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Solutions and Continuum Limits of Two Semi-Discrete Lattice Potential Korteweg-de Vries Equations*

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Abstract *We derive bilinear forms and Casoratian solutions for two semi-discrete potential Korteweg-de Vries equations. Their continuum limits go to the counterparts of the continuous potential Korteweg-de Vries equation.*

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1 Introduction

In recent decades the research of integrability of difference equations has received remarkable progress. Discrete integrable systems have shown deep connections with many branches of mathematics and physics. The famous lattice potential Korteweg-de Vries (lpKdV) equation[1−2] is also known as H1 equation in the Adler–Bobenko–Suris (ABS) list.[3] This equation is

$$(\widehat{\widetilde{u}} - u)(\widetilde{u} - \widehat{u}) = \alpha^2 - \beta^2\,, \tag{1}$$

where $u = u_{n,m}$, $\widetilde{u} = u_{n+1,m}$, $\widehat{u} = u_{n,m+1}$, $\widehat{\widetilde{u}} = u_{n+1,m+1}$, and lattice parameters α for direction n and β for direction m are arbitrary constants. With the transformation

$$u = \alpha n + \beta m + c + w\,, \tag{2}$$

where c is a constant, the lpKdV equation (1) is written as an alternative form:

$$(\widehat{\widetilde{w}} - w + \alpha + \beta)(\widetilde{w} - \widehat{w} + \alpha - \beta) = \alpha^2 - \beta^2\,. \tag{3}$$

It has two semi-discrete continuum limits, one is[4]

$$\partial_\tau(w_{n+1} + w_n) = 2\alpha(w_{n+1} - w_n) + (w_{n+1} - w_n)^2\,, \tag{4}$$

the other is[2,4]

$$\partial_\tau U_{\mathcal{N}} = \frac{U_{\mathcal{N}+1} - U_{\mathcal{N}-1}}{2\alpha + U_{\mathcal{N}+1} - U_{\mathcal{N}-1}}\,. \tag{5}$$

They are respetively known as the straight limit and skew limit of the lpKdV equation (3). In the full continuum limit both of them go to the potential KdV (pKdV) equation

$$W_t + 3(W_x)^2 - W_{xxx} = 0\,. \tag{6}$$

The lpKdV equation (3) has a bilinear form and N-soliton solutions in Casoratian form.[5] In this short paper we investigate bilinear forms and Casoratian solutions of the two semi-discrete potential KdV (sdpKdV) equations (4) and (5). Casoratian solutions will be proved based on the shift formulae of Casoratians developed in Ref. [5]. We will also show that on one side, these results agree with the continuum limits of bilinear form and solutions of the lpKdV equation, and on the other side, in continuum limits these results yield bilinear form and Wronskian solutions of the continuous pKdV equation. These investigations will be done in Sec. 3. Section 4 is for conclusions where we will give some remarks of the continuum limits of the ABS list.

2 Bilinear Form and Casoratian Solution of the lpKdV Equation

By the transformation

$$w = -\frac{g}{f}\,, \tag{7}$$

the lpKdV equation (3) is bilinearized as[5]

$$B_1 \equiv \widehat{g}\widetilde{f} - \widetilde{g}\widehat{f} + (\alpha - \beta)(\widehat{f}\widetilde{f} - f\widehat{\widetilde{f}}) = 0\,, \tag{8a}$$

$$B_2 \equiv g\widehat{\widetilde{f}} - \widehat{\widetilde{g}}f + (\alpha + \beta)(f\widehat{\widetilde{f}} - \widehat{f}\widetilde{f}) = 0\,, \tag{8b}$$

which is solved by Casoratians[5]

$$f = |\widehat{N-1}|\,, \quad g = |\widehat{N-2}, N|\,. \tag{9}$$

The Casoratians are defined as the following. The entry function is

$$\psi_i(n, m; h) = a_i^+ k_i^h (\alpha + k_i)^n (\beta + k_i)^m + a_i^- (-k_i)^h (\alpha - k_i)^n (\beta - k_i)^m\,, \tag{10}$$

where a_i^+, a_i^-, k_i are parameters, and the variable h serves as the column index in the Casoratians. By the above $\psi_i(n, m; h)$, the column vector $\psi(n, m; h)$ is defined by

$$\psi(n, m; h) = (\psi_1(n, m; h), \psi_2(n, m; h), \ldots, \psi_N(n, m; h))^{\mathrm{T}}\,, \tag{11a}$$

and then the Casoratians f and g together with their shorthand notations are defined by

$$f = |\psi(n, m; 0), \psi(n, m; 1), \ldots, \psi(n, m; N-1)|$$

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$$\equiv |0,1,\ldots,N-1| \equiv |\widehat{N-1}|\,, \tag{11b}$$

$$g = |\psi(n,m;0),\ldots,\psi(n,m;N-2),\psi(n,m;N)|$$
$$\equiv |0,1,\ldots,N-2,N| \equiv |\widehat{N-2},N|\,. \tag{11c}$$

3 Continuum Limits and N-Soliton Solutions

In this section, we will solve the sdpKdV equations (4) and (5) using bilinear approach. Solutions will be proved in Casoratian form. We will also investigate continuum limits of bilinear equations as well as Casoratians.

3.1 *Straight Continuum Limit*

Consider the straight continuum limit

$$m \to \infty\,,\ \beta \to \infty\,,\ \text{while } \frac{m}{\beta} = \tau - \tau_0 \sim O(1)\,, \tag{12}$$

where τ_0 is a constant. In this case, let $w_n =: w_n(\tau)$, then $\widehat{w}_n = w_n(\tau + 1/\beta)$ and $\widehat{\widetilde{w}} = w_{n+1}(\tau + 1/\beta)$. Substituting the Taylor expansions of $\widehat{w}$ and $\widehat{\widetilde{w}}$ at τ into Eq. (3), the leading term gives the semi-discrete equation (4).[4]

Making use of the transformation

$$w_n = -(\ln f_n)_\tau\,, \tag{13}$$

Eq. (4) can be bilinearized as

$$(D_\tau^2 - 2\alpha D_\tau) f_{n+1}\cdot f_n = 0\,, \tag{14}$$

where D is the well-known Hirota bilinear operator defined as[6]

$$D_x^m D_t^n a\cdot b = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n a(x,t)b(x',t')|_{x'=x,t'=t}\,.$$

As for solutions to Eq. (14), we have the following.

Proposition 1 The bilinear equation (14) is solved by the Casoratian

$$f = |\widehat{N-1}|\,, \tag{15}$$

which is composed by

$$\psi_i(n,\tau;h) = a_i^+ k_i^h(\alpha + k_i)^n\,\mathrm{e}^{k_i\tau} + a_i^-(-k_i)^h(\alpha - k_i)^n\,\mathrm{e}^{-k_i\tau}\,. \tag{16}$$

To prove this, we need some identities which can be generated by means of the following lemmas.

Lemma 1[7–8] Suppose that $\boldsymbol{B}$ is an $N\times(N-2)$ matrix and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, $\boldsymbol{d}$ are N-th order column vectors, then

$$|\boldsymbol{B},\boldsymbol{a},\boldsymbol{b}||\boldsymbol{B},\boldsymbol{c},\boldsymbol{d}| - |\boldsymbol{B},\boldsymbol{a},\boldsymbol{c}||\boldsymbol{B},\boldsymbol{b},\boldsymbol{d}| + |\boldsymbol{B},\boldsymbol{a},\boldsymbol{d}||\boldsymbol{B},\boldsymbol{b},\boldsymbol{c}| = 0\,. \tag{17}$$

Lemma 2[9] Suppose that Ξ is an $N\times N$ matrix with column vector set $\{\Xi_j\}$; Ω is an $N\times N$ operator matrix with column vector set $\{\Omega_j\}$ and each entry $\Omega_{j,s}$ being an operator. Then we have

$$\sum_{j=1}^N |\Omega_j * \Xi| = \sum_{j=1}^N |(\Omega^{\mathrm{T}})_j * \Xi^{\mathrm{T}}|\,, \tag{18}$$

where for any N-th order column vectors A_j and B_j we define

$$A_j \circ B_j = (A_{1,j}B_{1,j}, A_{2,j}B_{2,j}, \ldots, A_{N,j}B_{N,j})^{\mathrm{T}}\,,$$
$$|A_j * \Xi| = |\Xi_1,\ldots,\Xi_{j-1}, A_j\circ\Xi_j, \Xi_{j+1},\ldots,\Xi_N|\,.$$

With the help of Lemma 2, we take $\Omega_{j,s} = \partial_\tau^2$ or k_i^2 (in fact $\partial_\tau^2\psi_i(n,\tau;h) = \psi_i(n,\tau;h+2) = k_i^2\psi_i(n,\tau;h)$) and $\Xi = |\widehat{N-1}|$, we have an identity

$$\sum_{i=1}^N k_i^2|\widehat{N-1}| = -|\widehat{N-3},N-1,N| + |\widehat{N-2},N+1|\,.$$

Similarly,

$$\sum_{i=1}^N k_i^2|\widehat{N-2},\underset{\sim}{\psi}(N-2)|$$
$$= -|\widehat{N-4},N-2,N-1,\underset{\sim}{\psi}(N-2)| + |\widehat{N-3},N,\underset{\sim}{\psi}(N-2)| + |\widehat{N-1}| + \alpha^2|\widehat{N-2},\underset{\sim}{\psi}(N-2)|\,,$$

where we have made use of the relation

$$\underset{\sim}{\psi}(N) = \psi(N-1) - \alpha\psi(N-2) + \alpha^2\underset{\sim}{\psi}(N-2)\,.$$

Thus by the identity

$$\Big(\sum_{i=1}^N k_i^2|\widehat{N-1}|\Big)|\widehat{N-2},\underset{\sim}{\psi}(N-2)| = \Big(\sum_{i=1}^N k_i^2|\widehat{N-2},\underset{\sim}{\psi}(N-2)|\Big)|\widehat{N-1}|\,,$$

we get the following,

$$(-|\widehat{N-3},N-1,N| + |\widehat{N-2},N+1|)|\widehat{N-2},\underset{\sim}{\psi}(N-2)|$$
$$= |\widehat{N-1}|(-|\widehat{N-4},N-2,N-1,\underset{\sim}{\psi}(N-2)| + |\widehat{N-3},N,\underset{\sim}{\psi}(N-2)| + |\widehat{N-1}| + \alpha^2|\widehat{N-2},\underset{\sim}{\psi}(N-2)|)\,. \tag{19}$$

Now we start our proof for Proposition 1. We prove the down-tilde version of Eq. (14), i.e.,

$$f_{\tau\tau}\underset{\sim}{f} - 2f_\tau\underset{\sim}{f}_\tau + f\underset{\sim}{f}_{\tau\tau} + 2\alpha(f\underset{\sim}{f}_\tau - f_\tau\underset{\sim}{f}) = 0\,, \tag{20}$$

where we have omitted the suffix n and used tilde to denote shifts of n. From $\partial_\tau\psi(h) = \psi(h+1)$ and $\alpha\underset{\sim}{\psi}(h) = \psi(h) - \underset{\sim}{\psi}(h+1)$ we have

$$f_\tau = |\widehat{N-2},N|\,,$$
$$f_{\tau\tau} = |\widehat{N-3},N-1,N| + |\widehat{N-2},N+1|\,,$$
$$-\alpha^{N-2}\underset{\sim}{f} = |\widehat{N-2},\underset{\sim}{\psi}(N-2)|$$
$$-\alpha^{N-2}\underset{\sim}{f}_\tau = |\widehat{N-3},N-1,\underset{\sim}{\psi}(N-2)| - \alpha|\widehat{N-2},\underset{\sim}{\psi}(N-2)|\,,$$
$$-\alpha^{N-2}\underset{\sim}{f}_{\tau\tau} = |\widehat{N-4},N-2,N-1,\underset{\sim}{\psi}(N-2)| + |\widehat{N-3},N,\underset{\sim}{\psi}(N-2)| - \alpha^2|\widehat{N-2},\underset{\sim}{\psi}(N-2)| - |\widehat{N-1}| + 2\alpha^{N-1}\underset{\sim}{f}_\tau\,.$$

Then by substitution we have

$$(-\alpha^{N-2})\times[f_{\tau\tau}\underset{\sim}{f} - 2f_\tau\underset{\sim}{f}_\tau + f\underset{\sim}{f}_{\tau\tau} + 2\alpha(f\underset{\sim}{f}_\tau - f_\tau\underset{\sim}{f})]$$
$$= |\widehat{N-2},\underset{\sim}{\psi}(N-2)|(|\widehat{N-3},N-1,N| + |\widehat{N-2},N+1|) + |\widehat{N-1}|$$
$$\times(|\widehat{N-4},N-2,N-1,\underset{\sim}{\psi}(N-2)|$$

$$+|\widehat{N-3},N,\underset{\sim}{\psi}(N-2)|$$
$$-\alpha^2|\widehat{N-2},\underset{\sim}{\psi}(N-2)|-|\widehat{N-1}|)$$
$$-2|\widehat{N-2},N||\widehat{N-3},N-1,\underset{\sim}{\psi}(N-2)|\,.$$

Using the identity (19) to eliminate some terms and it then turns out that

$$(-\alpha^{N-2})\times[f_{\tau\tau}\underset{\sim}{f}-2f_\tau\underset{\sim}{f}_\tau+f\underset{\sim}{f}_{\tau\tau}+2\alpha(f\underset{\sim}{f}_\tau-f_\tau\underset{\sim}{f})]$$
$$=2|\widehat{N-2},\underset{\sim}{\psi}(N-2)||\widehat{N-3},N-1,N|$$
$$+2|\widehat{N-1}||\widehat{N-3},N,\underset{\sim}{\psi}(N-2)|$$
$$-2|\widehat{N-2},N||\widehat{N-3},N-1,\underset{\sim}{\psi}(N-2)|\,,$$

which is zero in light of Lemma 1. Thus the proof for Proposition 1 is completed.

Next, we will show that the bilinear form (14), transformation (13), solution (15) together with the Casoratian entry (16) agree with the continuum limits of those results of the lpKdV equation.

We start from the transformation (7) and Casoratians (9) with entry functions (10). Thanks to the fraction structure of Eq. (7), we can equivalently take the Casoratian entry function (10) as

$$\psi_i(n,m;h)=a_i^+k_i^h(\alpha+k_i)^n\Big(1+\frac{k_i}{\beta}\Big)^m$$
$$+a_i^-(-k_i)^h(\alpha-k_i)^n\Big(1-\frac{k_i}{\beta}\Big)^m\,. \tag{21}$$

Thus under the limit (12) the above ψ_i goes to Eq. (16) (after the scaling $a_i^\pm\,\mathrm{e}^{\mp k_i\tau_0}\mapsto a_i^\pm$), and the transformation (7) goes to

$$w_n=-\frac{g_n}{f_n}\,, \tag{22}$$

where

$$f_n=:f_n(\tau)=|\widehat{N-1}|\,,\quad g_n=:g_n(\tau)=|\widehat{N-2},N|\,, \tag{23}$$

with entries (16). Here and below we still use the notation ψ_i without confusion. Noting that Eq. (16) provides the relation

$$\partial_\tau\psi_i(n,\tau;h)=\psi_i(n,\tau;h+1) \tag{24}$$

by which we find

$$g_n=|\widehat{N-2},N|=f_{n,\tau}=\partial_\tau f_n\,, \tag{25}$$

Eq. (22) turns out to be

$$w_n=-(\ln f_n)_\tau\,, \tag{26}$$

i.e. the transformation (13).

Now we consider the continuum limit of the bilinear equation (8). Applying the continuum limit (12) on the bilinear form (8a), we get the following leading part,

$$O(1):\quad -g_{n+1}f_n+g_nf_{n+1}+f_nf_{n+1,\tau}-f_{n+1}f_{n,\tau}\,,$$
$$O\Big(\frac{1}{\beta}\Big):\quad g_{n,\tau}f_{n+1}-g_{n+1}f_{n,\tau}-\alpha(f_nf_{n+1,\tau}-f_{n+1}f_{n,\tau})$$
$$+\frac{1}{2}(f_nf_{n+1,\tau\tau}-f_{n+1}f_{n,\tau\tau})\,.$$

The first term yields $g_n-f_{n,\tau}=s(\tau)f_n$ where $s(\tau)$ is an arbitrary function of τ. However, following the continuum result (25) a reasonable choice is to take $s(\tau)=0$, which agrees with $g_n=f_{n,\tau}$. Thus the leading term becomes $O(1/\beta)$ and its coefficient gives nothing but the bilinear equation (14).

By a similar procedure from another bilinear equation (8b) we get the same semi-continuum limit result, i.e., Eq. (14). Besides, it is interesting that the bilinear form (14) is also the continuum limit of the following bilinear equation[4]

$$(\alpha+\beta)\widehat{\underset{\sim}{f}}\widetilde{f}+(\alpha-\beta)\underset{\sim}{f}\widehat{\widetilde{f}}=2\alpha f\widehat{f}\,, \tag{27}$$

which is derived from Cauchy matrix approach[4] and also acts as a bilinear form of Hirota's semi-discrete KdV equation.[10]

Next, let us recall the full-continuum results[4] of the nonlinear equation (4) under the limit

$$n\to\infty\,,\ \alpha\to\infty\,,\quad\text{while}\quad\frac{n}{\alpha}=\xi\sim O(\alpha^2)\,. \tag{28}$$

Take $w_n(\tau)=:\overline{\mathrm{W}}(\tau,\xi)=:W(x,t)$, and the continuous variables x and t to be

$$x=2(\tau+\xi)\,,\quad t=\frac{2\xi}{3\alpha^2}\,, \tag{29}$$

which suggests

$$\partial_\xi=2\partial_x+\frac{2}{3\alpha^2}\partial_t\,,\quad\partial_\tau=2\partial_x\,. \tag{30}$$

Then, firstly, substituting the Taylor expansion of $w_{n+1}(\tau)=\overline{\mathrm{W}}(\tau,\xi+1/\alpha)$ at ξ into Eq. (4), and secondly, replacing those derivatives of $\overline{\mathrm{W}}$ w.r.t. ξ and τ by Eq. (30), one can find the leading term of Eq. (4) is of $O(1/\alpha^2)$ and its coefficient provides the pKdV equation (6).

It is well known that the pKdV equation (6) has the bilinear form

$$(D_tD_x-D_x^4)f\cdot f=0 \tag{31}$$

via the transformation

$$W=-2(\ln f)_x\,, \tag{32}$$

and f can be formulated by a Wronskian w.r.t. x[8]

$$f=|\widehat{N-1}|_{[x]}=|\psi(x,t),\partial_x\psi(x,t),\ldots,\partial_x^{N-1}\psi(x,t)|\,, \tag{33}$$

where

$$\psi_i(x,t)=a_i^+\,\mathrm{e}^{\eta_i/2}+a_i^-\,\mathrm{e}^{-\eta_i/2}\,,\ \eta_i=k_ix+k_i^3t\,. \tag{34}$$

These results can be recovered from those counterparts of Eq. (4) through the continuum limit (28). In fact, to get Eq. (34), we first equivalently write Eq. (16) as

$$\psi_i(n,\tau;h)=a_i^+k_i^h\Big(\frac{\alpha+k_i}{\alpha-k_i}\Big)^{n/2}\mathrm{e}^{k_i\tau}$$
$$+a_i^-(-k_i)^h\Big(\frac{\alpha+k_i}{\alpha-k_i}\Big)^{-n/2}\mathrm{e}^{-k_i\tau}\,. \tag{35}$$

Then, under the limit (28) it goes to

$$\psi_i(x,t;h)=a_i^+k_i^h\,\mathrm{e}^{\eta_i/2}+a_i^-(-k_i)^h\,\mathrm{e}^{-\eta_i/2}\,,$$
$$\eta_i=k_ix+k_i^3t\,. \tag{36}$$

Note that for the vector $\psi(x,t;h)$ composed by the above ψ_i it holds $\psi(x,t;h)=2^h\partial_x\psi(x,t;0)$. This means, under the rule (28), we have

$$f=|\widehat{N-1}|\to 2^{N(N-1)/2}|\widehat{N-1}|_{[x]}\,,$$

and then noting the relation $\partial_\tau = 2\partial_x$ in Eq. (30), the transformation (13) goes to the transformation (32). Finally, applying the limit procedure (28)–(30) on the semi-discrete bilinear equation (14), from the leading term $O(1/\beta^2)$, we can recover the continuous bilinear equation (31).

3.2 *Skew Continuum Limit*

First, we apply the skew-change of variables $(n,m)\mapsto(\mathcal{N}=n+m,m)$ (cf. Refs. [2, 4, 11]) in Eq. (3) with the change

$$w_{n,m}=U_{n+m,m}=:U_{\mathcal{N},m}=:U\,,$$
$$\widetilde{w}=\widetilde{U}\,,\ \widehat{w}=\widehat{\widetilde{\widetilde{U}}}\,,\ \widehat{\widetilde{w}}=\widehat{\widetilde{\widetilde{\widetilde{U}}}}\,, \tag{37}$$

and consequently we have

$$(U-\widehat{\widetilde{\widetilde{\widetilde{U}}}}-\alpha-\beta)(\widetilde{U}-\widehat{\widetilde{\widetilde{U}}}+\alpha-\beta)=-\alpha^2+\beta^2\,. \tag{38}$$

Consider the limit[1,11]

$$m\to\infty,\ \varepsilon=\alpha-\beta\to 0,\ \text{while}\ \varepsilon m=\tau-\tau_0\sim O(1), \tag{39}$$

where τ_0 is a constant. Then by taking $U_{\mathcal{N}}=:U_{\mathcal{N}}(\tau)$, and employing Taylor expansions in Eq. (38), the leading term is of $O(\varepsilon)$ and it gives a semi-discrete equation[2,4]

$$\partial_\tau U_{\mathcal{N}+1}=\frac{U_{\mathcal{N}+2}-U_{\mathcal{N}}}{2\alpha+U_{\mathcal{N}+2}-U_{\mathcal{N}}}\,, \tag{40}$$

i.e. Eq. (5) after a down-tilde shift.

Next, let us solve Eq. (5) using bilinear approach. With the transformation

$$U_{\mathcal{N}}=-\frac{g_{\mathcal{N}}}{f_{\mathcal{N}}}\,, \tag{41}$$

from Eq. (5) one can separate out the following two bilinear equations,

$$D_\tau g\cdot f+f^2-\underset{\sim}{f}\widetilde{f}=0\,, \tag{42a}$$
$$\widetilde{g}\underset{\sim}{f}-\underset{\sim}{g}\widetilde{f}+2\alpha(f^2-\underset{\sim}{f}\widetilde{f})=0\,, \tag{42b}$$

where we have omitted the suffix $\mathcal{N}$ and used tilde to denote shifts of $\mathcal{N}$.

As for solutions, we have the following.

Proposition 2 The bilinear equation set (42) is solved by the Casoratians

$$f=|\widehat{N-1}|\,,\quad g=|\widehat{N-2},N|\,, \tag{43}$$

which are composed by

$$\psi_i(\mathcal{N},\tau;h)=a_i^+k_i^h(\alpha+k_i)^{\mathcal{N}}\,\mathrm{e}^{-\tau/(\alpha+k_i)}+a_i^-(-k_i)^h(\alpha-k_i)^{\mathcal{N}}\,\mathrm{e}^{-\tau/(\alpha-k_i)}\,. \tag{44}$$

Proof To prove Eq. (42a), it will be convenient to rewrite f and g as the Casoratians w.r.t. $\mathcal{N}$ and we use the suffix $[\mathcal{N}]$ to denote them; besides, we introduce a tilde to indicate sequential changes in the column index:

$$|\psi(\mathcal{N},\tau;0),\psi(\mathcal{N}+1,\tau;0),\ldots,\psi(\mathcal{N}+M,\tau;0),\ldots|\equiv|\widehat{M},\ldots|_{[\mathcal{N}]}\equiv|0,\widetilde{M},\ldots|_{[\mathcal{N}]}\,.$$

In Ref. [5] we have shown that the index variables can be changed among n,m,h and this does not change the value of $|\widehat{N-1}|$ for certain ψ.

With the above notations and noting the relations

$$\psi(h+1)=\widetilde{\psi}(h)-\alpha\psi(h)\,,\ \partial_\tau\psi(h)=-\underset{\sim}{\psi}(h)\,,$$

we have

$$f=|\widehat{N-1}|_{[\mathcal{N}]}\,, \tag{45a}$$
$$g=|\widehat{N-2},N|_{[\mathcal{N}]}-\alpha Nf\,, \tag{45b}$$
$$f_\tau=-|-1,\widetilde{N-1}|_{[\mathcal{N}]}\,, \tag{45c}$$
$$g_\tau=-|-1,\widetilde{N-2},N|_{[\mathcal{N}]}-f-\alpha Nf_\tau\,. \tag{45d}$$

Then

$$\begin{aligned}
&D_\tau g\cdot f+f^2-\underset{\sim}{f}\widetilde{f}\\
&=-|-1,\widetilde{N-2},N|_{[\mathcal{N}]}|\widehat{N-1}|_{[\mathcal{N}]}\\
&\quad+|\widehat{N-2},N|_{[\mathcal{N}]}|-1,\widetilde{N-1}|_{[\mathcal{N}]}\\
&\quad-|-1,\widehat{N-2}|_{[\mathcal{N}]}|\widetilde{N}|_{[\mathcal{N}]}\\
&=\frac{1}{2}\begin{vmatrix}\widetilde{N-2} & \mathbf{0} & -1 & 0 & N-1 & N\\ \mathbf{0} & \widetilde{N-2} & -1 & 0 & N-1 & N\end{vmatrix}=0\,,
\end{aligned}$$

and Eq. (42a) is proved.

Next we prove Eq. (42b). In this turn we go back to employ the usual Casoratians notations for f and g. Using the technique in Ref. [5], we have the following shift formulae,

$$-\alpha^{N-2}\underset{\sim}{f}=|\widehat{N-2},\underset{\sim}{\psi}(N-2)|\,, \tag{46a}$$
$$-\alpha^{N-2}\underset{\sim}{g}=|\widehat{N-3},N-1,\underset{\sim}{\psi}(N-2)|+\alpha^{N-1}\underset{\sim}{f}\,, \tag{46b}$$
$$\alpha^{N-2}\widetilde{f}=\Gamma\,|\widehat{N-2},\mathring{\psi}(N-2)|\,, \tag{46c}$$
$$\alpha^{N-2}\widetilde{g}=\Gamma\,|\widehat{N-3},N-1,\mathring{\psi}(N-2)|+\alpha^{N-1}\widetilde{f}\,, \tag{46d}$$

where $\Gamma=\prod_{i=1}^N\Gamma_j$, $\Gamma_i=\alpha^2-k_i^2$,

$$\mathring{\psi}(h)=(\mathring{\psi}_1(h),\mathring{\psi}_2(h),\ldots,\mathring{\psi}_N(h))^{\mathrm{T}}\,,$$
$$\mathring{\psi}_i(h)=\frac{1}{\Gamma_i}\widetilde{\psi}_i(h)\,.$$

Besides, making a down tilde on Eq. (46c) and using the relation (see Eq. (A.5b) in Ref. [5]) $2\alpha\underset{\sim}{\mathring{\psi}}(h)=\mathring{\psi}(h)+\underset{\sim}{\psi}(h)$, we have (see Eq. (A.6m) in Ref. [5])

$$2\alpha\cdot\alpha^{2(N-1)}f=\Gamma|\widehat{N-3},\underset{\sim}{\psi}(N-2),\mathring{\psi}(N-2)|\,. \tag{47}$$

By this trick we express the term f^2 in Eq. (42b) as

$$f^2=\frac{\Gamma}{2\alpha\cdot\alpha^{2(N-1)}}|\widehat{N-1}|\,|\widehat{N-3},\underset{\sim}{\psi}(N-2),\mathring{\psi}(N-2)|\,.$$

With these formulae in hand, a direct substitution yields

$$\begin{aligned}
&\frac{\alpha^{2(N-1)}}{\Gamma}[\widetilde{g}\underset{\sim}{f}-\underset{\sim}{g}\widetilde{f}+2\alpha(f^2-\underset{\sim}{f}\widetilde{f})]\\
&=-|\widehat{N-3},N-1,\mathring{\psi}(N-2)||\widehat{N-2},\underset{\sim}{\psi}(N-2)|\\
&\quad+|\widehat{N-3},N-1,\underset{\sim}{\psi}(N-2)||\widehat{N-2},\mathring{\psi}(N-2)|\\
&\quad+|\widehat{N-1}||\widehat{N-3},\underset{\sim}{\psi}(N-2),\mathring{\psi}(N-2)|=0\,.
\end{aligned}$$

Thus, the proof for Proposition 2 is completed. □

Next, we recover the bilinear form (42) and Casoratians in Proposition 2 using continuum limit.

First, under $(\mathcal{N}, m)$ notation we rewrite the Casoratian entry (10) as

$$\psi_i(\mathcal{N}, m; h) = a_i^+ k_i^h (\alpha + k_i)^{\mathcal{N}} \Big(1 - \frac{\varepsilon}{\alpha + k_i}\Big)^m + a_i^- (-k_i)^h (\alpha - k_i)^{\mathcal{N}} \Big(1 - \frac{\varepsilon}{\alpha - k_i}\Big)^m . \tag{48}$$

Under the limit (39) and after scaling $a_i^{\pm}\, \mathrm{e}^{\tau_0/(\alpha \pm k_i)} \mapsto a_i^{\pm}$, it goes to Eq. (44). This also means the Casoratians (43) and the transformation (41) are recovered from those of the lpKdV equation. Now we look at the bilinear equations (8). Applying the skew-change $(n, m) \mapsto (\mathcal{N} = n + m, m)$ on them we have

$$\widehat{\widetilde{g}}\widetilde{f} - \widetilde{g}\widehat{\widetilde{f}} + (\alpha - \beta)(\widehat{\widetilde{f}}\widetilde{f} - f\widehat{\widetilde{\widetilde{f}}}) = 0 , \tag{49a}$$

$$g\widehat{\widetilde{\widetilde{f}}} - \widehat{\widetilde{\widetilde{g}}}f + (\alpha + \beta)(f\widehat{\widetilde{\widetilde{f}}} - \widehat{\widetilde{f}}\widetilde{f}) = 0 , \tag{49b}$$

where now the tilde and hat are for the new coordinates $(\mathcal{N}, m)$. Then under the skew limit (39) and Taylor expansions, the coefficients of leading term of these two bilinear equations respectively give Eqs. (42a) and (42b).

Finally, we consider the full-continuous results under the limit

$$\mathcal{N} \to \infty,\ \alpha \to \infty , \quad \text{while} \quad \frac{\mathcal{N}}{\alpha} = \xi \sim O(\alpha^2) . \tag{50}$$

In this case, let $U_{\mathcal{N}}(\tau) = : \bar{\mathrm{W}}(\tau, \xi) = : W(x, t)$, where the continuous variables x and t are defined as the following (cf. Refs. [2, 4])

$$x = 2\Big(\xi + \frac{\tau}{\alpha^2}\Big), \quad t = 2\Big(\frac{\xi}{3\alpha^2} + \frac{\tau}{\alpha^4}\Big), \tag{51}$$

which suggest

$$\partial_\xi = 2\partial_x + \frac{2}{3\alpha^2}\partial_t , \quad \partial_\tau = \frac{2}{\alpha^2}\partial_x + \frac{2}{\alpha^4}\partial_t . \tag{52}$$

For nonlinear equation (4), by a similar procedure as described in the previous subsection, in Eq. (4) the leading term is of $O(1/\alpha^4)$ from which we get the continuous equation (6), i.e., the pKdV equation. To find the relation between bilinear equations, we need first to examine the continuum limit of Casoratian entries. From Eq. (44) by extracting the factor $((\alpha + k_i)/(\alpha - k_i))^{\mathcal{N}/2}\, \mathrm{e}^{-\alpha\tau/(\alpha^2 - \tau^2)}$ we write it as

$$\psi_i(\mathcal{N}, \tau; h) = a_i^+ k_i^h \Big(\frac{\alpha + k_i}{\alpha - k_i}\Big)^{\mathcal{N}/2} \mathrm{e}^{k_i\tau/(\alpha^2 - k_i^2)} + a_i^- (-k_i)^h \Big(\frac{\alpha + k_i}{\alpha - k_i}\Big)^{-\mathcal{N}/2} \times \mathrm{e}^{-k_i\tau/(\alpha^2 - k_i^2)} . \tag{53}$$

This does not change the bilinear equations (42). In other words, the Casoratians f and g defined in Eq. (43) with the above ψ_i still solve Eq. (42). This is due to gauge equivalence of the bilinear equations (42). Then, it is easy to check Eq. (53) go to Eq. (36) under the limit scheme (50) and the continuous variables definition (51). Besides, recalling the discussion at the end of Subsec. 3.1, here we actually have

$$f = |\widehat{N-1}| \to 2^{N(N-1)/2}|\widehat{N-1}|_{[x]} ,$$
$$g = |\widehat{N-2}, N| \to 2^{N(N-1)/2+1}|\widehat{N-2}, N|_{[x]} .$$

This gives the relation $g = 2f_x$ and also indicates that the transformation (32) is recovered from (41) in the comtinuum limit. Now, let us see what happens on the bilinear equations (42). The leading parts for Eq. (42a) are

$$O\Big(\frac{1}{\alpha^2}\Big): \quad 2g_x f - 2g f_x + 4f_x^2 - 4f f_{xx} ,$$
$$O\Big(\frac{1}{\alpha^4}\Big): \quad \frac{2}{3}(3g_t f - 3g f_t - 4f f_{xt} + 4f_x f_t - 2f f_{xxxx} + 8f_x f_{xxx} - 6f_{xx}^2) .$$

The first term automatically holds in light of the relation $g = 2f_x$ that we have just found. Then the leading term turns out to be of $O(1/\alpha^4)$ and its coefficient just provides the bilinear equation (31), which has a Wronskian solution $f = : f(x, t) = |\widehat{N-1}|_{[x]}$ with ψ_i given by Eq. (34). The bilinear equation (42b) has same result, i.e. it yields Eq. (31) as well.

4 Conclusions

In this paper we have investigated bilinear forms and Casoratian solutions for two sdpKdV equations (4) and (5). In fact, Casoratian verifications are not as easy as those of Wronskians. It is more difficult to get shift formulae of Casoratians than derivatives of Wronskians. In this paper, we made use of many shift formulae and some tricks developed in Ref. [5] to finish Casoratian verifications. We also examined continuum limits of bilinear forms and their solutions from fully discrete case to continuous case. All the results are compatible in continuum limits. Together with the relation of Lax pairs which has already checked in Ref. [4], we can have the following map for the fully discrete, semi-discrete and continuous potential KdV systems w.r.t. continuum limits:

Lax Pair	⟶	Lax Pair	⟶	Lax Pair
↑		↑		↑
lpKdV	⟶	**sdpKdV**	⟶	**pKdV**
↓		↓		↓
Bilinear lpKdV	⟶	Bilinear sdpKdV	⟶	Bilinear pKdV
↓		↓		↓
Casoratian	⟶	Casoratian	⟶	Wronskian

Fig. 1 Compatible continuum limits of the KdV systems.

Finally, let us give some remarks on the continuum limits of the ABS lattice equations. The lpKdV equation (1), i.e. H1 equation, is the simplest member in the ABS list. It is not easy to study the continuum limits of other members together with their solutions. There are some already known connections such as $\mathrm{H3}^0$, $\mathrm{Q1}^0$ and Q4 with their continuous counterparts. These connections are more or less based on superposition formulae of Bäcklund transformations of the continuous systems, but they are not as rich as the above map for the KdV system. From the point of view of the universal Sato theory, the starting point of a continuum limit could be the connection of soliton solutions, more precisely, the connection of dispersion relations. Miwa's transformation[12] discretizes the famous Sato theory[13−17] by directly adding discrete independent variables into the continuous dispersion relation. This means, when we start from a discrete integrable equation, to find a connection with its possible continuous counterpart, we need to compare discrete and continuous dispersion relations. However, for the ABS list, except Q4,[18] all other equations share the same discrete dispersion relation (i.e. same discrete plane wave factor).[5,19] In addition, all soliton equations of the ABS lattice equations have nonzero backgrounds.[5,19] These two points, same discrete dispersion relation and nonzero backgrounds, make unusual when one investigates connections with continuous systems for most of members in the ABS list. In fact, not only Lax pairs, bilinear forms and solutions, but also integrability characteristics, such as symmetries and their algebraic structures, Hamiltonian structures and conservation laws (e.g. Refs. [20–22]), are also interesting in continuum limits.

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