Commun. Theor. Phys. **65** (2016) 569–574 Vol. 65, No. 5, May 1, 2016

Exact Solutions of Schrödinger Equation with Improved Ring-Shaped Non-Spherical Harmonic Oscillator and Coulomb Potential

Akpan Ndem Ikot,[1,*] Ita O. Akpan,[2] T.M. Abbey,[1] and Hassan Hassanabadi[3]

[1]Theoretical Physics Group, Department of Physics, University of Port Harcourt, PMB 5323 Choba Port Harcourt, Nigeria

[2]Department of Physics, University of Calabar, Nigeria

[3]Department of Physics, University of Shadrood, Shahrood, Iran

(Received November 4, 2015; revised manuscript received December 4, 2015)

Abstract *We propose improved ring shaped like potential of the form,* $V(r,\theta)=V(r)+(\hbar^2/2Mr^2)[(\beta\sin^2\theta+\gamma\cos^2\theta+\lambda)/\sin\theta\cos\theta]^2$ *and its exact solutions are presented via the Nikiforov–Uvarov method. The angle dependent part* $V(\theta)=(\hbar^2/2Mr^2)[(\beta\sin^2\theta+\gamma\cos^2\theta+\lambda)/\sin\theta\cos\theta]^2$*, which is reported for the first time embodied the novel angle dependent (NAD) potential and harmonic novel angle dependent potential (HNAD) as special cases. We discuss in detail the effects of the improved ring shaped like potential on the radial parts of the spherical harmonic and Coulomb potentials.*

PACS numbers: 03.65.Ge, 02.30.Cp, 03.67.Lx

Key words: improved ring shaped like potential, Schrödinger equation, Nikiforov–Uvarov method

1 Introduction

The ring-shaped like potentials which are non-central potential have been the subject of studies in physics and chemistry, in recent times.[1−5] The occurrence of accidental degeneracy and hidden symmetry for the non-central potentials has been reported.[6−7] The ring-shaped like potential have many applications in the field of nuclear physics and quantum chemistry which are used in describing the interaction between the deformed pair of nuclei in physics and describing the ring-shaped molecular benzene in chemistry.[8−10] Thus, the exact solutions of the Schrödinger equation can be used to extract the quantum information in chemical and molecular physics.[11−12] In recent years, considerable efforts have been made by many authors to obtain the exact analytical solutions of the Schrödinger equation with ring-shaped potentials.[13−15] In both, relativistic and non-relativistic regime different kinds of ring-shaped like potential have been investigated such as the non-spherical harmonic oscillator (NHO),[16] the ring-shaped oscillator (RHO),[17] the double ring-shaped harmonic oscillator (DRHO)[18] among others. Dong *et al.*[19] studied a ring-shape non-spherical harmonic (RNHO) potential of the form,

$$V(r,\theta)=\frac{1}{2}M\omega^2r^2+\frac{\hbar^2\alpha}{2Mr^2}+\frac{\hbar^2\beta}{2Mr^2\sin^2\theta}\,, \tag{1}$$

and obtained the non-relativistic energy spectra and wave function. In Eq. (1), M and ω are the mass and the frequency of the particles respectively, α and β are the two potential parameters. Berkdemir[20] proposed the novel angle-dependent (NAD) potential as,

$$V(r,\theta)=V(r)+\frac{\hbar^2}{2Mr^2}\left(\frac{\gamma+\beta\sin^2\theta+\alpha\sin^4\theta}{\sin^2\theta\cos^2\theta}\right)\,, \tag{2}$$

where the potential $V(r)$ contains a Coulomb potential or a harmonic potential in addition to NAD potential. Motivated by the work of Berkdemir,[20] Zhang *et al.*[21] proposed the harmonic novel angle dependent (HNAD) potential of the form

$$V(r,\theta)=\frac{1}{2}M\omega^2r^2+\frac{\hbar^2}{2Mr^2}\left(\frac{\eta+A\cos^2\theta+B\cos^4\theta}{\sin^2\theta\cos^2\theta}\right)\,, \tag{3}$$

where Zhang *et al.*[21] simply replaced $\sin\theta$ by $\cos\theta$ in the numerator of Eq. (2) to obtain HNAD of Eq. (3), where η, A and B are the three potential parameters. Also investigated is the double ring-shaped oscillator (DRSO) of the form,[22]

$$V(r,\theta)=\frac{1}{2}M\omega^2r^2+\frac{\hbar^2}{2Mr^2}\left(\frac{b}{\sin^2\theta}+\frac{c}{\cos^2\theta}\right)\,, \tag{4}$$

where b and c are two potential parameter. It can be observed that when $c=0$ and $b=c=0$, the DRSO reduces to the ring shaped oscillator (RSO) and spherical oscillator (SO) respectively. Furthermore, Zhang[23] put forward a new RNHO as,

$$V(r,\theta)=\frac{1}{2}M\omega^2r^2+\frac{\hbar^2A}{2Mr^2}+\frac{\hbar^2B\cos\theta}{2Mr^2\sin^2\theta}\,, \tag{5}$$

and obtained the energy spectra and wave function for this potential. In Ref. [24], single ring shaped potentials are discussed using the universal associated Legendre polynomials. In Ref. [25], double ring shaped potentials are

---

[*] E-mail address: ndemikotphysics@gmail.com

discussed resulting the super-universal associated Legendre polynomials.

Motivated by the study of the ring-shaped-like potential, we attempt to propose an improved ring-shaped non-spherical harmonic (IRNHO) and improved ring-shaped Coulomb (IRC) potentials of the form,

$$V(r,\theta) = \frac{1}{2}M\omega^2 r^2 + \frac{1}{2}\frac{\hbar^2\eta}{Mr^2} + \frac{\hbar^2}{2Mr^2} \times \left(\frac{\beta\sin^2\theta + \gamma\cos^2\theta + \lambda}{\sin\theta\cos\theta}\right)^2 , \tag{6}$$

$$V(r,\theta) = -\frac{\beta}{r} + \frac{\hbar^2}{2Mr^2}\left(\frac{\beta\sin^2\theta + \gamma\cos^2\theta + \lambda}{\sin\theta\cos\theta}\right)^2 , \tag{7}$$

where β, γ and λ are the three adjustable potential parameters. The RNHO, NAD, HNAD, DRSO and the new HNAD potential are incorporated in IRNHO and are reduced as special cases depending on the values of the adjustable potential parameter as will be seen later. Similarly, the Coulomb plus NAD[20] and Makarov[26] potentials are special cases of IRC. Different analytical methods such as supersymmetric quantum mechanics (SUSYQM),[27] factorization method,[28] tridiagonaliztion method,[29] asymptotic iteration method (AIM)[30] and Nikiforov–Uvarov (NU) method[31] have been used to study the non-relativistic Schrödinger equation with ring-shaped like potentials.

The aim of the present work is to study the Schrödinger equation with IRNHO and IRC potentials using the NU method which to the best of our knowledge, has never been reported before in the available literature. We also deduce other familiar ring shaped-like potentials as special cases.

2 Nikiforov–Uvarov Method

The NU method can solve a second-order differential equation of the form[31]

$$\psi''_n(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'_n(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi_n(s) = 0 , \tag{8}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most of second degree, and $\tilde{\tau}(s)$ is a first-degree polynomial. To make the application of the NU method simpler and more direct, we introduce a more compact presentation of the idea. In order to do this, we rewrite Eq. (8) as follows[32]

$$\psi''_n(s) + \left(\frac{c_1 - c_2 s}{s(1-c_3 s)}\right)\psi'_n(s) + \left(\frac{-\xi_1 s^2 + \xi_2 s - \xi_3}{s^2(1-c_3 s)^2}\right)\psi_n(s) = 0 , \tag{9}$$

in which

$$\psi_n(s) = \phi(s) y_n(s) . \tag{10}$$

Comparing Eq. (8) with Eq. (9), we obtain the following identifications:

$$\tilde{\tau}(s) = c_1 - c_2 s , \quad \sigma(s) = s(1-c_3 s) , \qquad \tilde{\sigma}(s) = -\xi_1 s^2 + \xi_2 s - \xi_3 . \tag{11}$$

Following the NU method,[31–32] we obtain the following required parameters:

(i) The relevant constant:

$$\begin{aligned} &c_4 = \frac{1}{2}(1-c_1) , \quad c_5 = \frac{1}{2}(c_2 - 2c_3) , \\ &c_6 = c_5^2 + \xi_1 , \quad c_7 = 2c_4 c_5 - \xi_2 , \\ &c_8 = c_4^2 + \xi_3 , \quad c_9 = c_3 c_7 + c_3^2 c_8 + c_6 , \\ &c_{10} = c_1 + 2c_4 + 2\sqrt{c_8} , \\ &c_{11} = c_2 - 2c_5 + 2(\sqrt{c_9} + c_3\sqrt{c_8}) , \\ &c_{12} = c_4 + \sqrt{c_8} , \quad c_{13} = c_5 - (\sqrt{c_9} + c_3\sqrt{c_8}) . \end{aligned} \tag{12}$$

(ii) The essential polynomial functions:

$$\pi(s) = c_4 + c_5 s - [(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}] , \tag{13}$$

$$k = -(c_7 + 2c_3 c_8) - 2\sqrt{c_8 c_9} , \tag{14}$$

$$\tau(s) = c_1 + 2c_4 - (c_2 - 2c_5)s - 2[(\sqrt{c_9} + c_3\sqrt{c_8})s - \sqrt{c_8}] , \tag{15}$$

$$\tau'(s) = -2c_3 - 2(\sqrt{c_9} + c_3\sqrt{c_8}) < 0 . \tag{16}$$

(iii) The energy equation:

$$c_2 n - (2n+1)c_5 + (2n+1)(\sqrt{c_9} + c_3\sqrt{c_8}) + n(n-1)c_3 + c_7 + 2c_3 c_8 + 2\sqrt{c_8 c_9} = 0 . \tag{17}$$

(iv) The wave functions:

$$\rho(s) = s^{c_{10}}(1-c_3 s)^{c_{11}} , \tag{18}$$

$$\phi(s) = s^{c_{12}}(1-c_3 s)^{c_{13}} , \quad c_{12} > 0 , \quad c_{13} > 0 , \tag{19}$$

$$y_n(s) = P_n^{(c_{10},c_{11})}(1-2c_3 s) , \ c_{10} > -1 , \ c_{11} > -1 , \tag{20}$$

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}}(1-c_3 s)^{-c_{12}-c_{13}/c_3} \times P_n^{(c_{10}-1,\, c_{11}/c_3 - c_{10}-1)}(1-2c_3 s) , \tag{21}$$

where $P_n^{(\mu,\nu)}(x), \mu > -1, \nu > -1$, and $x \in [-1,1]$ are Jacobi polynomials with

$$P_n^{(\alpha,\beta)}(1-2s) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1(-n, 1+\alpha+\beta+n, \alpha+1, s) , \tag{22}$$

and $N_{n\kappa}$ is a normalization constant. Also, the above wave functions can be expressed in terms of the hypergeometric function via

$$\psi_{n\kappa}(s) = N_{n\kappa} s^{c_{12}}(1-c_3 s)^{c_{13}}\,{}_2F_1(-n, 1+c_{10}+c_{11}+n, c_{10}+1, c_3 s) , \tag{23}$$

where $c_{12} > 0$, $c_{13} > 0$ and $s \in [0, 1/c_3]$, $c_3 \neq 0$.

3 Variable Separation of Schröinger Equation in Spherical Coordinates

The Schröinger equation in three-dimensional is given as,

$$-\frac{\hbar^2}{2M}\nabla^2\psi(r,\theta,\varphi) + V(r,\theta)\psi(r,\theta,\varphi) = E\psi(r,\theta,\varphi) , \tag{24}$$

where $V(r,\theta)$ is the IRNHO or IRC potentials and ∇^2 is the Laplacian operator. In spherical coordinate, we write the wave function as,

$$\psi(r,\theta,\varphi) = \frac{R(r)}{r}H(\theta)\Phi(\varphi) , \tag{25}$$

and the Laplacian operator as,

$$\nabla^2 = \frac{1}{r^2\sin\theta}\Big[\sin\theta\frac{\partial}{\partial r}(r^2)\frac{\partial}{\partial r} + \frac{\partial}{\partial\theta}(\sin\theta)\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial^2}{\partial\varphi^2}\Big]\,, \tag{26}$$

and then separate variables of the Schröinger equation into three sets of differential equations,

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \Big[\frac{2M}{\hbar^2}(E - V(r)) - \frac{\Lambda}{r^2}\Big] = 0\,, \tag{27}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\sin\theta\frac{\mathrm{d}H(\theta)}{\mathrm{d}\theta}\Big) + \Big[\Lambda - \frac{m^2}{\sin^2\theta} - \Big(\frac{\beta\sin^2\theta + \gamma\cos^2\theta + \lambda}{\sin\theta\cos\theta}\Big)^2\Big]H(\theta) = 0\,, \tag{28}$$

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2} + m^2\Phi(\varphi) = 0\,, \tag{29}$$

where m^2 and Λ are separation constants, which are real and dimensionless. The solution of Eq. (29) is periodic and for bound state $\Phi(\varphi)$ satisfies the periodic boundary condition $\Phi(\varphi + 2\pi)$ and its solutions become,

$$\Phi(\varphi) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{\mathrm{i}m\varphi}, \quad m = 0, \pm1, \pm2, \ldots \tag{30}$$

4 Exact Solutions of IRNHO and IRC Potentials

4.1 *Solutions of Polar Angular Equation*

In order to get the solution of equation Eq. (28), we introduce a coordinate transformation of the form, $x = \cos^2\theta$ and Eq. (28) becomes

$$\frac{\mathrm{d}^2H(x)}{\mathrm{d}x^2} + \frac{1-3x}{2x(1-x)}\frac{\mathrm{d}H(x)}{\mathrm{d}x} + \frac{1}{4x^2(1-x)^2}\big\{-[(\beta-\gamma)^2 + \Lambda]x^2 + [\Lambda - m^2 - 2(\beta+\lambda)(\beta-\gamma)x] - (\beta+\lambda)^2\big\}H(x) = 0\,. \tag{31}$$

Now comparing Eq. (9) with Eq. (31), we can obtain the following relation for the NU formalism,

$$c_1 = \frac{1}{2}\,, \quad c_2 = \frac{3}{2}\,, \quad c_3 = 1\,, \quad \xi_1 = \frac{(\beta-\gamma)^2 + \Lambda}{4}\,, \quad \xi_2 = \frac{\Lambda - m^2 - 2(\beta+\lambda)(\beta-\gamma)}{4}\,, \quad \xi_3 = \frac{(\beta+\lambda)^2}{4}\,. \tag{32}$$

The remaining coefficients can be calculated from Eq. (12) as,

$$\begin{aligned}
&c_4 = \frac{1}{4}\,, \quad c_5 = -\frac{1}{4}\,, \quad c_6 = \frac{1}{16} + \frac{(\beta-\gamma)^2 + \Lambda}{4}\,, \quad c_7 = -\frac{1}{8} - \frac{\Lambda - m^2 - 2(\beta+\lambda)(\beta-\gamma)}{4}\,, \quad c_8 = \frac{1}{16} + \frac{(\beta+\lambda)^2}{4}\,, \\
&c_9 = \frac{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)}{4}\,, \quad c_{10} = \frac{1}{2}\Big(2 + \sqrt{1 + 4(\beta+\lambda)^2}\Big)\,, \\
&c_{11} = 2\Big(1 + \frac{1}{2}\sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)} + \frac{1}{4}\sqrt{1 + 4(\beta+\lambda)^2}\Big)\,, \quad c_{12} = \frac{1}{4}\Big(\sqrt{1 + 4(\beta+\lambda)^2}\Big)\,, \\
&c_{13} = -\frac{1}{4}\Big(1 + 2\sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)} + \sqrt{1 + 4(\beta+\lambda)^2}\Big)\,.
\end{aligned} \tag{33}$$

Using the energy equation (17) and the coefficients given in Eqs. (32) and (33), we obtain the relationship between the separation constant Λ and the non-negative integer $n = n_r$ as,

$$\begin{aligned}
\Lambda = {}& \Big(2n_r + 1 + \sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)}\Big) \\
&\times \Big(2n_r + 1 + \sqrt{(\beta-\gamma)^2 + (\beta+\lambda)^2 + m^2 + 2(\beta+\lambda)(\beta-\gamma)} + \sqrt{1 + 4(\beta+\lambda)^2}\Big) - 4[(\beta-\gamma)^2 + (\beta+\lambda)^2]\,.
\end{aligned} \tag{34}$$

Equation (34) is the contribution of the angle-dependent part of our proposed IRNHO and IRC which to the best of our knowledge has never been reported before in the literature. However, when the ring-shaped term potential in IRNHO disappeared by setting $\gamma = \beta = \lambda = 0$, then the constant of separation becomes $\Lambda = l(l+1)$, where $l = 2n_r + 1 + |m|\,, m = 0, 1, 2, \ldots$ In order to find the polar angular part of the wave function, we first find the weight function from Eq. (12) as,

$$\rho(\theta) = (\cos^2\theta)^{(1/2)(2+\sqrt{1+4(\beta+\lambda)^2})}(\sin^2\theta)^{2[1+(1/2)\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)}+(1/4)\sqrt{1+4(\beta+\lambda)^2}]}\,, \tag{35}$$

which leads the solutions of the first part of the angular wave function from Eq. (20) in terms of the Jacobi polynomial as,

$$y_n(\theta) = P_n^{[(1/2)(2+\sqrt{1+4(\beta+\lambda)^2}),2(1+(1/2)\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)}+(1/4)\sqrt{1+4(\beta+\lambda)^2})]}(1 - 2\cos^2\theta)\,. \tag{36}$$

From Eq. (19), we obtain the second part of the angular function as follow

$$\phi(\theta) = (\cos^2\theta)^{(1/4)(\sqrt{1+4(\beta+\lambda)^2})}(\sin^2\theta)^{-(1/4)(1+2\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)}+\sqrt{1+4(\beta+\lambda)^2})}\,. \tag{37}$$

Thus, the total angular wave function can be obtain namely, $H(\theta) = \phi_n(\theta)y_n(\theta)$ or from Eq. (23) in terms of the hypergeometric function as,

$$\begin{aligned} H_{lm}(\theta) = {} & A_n(\cos^2\theta)^{(1/4)(\sqrt{1+4(\beta+\lambda)^2})}(\sin^2\theta)^{c_{13}=-(1/4)(1+2\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)}+\sqrt{1+4(\beta+\lambda)^2})} \\ & \times \mathrm{F}_1\Big[-n, 4+\frac{1}{2}(\sqrt{1+4\,(\beta+\lambda)^2}) + (\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2\,(\beta+\lambda)\,(\beta-\gamma)} + \frac{1}{2}\sqrt{1+4\,(\beta+\lambda)^2}) \\ & + n; \frac{1}{2}(\sqrt{1+4(\beta+\lambda)^2}) + 2; \cos^2\theta\Big]\,, \end{aligned} \tag{38}$$

where A_n is the normalization constant.

4.2 *Solutions of the Radial Part of IRNHO*

In this section we will consider the radial part of the IRHNO and calculate the energy eigenvalues and the corresponding wave function. Substituting,

$$V(r) = \frac{1}{2}M\omega^2r^2 + \frac{1}{2}\frac{\hbar^2\eta}{Mr^2}\,, \tag{39}$$

into Eq. (27) with the transformation, $x = r^2$, we obtain

$$\frac{\mathrm{d}^2R(x)}{\mathrm{d}x^2} + \frac{1/2}{x}\frac{\mathrm{d}R(x)}{\mathrm{d}x} + \frac{1}{x^2}\Big[-\Big(\frac{M\omega}{2\hbar}\Big)^2x^2 + \Big(\frac{ME}{2\hbar^2}\Big)x - \frac{(\eta+\Lambda)}{4}\Big]R(x) = 0\,. \tag{40}$$

Now comparing Eq. (40) with the hypergeometric equation (Eq. (9)), we get physical constants,

$$c_1 = \frac{1}{2}\,,\quad c_2 = c_3 = 0\,,\quad \xi_1 = \Big(\frac{M\omega}{2\hbar}\Big)^2,\quad \xi_2 = \Big(\frac{ME}{2\hbar^2}\Big),\quad \xi_3 = \frac{(\eta+\Lambda)}{4}\,, \tag{41}$$

and the remaining constants can be calculated from Eq. (12) as

$$\begin{gathered} c_4 = \frac{1}{4}\,,\quad c_5 = 0\,,\quad c_6 = \Big(\frac{M\omega}{2\hbar}\Big)^2,\quad c_7 = -\Big(\frac{ME}{2\hbar^2}\Big),\quad c_8 = \frac{1}{16} + \frac{(\eta+\Lambda)}{4},\quad c_9 = \Big(\frac{M\omega}{2\hbar}\Big)^2, \\ c_{10} = 1 + \frac{1}{2}\sqrt{1+4(\eta+\Lambda)}\,,\quad c_{11} = \frac{1}{2} + 2\Big(\frac{M\omega}{2\hbar}\Big),\quad c_{12} = \frac{1}{4} + \frac{1}{2}\sqrt{1+4(\eta+\Lambda)}\,,\quad c_{13} = -\Big(\frac{M\omega}{2\hbar}\Big)\,. \end{gathered} \tag{42}$$

After a straight forward and little algebra, we obtain the energy eigenvalues using Eq. (41) and Eq. (42) as

$$E_{nl} = \hbar\omega\Big(2n + 1 + \frac{1}{2}\sqrt{1+4\eta+4\Lambda}\Big)\,. \tag{43}$$

Now using Eq. (34), we obtain the discrete energy eigenvalues as

$$E_{nl} = \hbar\omega(2n+1+\Omega)\,, \tag{44}$$

where,

$$\begin{aligned} \Omega = {} & \frac{1}{2}\surd\{1 + 4\eta + 4[(2n_r + 1 + \sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)}) \\ & \times (2n_r + 1 + \sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)} + \sqrt{1+4(\beta+\lambda)^2}) - 4((\beta-\gamma)^2+(\beta+\lambda)^2)]\}, \end{aligned} \tag{45}$$

and $n, n_r = 0, 1, 2, \ldots$ The corresponding radial wave function is obtained as follows,

$$R_{nl}(r) = N_{nl}(r^2)^{(1/4)+(1/2)\sqrt{1+4(\eta+\Lambda)}}\,\mathrm{e}^{-r^2}L_n^{\big((1/2)\sqrt{1+4(\eta+\Lambda)}\big)}(r^2)\,, \tag{46}$$

where N_{nl} is the normalization constant and $L_n^{\mu}(x)$ is the associated Laguerre polynomial.

4.3 *Solutions of the Radial Part of IRC*

Following the same approach for the non-spherical harmonic potential, then the Schrödinger equation for the Coulomb potential takes the following form,

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2} + \frac{1}{r^2}\left(\Big(\frac{2ME}{\hbar^2}\Big)r^2 + \Big(\frac{2M\beta}{\hbar^2}\Big)r - \Lambda\right)R(r) = 0\,. \tag{47}$$

Equation (47) can be solved completely by using the NU method as before. Thus, the energy spectrum for the Coulomb potential becomes,

$$E_{nl} = -\frac{2\beta}{\hbar^2}\Big(2n + 1 + \frac{1}{2}\sqrt{1+4\Lambda}\Big)^{-2}\,. \tag{48}$$

Substituting the values of Λ into Eq. (47), we obtain the complete solution for our proposed IRC as,

$$E_{nl} = -\frac{2\beta}{\hbar^2}(2n+1+\chi)^{-2}\,, \tag{49}$$

where,

$$\begin{aligned}\chi = \frac{1}{2}\surd\Big\{1+4\Big[&(2n_r+1+\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)})\\ &\times(2n_r+1+\sqrt{(\beta-\gamma)^2+(\beta+\lambda)^2+m^2+2(\beta+\lambda)(\beta-\gamma)}+\sqrt{1+4(\beta+\lambda)^2})-4((\beta-\gamma)^2+(\beta+\lambda)^2)\Big]\Big\}\end{aligned} \tag{50}$$

Also we will have

$$R_{nl}(r) = r^{l+(1/2)}\,\mathrm{e}^{-r}L_n^{2l}(2r)\,, \tag{51}$$

which presents the corresponding radial wave function.

5 Discussions

We have so far presented the energy spectra and wave functions for the IRNHO and IRC non central potentials. We now proceed to discuss the special cases of these complicated ring shaped like potentials.

5.1 *Novel Angle-Dependent (NAD) Potential*

As stated before Berkdemir[20] proposed the NAD potential of Eq. (2). If we take $\gamma = 0$ in IRNHO and IRC and setting $\beta^2 = A, 2\beta\lambda = B$ and $C = \lambda^2$, we obtain the NAD of the form,

$$V(r,\theta) = V(r) + \frac{\hbar^2}{2Mr^2}\left(\frac{C + A\sin^4\theta + B\sin^2\theta}{\sin^2\theta\cos^2\theta}\right)\,, \tag{52}$$

where $V(r)$ is the harmonic or Coulomb potential respectively. Equation (52) is the recently proposed Berkdemir[20] NAD potential deduced from the IRNHO or IRC. Substituting these parameters into Eq. (44) and Eq. (49), we obtain the energy spectra for the NAD potential with harmonic and Coulomb potential as,

$$E_{nl} = \hbar\omega\left(2n+1+\Omega^{\mathrm{NAD}}\right)\,, \tag{53}$$

where

$$\begin{aligned}\Omega^{\mathrm{NAD}} &= \frac{1}{2}\surd\big\{1+4\eta+4\big[(2n_r+1+\sqrt{2A+B+C+m^2+(2A+B)})\big(2n_r+1+\sqrt{2A+B+C+m^2+(2A+B)}\\ &\quad+\sqrt{(1+4(A+B+C))}\big)-4(2A+(B+C))\big]\big\}\,,\end{aligned} \tag{54}$$

$$E_{nl} = -\frac{2\beta}{\hbar^2}(2n+1+\chi^{\mathrm{NAD}})^{-2}\,, \tag{55}$$

$$\begin{aligned}\chi^{\mathrm{NAD}} &= \frac{1}{2}\surd\big\{1+4\big[(2n_r+1+\sqrt{2A+B+C+m^2+(2A+B)})(2n_r+1+\sqrt{2A+B+C+m^2+(2A+B)}\\ &\quad+\sqrt{(1+4(A+B+C))})-4(2A+(B+C))\big]\big\}\,.\end{aligned} \tag{56}$$

These results are consistent with those obtained by Berkdemir[20] if we use the transformation $x = \sin^2\theta$.

5.2 *The Harmonic Novel Angle Dependent (HNAD) Potential*

The HNAD potential is obtained from IRNHO by setting $\beta = 0$ and rescaling $\gamma^2 = A$, $2\gamma\lambda = B$, $\eta = 0$ and $\lambda^2 = C$ as,

$$V(r,\theta) = \frac{1}{2}M\omega^2r^2 + \frac{\hbar^2}{2Mr^2}\Big(\frac{C + A\cos^4\theta + B\cos^2\theta}{\sin^2\theta\cos^2\theta}\Big)\,. \tag{57}$$

Substituting the above parameters into Eq. (43), we obtain the energy level for HNAD as,

$$E_{nl} = \hbar\omega\left(2n+1+\frac{1}{2}\sqrt{1+4\Lambda}\right)\,, \tag{58}$$

where

$$\begin{aligned}\Lambda =&\frac{1}{2}\surd\{1+4[(2n_r+1+\sqrt{A+A+B+C+m^2-C})(2n_r+1+\sqrt{A+A+B+C+m^2-C}\\ &+\sqrt{1+4C})-4(A+C)]\}\,.\end{aligned} \tag{59}$$

This energy level is the same as those obtained by Zhang *et al.*[21]

6 Conclusions

In this article, we studied the Schrödinger equation with improved ring shaped like potential. We have calculated the energy eigenvalues and the corresponding wave function of the angle-dependent part completely using the parametric Nikiforov–Uvarov method. We also solved the harmonic potential and the Coulomb potentials plus the improved ring shaped like potential. It is shown that when $\gamma = 0$, our results are in good agreement with that of Berkdemir[20] and when $\beta = 0$ it reduces to that of Zhang *et al.*[21] Likewise, when the ring shaped like term vanishes, i.e. $\gamma = \beta = \lambda = 0$ then the results are in good agreement with Refs. [33–34]. Different ring shaped like potentials can be deduced from this new proposed improved ring shaped like potentials. Finally, our results can find many applications in nuclear physics and quantum chemistry such as cyclic benzene.

7 Acknowledgments

It is a pleasure for authors to thank the kind referee for his many useful comments on the manuscript.

References

[1] H. Hassanabadi, A.N. Ikot, and S. Zarrinkamar, Acta Phys. Polo. A **126** (2014) 647.

[2] H. Hartmann and D. Schuch, Int. J. Quant. Chem. **18** (1980) 125.

[3] A. Khare and R.K. Bhaduri, Am. J. Phys. **62** (1994) 1008.

[4] S.H. Dong, C.Y. Chen, and M. Lozada Cassou, Int. J. Quant. Chem. **102** (2005) 147.

[5] A.D. Alhaidari, J. Math. Phys: Math. Gen. **38** (2005) 3409.

[6] M. Aktas, J. Math. Chem. **49** (2011) 1831.

[7] M. Kibler and P. Winternitz, J. Phys. A: Math. Gen. **20** (1987) 4097.

[8] A.S. Zhedanov, J. Phys. A: Math. Gen. **26** (1993) 4633.

[9] C.Y. Chen, F.L. Lu, D.S. Sun, and S.D. Dong, Chin. Phys. B **22** (2013) 100302.

[10] C. Berkdemir and R. Sever, J. Math. Chem. **46** (2009) 1122.

[11] X.Q. Hu, G. Luo, Z.M. Wu, and L.B. Niu, Commun. Theor. Phys. **53** (2010) 242.

[12] Y.J. Xiao and Z.W. Long, Commun. Theor. Phys. **53** (2010) 54.

[13] F.L. Lu and C.Y. Chen, Chin. Phys. B **19** (2010) 100309.

[14] C.Y. Chen, C.L. Liu, and F.L. Lu, Phys. Lett. A **374** (2010) 1346.

[15] D.S. Sun, Y. You, F.L. Lu, C.Y. Chen, and S.H. Dong, Phys. Scr. **89** (2014) 045002.

[16] C.Y. Chen and Y.W. Liu, HEP & HP **23** (1999) 865.

[17] C. Quesne, J. Phys. A **21** (1988) 2093.

[18] C.Y. Chen, Y. You, X.H. Wang, and S.H. Dong, Phys. Lett. A **377** (2013) 1521.

[19] S.H. Dong, G.H. Sun, and M. Lozada Cassou, Phys. Lett. A **328** (2004) 299.

[20] C. Berkdemir, J. Math. Phys. **46** (2009) 13.

[21] M.C. Zhang, G.H. Sun, and S.H. Dong, Phys. Lett. A **374** (2010) 704.

[22] S.H. Dong, G.H. Sun, and M. Lozada Cassou, Phys. Lett. A **340** (2005) 94.

[23] M.C. Zhang, Int. J. Theor. Phys. **48** (2009) 2625.

[24] C.Y. Chen, Y. You, F.L. Lu, D.S. Sun, and S.H. Dong, Commun. Theor. Phys. **62** (2014) 331.

[25] C.Y. Chen, F.L. Lu, D.S. Sun, Y. You, and S.H. Dong, App. Math. Lett. **40** (2015) 90.

[26] A.A. Mararov, *et al.,* Nuovo Cimento A **52** (1967) 1061.

[27] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. **251** (1995) 267.

[28] S.H. Dong, *Factorization Method in Quantum Mechanics*, Springer, Dordrecht (2007).

[29] M.C. Zhang and G.Q.H. Fu, Ann. Phys. **327** (2012) 841.

[30] H. Ciftci, R.L. Hall, and N. Saad, J. Phys. A: Math. Gen. **38** (2005) 1147.

[31] A.F. Nikiforov and V.B. Uvarov, *Special Functions of Mathematical Physics*, Birkhauser, Basel (1988).

[32] C. Tezcan and R. Sever, Int. J. Theor. Phys. **48** (2009) 337.

[33] C. Berkdemir, A. Berkdemir, and J. Han, Chem. Phys. Lett. **417** (2006) 326.

[34] F. Buyukkilic, H. Egrifes, and Demirhan, Theor. Chem. Acc. **98** (1997) 191.