge theory,^[3] which also happens to be a tractable field theoretic model for the Hodge theory (see, e.g. [6]).

In our present endeavor, we have considered the 2D QED with a background spacetime manifold which is flat, Minkowskian and commutative in nature. We have demonstrated explicitly the equivalence between the (anti)commutators at the field level and at the level of creation and annihilation operators that appear in the normal mode expansion of the dynamical fields. This equivalence breaks down in the case of 2D Minkowskian spacetime that is taken to be noncommutative. In fact, in a very interesting piece of recent work,^[15] it has been

clearly demonstrated that the results are completely different when one exploits the coordinate coherent states approach for the discussion of the Unruh effect and Hawking radiation by adopting two different kinds of quantization procedures (because it is well-known that, in context of the noncommutative field theories, the canonical and non-canonical quantization procedures do exist). However, in our present investigation on the 2D QED, we have not adopted the coordinate coherent states approach and our entire discussion is confined to the commutative space-time only. Furthermore, our novel approach of quantization procedure is valid only for the special class of gauge field theoretic models which present tractable examples of Hodge theory.

Symmetry principles, as is well-known, have already played decisive roles in the developments of modern theoretical physics. We firmly believe that the key aspects of symmetry principles, highlighted in our present investigation, can be generalized to the description of higher p -form ($p \geq 2$) gauge fields that appear in the excitations of the (super)strings. Thus, our present endeavor should be taken as our modest step towards our main goal of

studying various aspects of the free 4D Abelian 2-form (see, e.g. [3]) and higher p -form ($p \geq 3$) gauge theories within the framework of BRST formalism. In this context, it is gratifying to point out that we have already applied the idea of our present work to the free 4D Abelian 2-form gauge theory and derived the correct basic canonical brackets amongst the creations and annihilation operators by exploiting the continuous symmetries of the theory.^[3] We hope to pursue, some of the above mentioned issues (especially related with the higher p -form ($p \geq 3$) gauge theories), in the future, too.^[16]

Appendix

We establish here the consistency between the anticommutators of various types amongst the creation and annihilation operators of the Dirac fields and the ones that exist between the field variables themselves (i.e. $\{\psi(x, t), \psi(y, t)\} = 0$, $\{\psi^\dagger(x, t), \psi^\dagger(y, t)\} = 0$, $\{\psi(x, t), \psi^\dagger(y, t)\} = -\delta(x - y)$). To this end in mind, it can be seen that the l.h.s. of the last anticommutator can be expressed, using expansions (15) and (16), as

$$\begin{aligned} \{\psi(x, t), \psi^\dagger(y, t)\} = & \frac{1}{2\pi} \int \frac{dk dk'}{\sqrt{(2k_0 2k'_0)}} \sum_\alpha \sum_\beta [\{b^\alpha(k), (b^\beta)^\dagger(k')\} u^\alpha(k) (u^\beta)^\dagger(k') \exp[i(k_0 - k'_0)t - i(kx - k'y)] \\ & + \{(d^\alpha)^\dagger(k), d^\beta(k')\} v^\alpha(k) (v^\beta)^\dagger(k') \exp[-i(k_0 - k'_0)t + i(kx - k'y)] \\ & + \{(d^\alpha)^\dagger(k), (b^\beta)^\dagger(k')\} v^\alpha(k) (u^\beta)^\dagger(k') \exp[-i(k_0 + k'_0)t + i(kx + k'y)] \\ & + \{b^\alpha(k), d^\beta(k')\} u^\alpha(k) (v^\beta)^\dagger(k') \exp[i(k_0 + k'_0)t - i(kx + k'y)]], \end{aligned} \quad (A1)$$

where we have not taken $t = 0$ for the sake of generality and transparency. It is obvious, from this canonical bracket, that the r.h.s. of the above equation is the Dirac delta-function [cf. (29)]. A comparison with the definition of the delta-function implies that the following anticommutators are true

$$\{b^\alpha(k), d^\beta(k')\} = 0, \quad \{(b^\alpha)^\dagger(k), (d^\beta)^\dagger(k')\} = 0. \quad (A2)$$

The above conclusion is drawn because of the exponentials that are present in the Dirac delta-function (29) (i.e. r.h.s) and (A1) (i.e. l.h.s.). It can be easily seen that the exponentials in third and fourth terms of (A1) can not produce $\delta(x - y)$ for k and k' being positive definite.

Furthermore, if we assume the other remaining brackets to be (62), we obtain the following expression for (A1), namely;

$$\begin{aligned} -\frac{1}{2\pi} \int \frac{dk dk'}{\sqrt{(2k_0 2k'_0)}} \sum_\alpha \delta(k - k') \\ \times (u^\alpha(k) (u^\alpha)^\dagger(k') \exp[-ik(x - y)] \\ + v^\alpha(k) (v^\alpha)^\dagger(k') \exp[+ik(x - y)]), \end{aligned} \quad (A3)$$

where we have taken into account the fact that $k_0 = k'_0$ because of the presence of the $\delta(k - k')$ (implying $e^{i(k_0 - k'_0)t} \delta(k - k') = \delta(k - k')$). Inserting $(\gamma^0)^2 = I$ ap-

propriately and using (17), we obtain the following

$$\begin{aligned} -\frac{1}{2\pi} \int \frac{dk}{2k_0} [(k_0 \gamma_0 - k \gamma_1 + m) \gamma_0 \exp[-ik(x - y)] \\ + (k_0 \gamma_0 - k \gamma_1 - m) \gamma_0 \exp[+ik(x - y)]]. \end{aligned} \quad (A4)$$

Changing $k \rightarrow -k$ in the second term, we obtain the following final expression

$$-\int \frac{dk}{(2\pi)(2k_0)} (2k_0) \exp[-ik(x - y)], \quad (A5)$$

which is nothing other than the Dirac delta-function.

Similarly, it can be checked that there is an absolute consistency between the canonical anticommutators that emerge from the Lagrangian density (2):

$$\{\psi(x, t), \psi(y, t)\} = \{\psi^\dagger(x, t), \psi^\dagger(y, t)\} = 0, \quad (A6)$$

and the following anticommutators amongst the creation and annihilation operators that ensue due to the mode expansions of the above fields, namely

$$\begin{aligned} \{b^\alpha(k), b^\beta(k')\} = 0, \quad \{d^\alpha(k), d^\beta(k')\} = 0, \\ \{b^\alpha(k), (d^\beta)^\dagger(k')\} = 0, \quad \{(b^\alpha)^\dagger(k), (b^\beta)^\dagger(k')\} = 0, \\ \{(d^\alpha)^\dagger(k), (d^\beta)^\dagger(k')\} = 0, \quad \{(b^\alpha)^\dagger(k), d^\beta(k')\} = 0. \end{aligned} \quad (A7)$$

Thus, we have demonstrated an absolute consistency between the canonical brackets consisting of $\psi(x, t)$ and

$\psi^\dagger(x, t)$ and the corresponding canonical brackets (anti-commutators) existing amongst the creation and annihilation operators that appear in the mode expansions of the above fields. In other words, we have established clearly the equivalence between the canonical brackets at the field level and the corresponding brackets at the level of the fermionic creation and annihilation operators, which ap-

pear in the normal mode expansions of the fermionic Dirac fields.

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