

Two-Dimensional Linear Dependencies on the Coordinate Time-Dependent Interaction in Relativistic Non-Commutative Phase Space

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Abstract In this paper, the Non-Commutative phase space and Dirac equation, time-dependent Dirac oscillator are introduced. After presenting the desire general form of a two-dimensional linear dependency on the coordinate time-dependent potential, the Dirac equation is written in terms of Non-Commutative phase space parameters and solved in a general form by using Lewis–Riesenfeld invariant method and the time-dependent invariant of Dirac equation with two-dimensional linear dependency on the coordinate time-dependent potential in Non-Commutative phase space has been constructed, then such latter operations are done for time-dependent Dirac oscillator. In order to solve the differential equation of wave function time evolution for Dirac equation and time-dependent Dirac oscillator which are partial differential equation some appropriate ordinary physical problems have been studied and at the end the interesting result has been achieved.

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1 Introduction

The interaction which depends on coordinate linearly, has been interesting term in the quantum systems. In particular, the relativistic linear interaction, which is called the relativistic oscillator due to the similarity with the nonrelativistic harmonic oscillator, has been the subject of many successful theoretical studies.^[1–2] On the other hand, the time dependent potentials have been an appealing choice for the study of dynamics of quantum systems.^[3–5] In order to solve such Hamiltonians, various techniques have been proposed including the path integral, second quantization, and dynamical invariant.^[3–5] The third point which is of essential use in the present study is the Non-Commutative (NC) formulation of quantum mechanics. The NC formulation is motivated by fundamental theories such as quantum gravity and string theory.^[6–8] Till now, the NC formalism has been incorporated with important physical concepts and tools such as matrix theory,^[9] quantum Hall effect,^[10] quantum gravity,^[11] Aharonov–Bohm effect,^[12] Aharonov–Casher effect^[13] and etc.^[14–22] In the technical jargon, the Non-Commutative space is referred to the case where the position operators do not commute with each other and the Non-Commutative phase-space (NCPS) is recognized as the case where neither the momenta nor the positions commute. Such a space has interesting property and algebra for example we refer readers to some article in which a free particle has been studied in different situations, these accessible in Refs. [23–25] Dirac oscillator is system which is constituted by a relativistic fermion that is subjected to a linear vector potential. Moshinsky and Szczepanoak

named this system as Dirac oscillator for first time because it resembles the non-relativistic harmonic oscillator with an extremely strong spin-orbit coupling term.^[26–29] This system has strong potential for both theory and application for example when its non-relativistic limit is assumed, the equation which describes a harmonic oscillator in the presence of a considerable spin-orbit coupling is the associated Klein–Gordon equation or it is solvable exactly in one and two, three dimensions.^[26–27,30] For the first time it was studied by Ito *et al.*^[28] whom considered a Dirac equation in which the ordinary momentum \vec{P} was replaced by $\vec{P} - im\beta\omega\vec{r}$ where \vec{r} was the position vector and m was the mass of particle, ω was the frequency of the oscillator.^[31–33] Applications of Dirac oscillator can be seen in different situations that have attracted many interests. If we want to indicate to some of these efforts we can refer readers for theoretical jobs in Refs. [34–44], some practical cases of this topic in nuclear and sub-nuclear physics are mentioned in Refs. [45–49] and in the field of quantum optic we can check Refs. [50–53]. Readers should not imagine that this topic exists only on the papers! Realization of Dirac oscillator was studied experimentally in Ref. [54]. Our goal in this paper is combing of Non-Commutative phase space concepts and formalism, relativistic quantum mechanics by solving the time-dependent Dirac equation and oscillator in the NCPS via the Lewis–Riesenfeld invariant method that at the first sight sounds complicated and complex but it will be done in a simple way. In what follows, we first introduce Non-Commutative formalism. Then, in Sec. 3, a desire general form of two-dimensional linear dependency

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on the coordinate time-dependent Dirac equation is presented. In Sec. 4, time-dependent Dirac oscillator comes to this paper, in Sec. 5, some ordinary problems are studied for Dirac equation, in Sec. 6, the latter is done for Dirac oscillator then finally, and the conclusion appears.

2 The Non-Commutative Formalism

In the two-dimensional commutative space, the coordinates x_j and p_j satisfy the usual canonical commutation relation

$$\begin{aligned} [x_k, p_j] &= i\hbar\delta_{kj}, \\ [p_k, p_j] &= [x_k, x_j] = 0, \quad (k, j = 1, 2). \end{aligned} \quad (1)$$

In the NC phase-space, however, neither the position nor the momentum operators commute and Refs. [2–4],

$$\begin{aligned} [\hat{x}, \hat{y}] &= i\theta, \quad [\hat{p}_x, \hat{p}_y] = i\eta, \\ [\hat{x}_k, \hat{p}_j] &= i\hbar_{\text{eff}}\delta_{kj}, \quad k, j = 1, 2, \\ \hbar_{\text{eff}} &= \hbar(1 + \zeta), \quad \left(\zeta = \frac{\theta\eta}{4\hbar^2}\right), \end{aligned} \quad (2)$$

where θ and η are NC parameters of the NCPS. Obviously, for $\eta = \theta = 0$, we recover the ordinary commutative space. To map the NCPS into the commutative case, we can use the linear transformations^[3]

$$\hat{x} = x - \frac{1}{2\hbar}\theta p_y, \quad \hat{y} = y + \frac{1}{2\hbar}\theta p_x, \quad (3a)$$

and

$$\hat{p}_x = p_x + \frac{1}{2\hbar}\eta p_y, \quad \hat{p}_y = p_y - \frac{1}{2\hbar}\eta p_x. \quad (3b)$$

3 Time-Dependent Dirac Equation

Let us now consider a Dirac particle in the desire general form of a two-dimensional linear dependency on the coordinate time-dependent potential which is defined as follow^[3]

$$V(\hat{x}, \hat{y}, t) = \zeta_1 f_1(t)\hat{x} + \zeta_2 f_2(t)\hat{y}, \quad (4)$$

where $f_1(t)$ and $f_2(t)$ are arbitrary functions of time and ζ_j , ($j = 1, 2$) are constants. The corresponding Hamiltonian of this system is

$$\hat{H}(t) = c\alpha_x \hat{p}_x + c\alpha_y \hat{p}_y + \beta c^2 m + \zeta_1 f_1(t)\hat{x} + \zeta_2 f_2(t)\hat{y}, \quad (5)$$

where Pauli matrices are

$$\begin{aligned} \alpha_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \\ \beta &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \quad (6)$$

Substitute Eq. (3) into Eq. (5) gives^[3]

$$\begin{aligned} \hat{H}(t) &= \left(\alpha_x + \frac{f_2(t)\theta}{2}\right)p_x + \left(\alpha_y - \frac{f_1(t)\theta}{2}\right)p_y \\ &+ \left(f_1(t) - \frac{\eta\alpha_y}{2}\right)x + \left(f_2(t) + \frac{\eta\alpha_x}{2}\right)y + \beta m, \\ (\hbar = c = \zeta_1 = \zeta_2 = 1). \end{aligned} \quad (7)$$

In order to gain an understanding of time evolution for such a system, we should investigate the below relation

$$i\frac{\partial\Psi(x, y, t)}{\partial t} = H(t)\Psi(x, y, t),$$

where $\Psi(x, y, t)$ is the wave function of Dirac Hamiltonian. In our calculations, we will use the Lewis–Riesenfeld invariant method, which assumes the existence of a quantum invariant $I(t)$ satisfying:^[4]

$$\frac{dI}{dt} = \frac{1}{i}[I(t), H(t)] + \frac{\partial I}{\partial t} = 0, \quad (\hbar = 1). \quad (8)$$

By applying Eq. (8) on $\Psi(x, y, t)$ from the left side, we obtain

$$\left\{\frac{1}{i}(I(t)H(t) - H(t)I(t)) + \frac{\partial I(t)}{\partial t}\right\}\Psi(x, y, t) = 0,$$

or

$$i\frac{\partial(I(t)\Psi(x, y, t))}{\partial t} = H(t)(I(t)\Psi(x, y, t)), \quad (9)$$

Eq. (9) indicates that acting of $I(t)$ on $\Psi(x, y, t)$ leads to another solution of Dirac equation. For our purpose, we consider the linear invariant $I(t)$ as^[2,4]

$$I(t) = A_1(t)p_x + B_1(t)x + A_2(t)p_y + B_2(t)y + C(t). \quad (10)$$

Here $A_1(t)$, $B_1(t)$, $A_2(t)$, $B_2(t)$ and $C(t)$ are arbitrary time-dependent functions. By substituting Eq. (10) and Eq. (5) into Eq. (8), the invariant operator can be obtained as^[3]

$$\begin{aligned} I(t) &= a_1 p_x + a_2 p_y + \frac{\eta}{2}(a_2 x - a_1 y) \\ &+ \int \left(1 - \frac{\eta\theta}{4}\right)(a_1 f_1(t) + a_2 f_2(t))dt, \end{aligned} \quad (11)$$

where a_1 and a_2 are arbitrary time-dependent functions. The Eigen function of $I(t)$ has the form^[3]

$$\begin{aligned} \phi_\lambda(x, y, t) &\propto \exp[\mu_1(t)x + \mu_2(t)y + \mu_3(t)x^2 \\ &+ \mu_4(t)y^2]. \end{aligned} \quad (12)$$

And $\mu_j(t)$, ($j = 1, 2, 3, 4$) are arbitrary time-dependent functions. It is obvious from Eq. (9) that if Ψ is a solution of the time-dependent Dirac equation, so is therefore, $\phi = \lambda\Psi$ is an Eigen function of $I(t)$. Actually we have used the fact which is written in terms of mathematic in (9) and since this relation is an Eigen value relation we have done such and define ϕ as above. By this assumption and definition we are in a position to present and solve some ordinary problems which are common in physics.

4 Time-Dependent Dirac Oscillator with the Desire Potential

The Dirac oscillator in presence of a constant magnetic field and time-dependent angular frequency is^[26]

$$\hat{H} = c\vec{\alpha} \cdot \left[\left(\vec{P} - \frac{e}{c}\vec{A}\right) - im\omega(t)\vec{\beta} \cdot \vec{r}\right] + mc^2\beta, \quad (13)$$

where $A = (-(B/2)y, (B/2)x)$. By adding $V(x, y, t) = f_1(t)x + f_2(t)y$ to the Hamiltonian we obtain

$$\begin{aligned} H(t) &= \alpha_x p_x + \alpha_y p_y + \left(f_1(t) - im\omega(t)\alpha_{x\beta} - \alpha_y \frac{B}{2}\right)x \\ &+ \left(f_2(t) + \alpha_x \frac{B}{2} - im\omega(t)\alpha_{y\beta}\right)y + m\beta, \\ (e = \hbar = c = 1). \end{aligned} \quad (14)$$

For this system the invariant is suggested as follow

$$I(t) = A_1(t)p_x + B_1(t)x + A_2(t)p_y + B_2(t)y, \quad (15)$$

where $I(t)$ satisfies relation (8) and (9). Substituting (14) and (15) into (8), we can show that

$$\begin{aligned}
& \{[A_1(t), \alpha_x]\}p_x^2 + \{[A_1(t), \alpha_y] + [A_2(t), \alpha_x]\}p_x p_y \\
& - iA_1(t)\{f_1(t) + \alpha_x \beta m \omega(t)\} + \{[A_1(t), -im\omega(t)\alpha_x]\}\beta x p_x + \{[A_1(t), -im\omega(t)\beta]\}\alpha_x x p_x \\
& + \left\{ \left[A_1(t), \frac{-B}{2}\alpha_y \right] + \left[A_1(t), \frac{B}{2}\alpha_x \right] + [A_1(t), m\beta] + [C(t), \alpha_x] + i\dot{A}_1(t) \right\} p_x \\
& + \{[A_1(t), -im\omega(t)\alpha_y]\}\beta y p_x + \{[A_1(t), -im\omega(t)\beta]\}\alpha_x y p_x + iB_1(t)\{\alpha_x\} \\
& + \{B_1(t), \alpha_x\}\beta x p_x + \{[B_1(t), \alpha_y]\}x p_y + \{[B_1(t), -im\omega(t)\alpha_x]\}\beta x^2 + \{[B_1(t), -im\omega(t)\beta]\}\alpha_x x^2 \\
& + \left\{ [B_1(t), m\beta] + \left[B_1(t), \frac{B}{2}\alpha_x \right] + \left[B_1(t), \frac{-B}{2}\alpha_y \right] + i\dot{B}_1(t) \right\} x \\
& + \{[B_1(t), -im\omega(t)\alpha_x] + [A_2(t), -im\omega(t)\alpha_x] + [B_2(t), -im\omega(t)\alpha_x]\}\beta y x \\
& + \{[B_1(t), -im\omega(t)\beta] + [A_2(t), -im\omega(t)\beta] + [B_2(t), -im\omega(t)\beta]\}\alpha_x y x \\
& + \{[A_2(t), \alpha_y]\}p_y^2 + \{[A_2(t), -im\omega(t)\alpha_x]\}\beta x p_y + \{[A_2(t), -im\omega(t)\beta]\}\alpha_x x p_y \\
& + \left\{ \left[A_2(t), \frac{-B}{2}\alpha_y \right] + \left[A_2(t), \frac{B}{2}\alpha_x \right] + [A_2(t), m\beta] + [C(t), \alpha_y] + i\dot{A}_2(t) \right\} p_y \\
& + \{[B_2(t), \alpha_x]\}y p_x + \left\{ \left[B_2(t), \frac{-B}{2}\alpha_y \right] + \left[B_2(t), \frac{B}{2}\alpha_x \right] + [B_2(t), m\beta] + i\dot{B}_2(t) \right\} y \\
& + \{[B_2(t), -im\omega(t)\alpha_y]\}\beta y^2 + \{[B_2(t), -im\omega(t)\beta]\}\alpha_y y^2 + \{[C(t), -im\omega(t)\alpha_x]\}\beta x \\
& + \{[C(t), -im\omega(t)\beta]\}\alpha_x x + \{[C(t), -im\omega(t)\alpha_y]\}\beta y + \{[C(t), -im\omega(t)\beta]\}\alpha_y y \\
& + \left\{ [C(t), m\beta] + \left[C(t), \frac{B}{2}\alpha_x \right] + \left[C(t), \frac{-B}{2}\alpha_y \right] + i\dot{C}(t) \right\} = 0.
\end{aligned}$$

This implies that

$$\{[A_1(t), \alpha_x]\} = 0, \quad (16a)$$

$$\{[A_1(t), -im\omega(t)\alpha_y]\} = 0, \quad (16b)$$

$$\{[A_1(t), -im\omega(t)\alpha_x]\} = 0, \quad (16c)$$

$$\{[A_1(t), -im\omega(t)\beta]\} = 0, \quad (16d)$$

$$\{[A_2(t), \alpha_y]\} = 0, \quad (17a)$$

$$\{[A_2(t), -im\omega(t)\alpha_x]\} = 0, \quad (17b)$$

$$\{[A_2(t), -im\omega(t)\beta]\} = 0, \quad (17c)$$

$$\{[B_1(t), -im\omega(t)\beta]\} = 0, \quad (18a)$$

$$\{[B_1(t), \alpha_x]\} = 0, \quad (18b)$$

$$\{[B_1(t), \alpha_y]\} = 0, \quad (18c)$$

$$\{[B_1(t), -im\omega(t)\alpha_x]\} = 0, \quad (18d)$$

$$iB_1(t)\{\alpha_x\} = 0, \quad (18e)$$

$$\{[B_2(t), \alpha_x]\} = 0, \quad (19a)$$

$$\{[B_2(t), -im\omega(t)\alpha_y]\} = 0, \quad (19b)$$

$$\{[B_2(t), -im\omega(t)\beta]\} = 0, \quad (19c)$$

$$\{[C(t), -im\omega(t)\alpha_x]\} = 0, \quad (20a)$$

$$\{[C(t), -im\omega(t)\alpha_y]\} = 0, \quad (20b)$$

$$\{[C(t), -im\omega(t)\beta]\} = 0, \quad (20c)$$

$$\{[A_1(t), \alpha_y] + [A_2(t), \alpha_x]\} = 0, \quad (21)$$

$$-iA_1(t)\{f_1(t) + \alpha_x \beta m \omega(t)\} = 0, \quad (22)$$

$$\begin{aligned}
& \left\{ \left[A_1(t), \frac{-B}{2}\alpha_y \right] + \left[A_1(t), \frac{B}{2}\alpha_x \right] + [A_1(t), m\beta] \right. \\
& \left. + [C(t), \alpha_x] + i\dot{A}_1(t) \right\} = 0, \quad (23a)
\end{aligned}$$

$$\left\{ [B_1(t), m\beta] + \left[B_1(t), \frac{B}{2}\alpha_x \right] + \left[B_1(t), \frac{-B}{2}\alpha_y \right] \right.$$

$$\left. + i\dot{B}_1(t) \right\} = 0, \quad (23b)$$

$$\begin{aligned}
& \{[B_1(t), -im\omega(t)\alpha_x] + [A_2(t), -im\omega(t)\alpha_x] \\
& + [B_2(t), -im\omega(t)\alpha_x]\} = 0, \quad (23c)
\end{aligned}$$

$$\begin{aligned}
& \{[B_1(t), -im\omega(t)\beta] + [A_2(t), -im\omega(t)\beta] \\
& + [B_2(t), -im\omega(t)\beta]\} = 0, \quad (23d)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[A_2(t), \frac{-B}{2}\alpha_y \right] + \left[A_2(t), \frac{B}{2}\alpha_x \right] + [A_2(t), m\beta] \right. \\
& \left. + [C(t), \alpha_y] + i\dot{A}_2(t) \right\} = 0, \quad (23e)
\end{aligned}$$

$$\begin{aligned}
& \left\{ \left[B_2(t), \frac{-B}{2}\alpha_y \right] + \left[B_2(t), \frac{B}{2}\alpha_x \right] \right. \\
& \left. + [B_2(t), m\beta] + i\dot{B}_2(t) \right\} = 0, \quad (23f)
\end{aligned}$$

$$\begin{aligned}
& \left\{ [C(t), m\beta] + \left[C(t), \frac{B}{2}\alpha_x \right] + \left[C(t), \frac{-B}{2}\alpha_y \right] + i\dot{C}(t) \right\} \\
& = 0. \quad (23g)
\end{aligned}$$

From relations (18), (16), and (2) we can get to

$$A_1(t) = a_0, \quad B_1(t) = b_0, \quad (24)$$

where a_0 and b_0 are constant. And from (17a) and (17b), (17c) we can understand that

$$A_2(t) = a_1(t) + a_2(t)\alpha_x + a_3(t)\alpha_y + a_4(t)\beta, \quad (25)$$

where a_j ($j = 1, 2, 3, 4$) is an arbitrary function of time. From (19a) and (19b), (19c) we can find out that

$$B_2(t) = b_1(t) + b_2(t)\alpha_x + b_3(t)\alpha_y + b_4(t)\beta, \quad (26)$$

where b_j ($j = 1, 2, 3, 4$) is an arbitrary function of time. Finally, from (20a) and (20b), (20c) it can be obtained that

$$C(t) = c_1(t) + c_2(t)\alpha_x + c_3(t)\alpha_y + c_4(t)\beta, \quad (27)$$

where c_j ($j = 1, 2, 3, 4$) is an arbitrary function of time. So the Invariant operator is

$$I(t) = a_0 p_x + b_0 x + [a_1(t) + a_2(t)\alpha_x + a_3(t)\alpha_y + a_4(t)\beta] p_y + [b_1(t) + b_2(t)\alpha_x + b_3(t)\alpha_y + b_4(t)\beta] y + c_1(t) + c_2(t)\alpha_x + c_3(t)\alpha_y + c_4(t)\beta. \quad (28)$$

Now this is a good time to study some special cases for Dirac equation and Dirac oscillator.

5 The Solutions of Problem for Some Special Cases for Dirac Equation

Introducing^[3]

$$\Psi(x, y, t) = \chi(t)\phi_\lambda(x, y, t), \quad (29)$$

where

$$\chi(t) = \begin{pmatrix} \chi_1(t) \\ \chi_2(t) \end{pmatrix}.$$

Combination of Eqs. (9) and (29) in commutative space yields^[3]

$$\frac{d\mu_1(t)}{dt} = -if_1(t) - 2\mu_3(t), \quad (30a)$$

$$\frac{d\mu_2(t)}{dt} = -if_2(t), \quad (30b)$$

$$\frac{d\mu_3(t)}{dt} = \frac{d\mu_4(t)}{dt} = 0, \quad (30c)$$

$$\frac{d\chi(t)}{dt} = -(\alpha_x\mu_1(t) + \alpha_y\mu_2(t) + i\beta m)\chi(t). \quad (30d)$$

Let us now consider some physically motivated cases.

5.1 The Case of $f_1(t) = f_2(t) = c_0 t$

This choice is useful to represent some electric fields in some ordinary problems. Here, c_0 is constant.

From Eqs. (14) we have

$$i \frac{d\mu_1(t)}{dt} = c_0 t - 2i\mu_3(t), \quad (31a)$$

$$i \frac{d\mu_2(t)}{dt} = c_0 t, \quad (31b)$$

$$\frac{d\mu_3(t)}{dt} = \frac{d\mu_4(t)}{dt} = 0, \quad (31c)$$

$$\frac{d\chi(t)}{dt} = i[\alpha_x(c_0 t^2 + c_1 t + c_2) + \alpha_y(c_0 t^2 + d_1) + \beta m]\chi(t). \quad (31d)$$

Equations (31a) and (31b) give

$$i\mu_1(t) = c_0 t^2 + c_1 t + c_2, \quad (32a)$$

$$i\mu_2(t) = c_0 t^2 + d_1, \quad (32b)$$

where c_j ($j = 0, 1, 2$) and d_j are constants, and Eq. (31d) gives

$$\chi(t) = \exp[i(\alpha_x e(t) + \alpha_y f(t) + \beta m t)]. \quad (33)$$

With

$$e(t) = c'_0 t^3 + c'_1 t^2 + c'_2 t + c'_3, \quad (34a)$$

$$f(t) = d'_0 t^3 + d'_1 t + d'_2. \quad (34b)$$

where c'_j ($j = 0, 1, 2, 3$) and d'_k ($k = 0, 1, 2$) are constants. By expanding $\chi(t)$ we have

$$\chi(t) = (\cos[e(t)] + i\alpha_x \sin[e(t)])(\cos[f(t)] + i\alpha_y \sin[f(t)])(\cos(t) + i\beta \sin(t)), \quad (35)$$

where we use the well-known properties

$$\alpha_j^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_j^{2n+1} = \alpha_j, \\ \beta^{2n} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \beta^{2n+1} = \beta. \quad (36)$$

And “ n ” is an integer and $j = x, y$.

5.2 The Case of $f_1(t) = f_2(t) = d_0 t^2$

To express some phenomena which are similar to harmonic oscillators this form is an appropriate shape. In this case, from Eq. (30d)

$$\chi(t) = (\cos[e(t)] + i\alpha_x \sin[e(t)])(\cos[f(t)] + i\alpha_y \sin[f(t)])(\cos(t) + i\beta \sin(t)), \quad (37)$$

with

$$e(t) = q_0 t^4 + q_1 t^2 + q_2 t + q_3, \quad (38)$$

$$f(t) = w_0 t^4 + w_1 t + w_2. \quad (39)$$

where q_j ($j = 0, 1, 2, 3$) and w_k ($k = 0, 1, 2$) are constants.

5.3 The Case of $f_1(t) = f_2(t) = g_0 t^s$

After getting accustomed to linear and quadratic form of time dependency, it can be collected in a general form as $g_0 t^s$, in this case, we obtain from Eq. (30d)

$$\chi(t) = (\cos[e(t)] + i\alpha_x \sin[e(t)])(\cos[f(t)] + i\alpha_y \sin[f(t)])(\cos(t) + i\beta \sin(t)), \quad (40)$$

where

$$e(t) = r_0 t^{s+2} + r_1 t^2 + r_2 t + r_3, \quad (41)$$

$$f(t) = u_0 t^{s+2} + u_1 t + u_2. \quad (42)$$

where r_j ($j = 0, 1, 2, 3$) and u_k ($k = 0, 1, 2$) are constants.

5.4 The Case of $f_1(t) = f_2(t) = h_0 e^{h_0' t}$

In physics we have enough problems which need exponential form to express them by means of math such as periodic potentials, short range potentials and etc. Therefore, from Eq. (30d), we simply find

$$\chi(t) = (\cos[e(t)] + i\alpha_x \sin[e(t)])(\cos[f(t)] + i\alpha_y \sin[f(t)])(\cos(t) + i\beta \sin(t)). \quad (43)$$

And

$$e(t) = p_0 e^t + p_1 t^2 + p_2 t + p_3, \quad (44)$$

$$f(t) = k_0 e^t + k_1 t + k_2. \quad (45)$$

where p_j ($j = 0, 1, 2, 3$) and k_l ($l = 0, 1, 2$), d_0 , g_0 , h_0 , h_0' are constants. Here, indicating to an important point is worthwhile that we set $\hbar = c = \zeta_1 = \zeta_2 = 1$ to provide simplicity in calculations but in Sec. 5 we write the functions in general form. The constant coefficients represent all other coefficients which we can deal in problems.

6 Dirac Oscillator for Two General Special Cases

In present section we are going to do like what we have done in the later section but for Dirac oscillator. In order to make problem simpler we set the angular momentum being a constant with respect the time. So substituting Eqs. (29), (14), (12) into the (9) the below relation will appear

$$\begin{aligned} i \frac{d\mu_1(t)}{dt} &= f_1(t) - \left(2i\alpha_x\mu_3(t) + im\omega\alpha_x\beta + \frac{B}{2}\alpha_y\right), \\ i \frac{d\mu_2(t)}{dt} &= f_2(t) - \left(2i\alpha_y\mu_4(t) + im\omega\alpha_y\beta - \frac{B}{2}\alpha_x\right), \\ \frac{d\mu_3(t)}{dt} &= \frac{d\mu_4(t)}{dt} = 0, \\ i \frac{d\chi(t)}{dt} &= (-i\alpha_x\mu_1(t) - i\alpha_y\mu_2(t) + m\beta)\chi(t), \end{aligned} \quad (46)$$

or

$$\begin{aligned} i \frac{d\mu_1(t)}{dt} &= f_1(t) - \left(2i\alpha_x\mu_3 + im\omega\alpha_x\beta + \frac{B}{2}\alpha_y\right), \\ i \frac{d\mu_2(t)}{dt} &= f_2(t) - \left(2i\alpha_y\mu_4 + im\omega\alpha_y\beta - \frac{B}{2}\alpha_x\right), \\ \mu_3 &= \text{constant}, \quad \mu_4 = \text{constant}, \\ \chi(t) &= \exp\left\{-\int [im\beta + \alpha_x\mu_1(t) + \alpha_y\mu_2(t)]dt\right\}. \end{aligned} \quad (47)$$

Because we explain reasons of using some special cases which have been used in last section, here we restrict ourselves to two general cases that they can reduce to the details which are explained in the last section.

Case 1 $f_1(t) = f_2(t) = t_0 t^s$.

Here t_0 is a constant. It is obvious that choosing $s = 1$ linear cases and for $s = 2$ oscillator cases are obtained. Briefly “s” has been used in order to generalization. In this case (47) turns to

$$\mu_1(t) = i \left[\left(2i\mu_3\alpha_x + \frac{B\alpha_y}{2} + im\omega\alpha_x\beta\right)t - t'_0 t^{s+1} \right],$$

$$\begin{aligned} \mu_2(t) &= i \left[\left(2i\mu_4\alpha_y - \frac{B\alpha_x}{2} + im\omega\alpha_y\beta\right)t - t'_0 t^{s+1} \right], \\ \chi(t) &= \exp[-i(f(t)\alpha_x + g(t)\alpha_y + m\beta t)], \\ f(t) &= \left[\left(2i\mu_3\alpha_x + \frac{B\alpha_y}{2} + im\omega\alpha_x\beta\right)\frac{t^2}{2} - t''_0 t^{s+2} \right], \\ g(t) &= \left[\left(2i\mu_4\alpha_y - \frac{B\alpha_x}{2} + im\omega\alpha_y\beta\right)\frac{t^2}{2} - t''_0 t^{s+2} \right], \end{aligned} \quad (48)$$

where t'_0 and t''_0 are the new constants.

Case 2 $f_1(t) = f_2(t) = c_0 e^{d_0 t}$.

This form of the functions as we mentioned is appropriated to use as if we deal with functions whose behaviors are like short-rang or periodic functions. In this case c_0 and d_0 are constant. Like the last one (47) can change to

$$\begin{aligned} \mu_1(t) &= i \left[\left(2i\mu_3\alpha_x + \frac{B\alpha_y}{2} + im\omega\alpha_x\beta\right)t - c'_0 e^{d_0 t} \right], \\ \mu_2(t) &= i \left[\left(2i\mu_4\alpha_y - \frac{B\alpha_x}{2} + im\omega\alpha_y\beta\right)t - c'_0 e^{d_0 t} \right], \\ \chi(t) &= \exp[-i(f'(t)\alpha_x + g'(t)\alpha_y + m\beta t)], \\ f'(t) &= \left[\left(2i\mu_3\alpha_x + \frac{B\alpha_y}{2} + im\omega\alpha_x\beta\right)\frac{t^2}{2} - c''_0 e^{d_0 t} \right], \\ g'(t) &= \left[\left(2i\mu_4\alpha_y - \frac{B\alpha_x}{2} + im\omega\alpha_y\beta\right)\frac{t^2}{2} - c''_0 e^{d_0 t} \right], \end{aligned} \quad (49)$$

with the new constants c'_0 and c''_0 .

7 Conclusions

The Dirac equation and Dirac oscillators with linear dependency on the coordinate time-dependent potential are studied in NC phase-space. To report the solutions, the Lewis–Riesenfeld invariant method is used in the calculations. The problem is considered for the physically motivated cases of power and exponential terms and the analytical solutions are reported in each case and at the end it is shown that in this Dirac equation and Dirac oscillator the results that derived for special cases, are some functions those are similar to Euler rotations functions but the difference is in this article the parameter is time not angle.

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