

Analytical Derivations of Single-Particle Matrix Elements in Nuclear Shell Model

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(Received February 1, 2016; revised manuscript received May 3, 2016)

Abstract We present analytical method to calculate single particle matrix elements used in atomic and nuclear physics. We show seven different formulas of matrix elements of the operator $f(r)d_r^m$ where $f(r) = r^\mu, r^\mu j_J(qr), V(r)$ corresponding to the Gaussian and the Yukawa potentials used in nuclear shell models and nuclear structure. In addition, we take into account a general integral formula of the matrix element $\langle n' l' | f(r) d_r^{(m)} | n l \rangle$ that covers all seven matrix elements obtained analytically.

PACS numbers: 21.90.+f, 21.60.Cs, 23.40.Hc

Key words: nuclear structure, shell model, nuclear matrix elements and nuclear structure

1 Introduction

Widely usage of the harmonic oscillator wave function (HOWF) in different calculations of nuclear and atomic physics belongs to the fact that their matrix elements can be found analytically without exposing to the numerical and approximation methods. Therefore, the calculations of the matrix elements of HOWF cover a wide range in the theoretical problems of nuclear physics. Especially those appear in the nuclear electron scattering and nuclear reactions besides the theoretical approximation techniques used to their explanation.^[1–2]

For the above mentioned problems, the matrix elements of HOWF contribute to two main fields: The first one directly comes from the diagonalization of nuclear Hamiltonian of the phenomenological two-body interaction such as the modified surface delta (MSD) and the M3Y interactions.^[3–4] Therefore, the Hamiltonian can be reduced to coefficients, 3-j symbol, 6-j symbol, 9-j symbol and Brody–Moshinsky coefficients besides to a radial matrix element representing Gaussian and Yukawa potentials.^[4] We calculate and derive them using our formalism in this work. Building of the Hamiltonian of such phenomenological two-body interactions and determination of their matrix elements can be considered as a central tool to solve the nuclear many-body problems. This may be performed using perturbation theory such as Tamm–Daccoff approximation (TDA) and random phase approximation (RPA).^[1–2,5] In the case of nuclear many-body problems of Hartree–Fock (HF), TDA, RPA, we have to calculate the matrix elements of the kinetic energy that can be written in terms of the matrix elements of the second derivative of the HOWF.^[2]

The second aspect refers to the calculations of

longitudinal- and transverse-form factors and the reduced probability density of the excited nuclear states.^[3] In theory of nuclear electron scattering, the matrix elements of the multiple operators have been used in two different forms: The first form is used by Donnely–Walecka–Haxton,^[6–7] and the second form is derived and proposed by Brown.^[8] In both forms, we need to use the single-particle matrix elements of the multiple operators corresponding to the HOWF. This can be abbreviated to the matrix elements of the spherical Bessel functions.^[9] The spherical Bessel functions play an obvious role in the calculations of the longitudinal form factor, but they do not have role in the calculations of the reduced transition probability matrix elements. While in the matrix elements of the transverse form factors, the first derivative of HOWF appears.^[9]

In the review of literature, researchers have used matrix elements of the Harmonic oscillator (HO) and derived mathematical formulation to calculate those matrix elements. Haxton *et al.*^[10] have evaluated analytical expression of the HO single-particle matrix elements of the multiple operators in terms of finite polynomials. Brody and Moshinsky^[11] derived the one-body and the reducible two-body matrix elements of the HOWF. On the other hand, the derivations of the single-particle form factors are proposed by de Forest and Walecka^[12] using the iterative method of the HO wave functions.

The goal of the present study is to derive analytical formulations of the HOWF and their first and second derivatives. This may serve as a basis to investigate more physical observables in nuclear and atomic physics. The paper is organized as following: In Sec. 2, we try to simplify the HOWF for finding its first and second derivatives. In

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Sec. 3, the standard integrals will be shown. Section 4 covers the results and discussion.

2 General Forms of Matrix Elements of HOWF and Its Derivatives

Simplification of the harmonic oscillator wave function and its derivatives can be considered as a first step in the derivation of the matrix elements. Starting from the representation of the harmonic oscillator wave function $U_{nl}(r)$ in terms of the length parameter (b), we obtain^[4]

$$U_{nl}(y) = \frac{1}{\sqrt{b^3}} \sum_{k=0}^{n-1} H_k^{nl} S_{kl}(y), \quad r = yb, \quad l \geq 0, \quad n \geq 1, \quad (1)$$

with H_k^{nl} being a constant and $S_{kl}(y)$ a function given by:

$$H_k^{nl} = \frac{(-1)^k \sqrt{2(n-1)! \Gamma(n+l+1/2)}}{k! \Gamma(n-k) \Gamma(k+l+3/2)}, \quad (2)$$

$$S_{kl}(y) = e^{-y^2/2} y^{2k+l}. \quad (3)$$

The orthonormality condition for the harmonic oscillator wave function is independent on the length parameter and gives

$$\sum_{k=0}^{n-1} \sum_{k'=0}^{n'-1} H_k^{nl} H_{k'}^{n'l'} \int_0^\infty S_{kl}(y) S_{k'l'}(y) y^2 dy = \delta_{nn'} \delta_{ll'}. \quad (4)$$

A part of the our present study demonstrates the first and second derivatives of the wave function $U_{nl}(r)$, $d_y^{(m)} U_{nl}(r) = b^m d_r^{(m)} U_{nl}(r)$, where $m = 1, 2$ and $d_r^{(m)} = d^m / dr^m$ is the m -th derivative of $U_{nl}(r)$ with respect to variable r .

Taking into consideration the fact that the derivatives of the first and last terms of $S_{kl}(y)$ are not zero, the derivative $d_y^{(m)} U_{nl}(y)$ with $m = 1, 2$ is given by:

$$d_y^{(m)} U_{nl}(y) = \frac{1}{\sqrt{b^3}} \sum_{k=0}^{n-1} H_k^{nl} d_y^{(m)} S_{kl}(y), \quad (5)$$

where the derivative, $d_y^{(m)} S_{kl}(y)$, can be calculated directly from Eq. (3). It can be written in terms of the

function itself $S_{kl}(y)$,

$$d_y^{(1)} S_{kl}(y) = \left(\frac{(2k+l)}{y} - y \right) S_{kl}(y), \quad (6)$$

for $m = 1$, and

$$d_y^{(2)} S_{kl}(y) = \left[\frac{(2k+l)(2k+l-1)}{y^2} (1 + 2(2k+l)) + y^2 \right] \times S_{kl}(y), \quad (7)$$

for $m = 2$.

In the present work, we find analytical derivatives to compute the matrix elements $\langle n' l' | f(r) d_r^{(m)} | n l \rangle$, for $m = 0, 1, 2$, in the cases $f(r) = r^\mu$, $r^\mu j_J(qr)$ and $V(r)$, with $j_J(qr)$ being the spherical Bessel function with the transferred linear momentum q , and $V(r)$ the Gaussian and Youkawa potentials.

In the form of integral representation, the matrix elements $\langle n' l' | f(r) d_r^{(m)} | n l \rangle$ are governed by

$$\langle n' l' | f(r) d_r^{(m)} | n l \rangle = b^{3-m} \int_0^\infty U_{n'l'}(y) f(by) y^2 \times d_y^{(m)} U_{nl}(y) dy. \quad (8)$$

In fact, Eq. (8) can be directly computed using *Mathematica* programming. Therefore, we try to check the results of our derived formulas by comparing them to those of Eq. (8) using *Mathematica 9*. Below, we start the derivations by expanding the left-hand-side of Eq. (8) to cover all possible cases of m and $f(r)$. Using Eq. (1) and Eq. (5) we obtain

$$\langle n' l' | f(r) d_r^{(m)} | n l \rangle = \frac{1}{b^m} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k'k}^{n'l'nl} \int_0^\infty S_{k'l'}(y) \times f(by) y^2 d_y^{(m)} S_{kl}(y) dy, \quad (9)$$

where $G_{k'k}^{n'l'nl} = H_k^{nl} H_{k'}^{n'l'}$ obtaining from Eq. (2) as demonstrated here

$$G_{k'k}^{n'l'nl} = \frac{2(-1)^{k+k'} \sqrt{(n-1)!(n'-1)!} \sqrt{\Gamma(n+l+1/2)\Gamma(n'+l'+1/2)}}{k!k'!\Gamma(n-k)\Gamma(n'-k')} \frac{\Gamma(k+l+3/2)\Gamma(k'+l'+3/2)}{\Gamma(k+l+3/2)\Gamma(k'+l'+3/2)}. \quad (10)$$

As a first simplification, we assume $m = 0$ and using Eq. (3). Consequently, Eq. (9) can be modified to

$$\langle n' l' | f(r) | n l \rangle = \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k'k}^{n'l'nl} \int_0^\infty e^{-y^2} y^{2k'+l'+2k+l+2} f(by) dy. \quad (11)$$

The second simplification of Eq. (9) is to find the matrix elements of $f(r)$ for the first derivative when $m = 1$. The matrix elements are obtained by putting both functions $S_{kl}(y)$ and $d_y^{(1)} S_{kl}(y)$ from Eq. (3) and Eq. (6), respectively, into Eq. (9). It yields

$$\langle n' l' | f(r) d_r^{(1)} | n l \rangle = \frac{1}{b} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k'k}^{n'l'nl} \int_0^\infty e^{-y^2} y^{2k'+l'+2k+l+2} f(yb) \left(\frac{2k+l}{y} - y \right) dy. \quad (12)$$

Similarly, the final step for simplifying Eq. (9) occurs in the case of second derivative ($m = 2$), by putting Eq. (3) and Eq. (7) into Eq. (9) we find:

$$\langle n' l' | f(r) d_r^{(2)} | n l \rangle = \frac{1}{b^2} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k'k}^{n'l'nl} \int_0^\infty e^{-y^2} y^{2k'+l'+2k+l+2} f(yb) \left[\frac{(2k+l)(2k+l-1)}{y^2} + y^2 - [1 + 2(2k+l)] \right] dy. \quad (13)$$

In next section, we demonstrate analytical methods to calculate the matrix elements shown in Eq. (11) and Eq. (13) in which $f(r) = r^\mu r^\mu j_J(qr)$ and $V(r)$ define the Gaussian and Yukawa potentials. We also focus on calculating all forms of matrix elements of $f(r)$ in Eq. (11). In addition, $f(r) = r^\mu r^\mu j_J(qr)$ is utilized in Eq. (12). Finally, We only use $f(r) = r^\mu$ in Eq. (13).

3 Single-Particle Matrix Elements with- and without-Derivative

In nuclear electron scattering, the multiple operators of the longitudinal, and the transverse form factors can be defined in terms of spherical Bessel functions with the scalar and vector spherical harmonics.^[9] The matrix elements of the spherical harmonics (angular-dependent part) can be reduced to some coefficients including three-j symbols. But in the radial part, the matrix elements contained spherical Bessel functions can be determined from integration. To calculate such matrix elements, we need to propose the standard integration ($I^J(\rho, \alpha, \nu, c) = \int_0^\infty e^{-\alpha^2 x^2} x^\rho j_\nu(cx) dx$) which can be evaluated in terms of the confluent hypergeometric functions, ${}_1F^1(a, b, z)$, as shown below:^[13]

$$I^J(\rho, \alpha, \nu, c) = \frac{\sqrt{\pi} \Gamma\left(\frac{\rho+\nu+1}{2}\right) \left(\frac{c}{2\alpha}\right)^\nu}{4 \Gamma(\nu+3/2) \alpha^{\rho+1}} e^{-(c/2\alpha)^\nu} {}_1F^1\left(\frac{\nu-\rho+2}{2}, \nu+\frac{3}{2}, \frac{c^2}{4\alpha^2}\right). \quad (14)$$

From Eq. (1), Eq. (2), and Eq. (3), we can clearly see that both y^l and $e^{-y^2/2}$ come from the asymptotic solutions if $y \rightarrow 0$ and $y \rightarrow \infty$, respectively. The integration of their multiplication ($I^G(\rho, \alpha, s) = \int_0^\infty e^{-\alpha^2 x^2 - s x} x^\rho dx$) can be simplified to^[13]

$$I^G(\rho, \alpha, s) = \frac{e^{s^2/4\alpha^2}}{\alpha^{\rho+1}} \int_{s/2\alpha}^\infty e^{-x^2} \left(x - \frac{s}{2\alpha}\right)^\rho dx. \quad (15)$$

Now, one can use the binomial expansion to expand the term of $(x - s/2\alpha)^\rho$ achieving

$$I^G(\rho, \alpha, s) = \frac{e^{s^2/4\alpha^2}}{\alpha^{\rho+1}} \sum_{\lambda=0}^{\rho} \frac{\rho!}{\lambda!(\rho-\lambda)!} \left(-\frac{s}{2\alpha}\right)^{\rho-\lambda} \int_{s/2\alpha}^\infty e^{-x^2} x^\lambda dx. \quad (16)$$

The integral of Eq. (16) can be expressed in terms of the upper-incomplete gamma function or a complete gamma function and a confluent hypergeometric function^[13]

$$I^G(\rho, \alpha, s) = \frac{e^{\beta^2}}{\alpha^{\rho+1}} \sum_{\lambda=0}^{\rho} \frac{\rho!}{\lambda!(\rho-\lambda)!} (-\beta)^{\rho-\lambda} \left[\frac{1}{2} \Gamma\left(\frac{\lambda+1}{2}\right) - \frac{e^{-\beta^2}}{1+\lambda} \beta^{1+\lambda} {}_1F^1\left(1, \frac{3+\lambda}{2}, \beta^2\right) \right]. \quad (17)$$

Herein, $\beta = s/2\alpha$, and both $\Gamma((\lambda+1)/2, \beta^2)$ and $\Gamma(\lambda+1)/2$ are upper incomplete and complete gamma functions, respectively.

In a special case, if the coefficient of x presented in Eq. (16) is zero ($s = 0$). Consequently, both relations shown in Eq. (14) and Eq. (17) are equal if we assume $\nu = c = 0$ and $j_0(0) = 1$ in Eq. (14). As a result, the confluent hypergeometric functions equals one (${}_1F^1((2-\rho)/2, 3/2, 0) = 1$) in Eq. (14), then using it to find the integral $I^G(\rho, \alpha, 0)$ presented in Eq. (17) in terms of the integral $I^J(\rho, \alpha, 0, 0)$,

$$I^G(\rho, \alpha, 0) = \frac{\alpha^{-(1+\rho)}}{2} \Gamma\left(\frac{1+\rho}{2}\right). \quad (18)$$

To derive the relations for determining the matrix elements, we first start from a simplest case of Eq. (11) where $f(r) = r^\mu$ or $f(r) = r^\mu j_0(0)$ with $j_0(0) = 1$. One thus can write the matrix element of $\langle n' l' | r^\mu | n l \rangle$ displayed in Eq. (11) in terms of $I^G(\rho, \alpha, 0)$

$$\langle n' l' | r^\mu | n l \rangle = \frac{b^\mu}{2} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k'k}^{n'l'nl} \Gamma\left(\frac{2k'+l'+2k+l+\mu+3}{2}\right). \quad (19)$$

Now, we derive the matrix elements of the Gaussian and the Yukawa potentials, we begin from their general definitions

$$V^G(r, t_1 t_2) = t_1 e^{-t_2 r^2}, \quad V^Y(r, t_1 t_2) = t_1 t_2 \frac{e^{-r/t_2}}{r}. \quad (20)$$

We assume $f(yb) = V^G(yb, t_1, t_2)$ and plugging it into Eq. (11) to find the matrix elements of the Gaussian potential. In addition, we use Eq. (18) to introduce $I^G(\rho, \alpha, s)$, then one can obtain $\alpha = 1 + b^2 t_2$, $s = 0$, and $\rho = 2k' + l' + 2k + l + 2$. Using the above mentioned steps, the matrix elements of the Gaussian potential are governed by

$$\langle n' l' | V^G(r, t_1, t_2) | n l \rangle = \frac{t_1}{2} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k'k}^{n'l'nl} (1 + b^2 t_2)^{-(1+\rho)/2} \Gamma\left(\frac{2k'+l'+2k+l+3}{2}\right). \quad (21)$$

Similarly, the matrix elements of the Yukawa potential are achieved using $f(yb) = V^Y(y, b, t_1 t_2)$. We have also found $I^G(\rho, \alpha, s)$ from Eq. (17) with $\alpha = 1, s = b/t_2$ and $\rho = 2k' + l' + 2k + l + 1$. We thus obtain

$$\langle n' l' | V^Y(r, t_1, t_2) | n l \rangle = \frac{t_1 t_2}{b} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k' k}^{n' l' n l} \sum_{\lambda=0}^{\rho} \frac{\rho! (-\beta)^{\rho-\lambda}}{\lambda! (\rho-\lambda)!} \left[\frac{e^{\beta^2}}{2} \Gamma\left(\frac{\lambda+1}{2}\right) - \frac{\beta^{1+\lambda}}{1+\lambda} {}_1F^1\left(1, \frac{3+\lambda}{2}, \beta^2\right) \right], \quad (22)$$

with $\beta = s/2 = b/2t_2$.

We should notice that some times in nuclear interactions such as M3Y-interaction, $f(r)$ contains an exponential function. In that case, we need to find the matrix elements of $\langle n' l' | f(r) | n l \rangle$ in the relative coordinates from the transformation $f(yb) \rightarrow f(\sqrt{2} yb)$ of Eq. (11).^[14] It shows that $b \rightarrow \sqrt{2}b$ in the first row of Eq. (22).

Here, we demonstrate the radial matrix elements indicating the electron scattering. The radial part is $f(r) = r^\mu j_J(qr)$ obtained by putting $f(yb) = y^\mu b^\mu j_J(qyb)$ into Eq. (11). Furthermore, the confluent hypergeometric function shown in Eq. (14) is plugged into Eq. (11) with parameters $\alpha = 1, c = qb, \nu = J$ and $\rho = 2k' + l' + 2k + l + \mu + 2$. Consequently, we get

$$\langle n' l' | r^\mu j_J(qr) | n l \rangle = \frac{\sqrt{\pi} b^\mu \beta^J}{4 \Gamma(J + 3/2)} e^{-\beta^2} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k' k}^{n' l' n l} \Gamma\left(\frac{1+J+\rho}{2}\right) {}_1F^1\left(\frac{J-\rho+2}{2}, J + \frac{3}{2}, \beta^2\right), \quad (23)$$

where $\beta = q b/2$.

Finally, we present the derivatives of Eq. (12) for the following three situations: The first derivative of $f(yb) = y^\mu b^\mu j_J(qyb)$ with $m = 1, f(yb) = y^\mu b^\mu$ with $m = 1$, and the second derivative of $f(yb) = y^\mu b^\mu$ with $m = 2$.

The matrix elements of $\langle n' l' | r^\mu j_J(qr) d_r^{(1)} | n l \rangle$ presented in Eq. (12) can be obtained using the confluent hypergeometric functions (Eq. (14)). In which, $I^J(\rho_1, \alpha, \nu, c)$ and $I^J(\rho_2, \alpha, \nu, c)$ are used with parameters $\alpha = 1, \nu = J$, and $c = qb$.

$$\begin{aligned} \langle n' l' | r^\mu j_J(qr) d_r^{(1)} | n l \rangle &= \frac{\sqrt{\pi} b^{\mu-1} \beta^J}{4 \Gamma(J + 3/2)} e^{-\beta^2} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k' k}^{n' l' n l} \left[(2k+l) \Gamma\left(\frac{1+J+\rho_1}{2}\right) {}_1F^1\left(\frac{J-\rho_1+2}{2}, J + \frac{3}{2}, \beta^2\right) \right. \\ &\quad \left. - \Gamma\left(\frac{1+J+\rho_2}{2}\right) {}_1F^1\left(\frac{J-\rho_2+2}{2}, J + \frac{3}{2}, \beta^2\right) \right], \end{aligned} \quad (24)$$

herein, $\rho_1 = 2k' + l' + 2k + l + \mu + 1, \rho_2 = \rho_1 + 2$, and $\beta = c/2 = q b/2$.

In similar way, the first derivative ($m = 1$), $f(yb) = y^\mu b^\mu$, and the integration presented in Eq. (18) are utilized to obtain

$$\langle n' l' | r^\mu d_r^{(1)} | n l \rangle = \frac{b^{\mu-1}}{2} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k' k}^{n' l' n l} \left[(2k+l) \Gamma\left(\frac{\rho_1+1}{2}\right) - \Gamma\left(\frac{\rho_2+1}{2}\right) \right], \quad (25)$$

herein, $\rho_1 = 2k' + l' + 2k + l + \mu + 1$ and $\rho_2 = \rho_1 + 2$.

Then, we consider the second derivative ($m = 2$) to find its matrix elements using Eq. (18) three times in Eq. (13)

$$\begin{aligned} \langle n' l' | r^\mu d_r^{(2)} | n l \rangle &= \frac{b^{\mu-2}}{2} \sum_{k'=0}^{n'-1} \sum_{k=0}^{n-1} G_{k' k}^{n' l' n l} \left[(2k+l)(2k+l-1) \Gamma\left(\frac{\rho_1+1}{2}\right) (1+2(2k+l)) \right. \\ &\quad \left. \times \Gamma\left(\frac{\rho_2+1}{2}\right) + \Gamma\left(\frac{\rho_3+1}{2}\right) \right], \end{aligned} \quad (26)$$

where $\rho_1 = 2k' + l' + 2k + l + \mu, \rho_2 = \rho_1 + 2$, and $\rho_3 = \rho_1 + 4$.

4 Summary and Conclusions

The harmonic oscillator wave function $U_{nl}(y)$ can be written in terms of length parameter (b) depending on the quantized harmonic oscillator energy ($\hbar\omega$). The quantized energy is different for various nuclei (mass number). Therefore, the length parameter varies with the mass number (A). We should mention that the length parameter is fixed at $b = 1.761$ fm in our study.

When $f(r)$ includes spherical Bessel functions, $j_J(qr)$, the value of transferred momentum (q) is confined in the ranges of $(1.5-3.5)$ fm⁻¹ with step size being 0.5. We have considered that the ranges of the main quantum numbers (n and n') and the orbital quantum numbers (l and l') of the OXBASH sequence are $1 \leq (n, n') \leq 5$ and $0 \leq (l, l') \leq 8$.

The standard integrals are utilized to derive the relations that calculate the matrix elements. $\langle n' l' | f(r) d_r^{(m)} | n l \rangle$. To check the validity of our derivation we have used the integration method for the derived matrix elements presented in Eq. (8). In fact, the results of both integration and derivation methods were perfectly consistent and conceit.

In Table 1, we demonstrate the values of $\langle n' l' | f(r) d_r^{(m)} | n l \rangle$ for different situations. One can see more data in the supplementary materials. The simplest form of the derived matrix element is $\langle n' l' | f(r) | n l \rangle$ where $f(r) = r^\mu$ and $\mu \geq 0$.

This form of the matrix elements is widely used to investigate the reduced transition probability of excited nuclear states for different values of μ . However, there is a special case when $\mu = 2$, $n' = n$, and $l' = l$, the matrix elements are used to find the root-mean square of the nuclei corresponding to the harmonic oscillator wave-functions. It can be compared with its direct value of $b\sqrt{2n+l-1/2}$, where the length parameter is calculated from $b = \sqrt{\hbar/(m\omega)}$.^[2-3] On the other hand, the same matrix elements, $\langle n'l'|r^2|nl\rangle$, can be used to determine eigenvalues of the harmonic oscillator potential, $V(r) = m\omega^2 r^2/2$.

The eigenvalues may be represented in terms of the energy steps ($\hbar\omega$) and the length parameter (b) of the harmonic oscillator such as $\langle nl|V(r)|nl\rangle = \langle nl|r^2|nl\rangle \hbar\omega/2b^2$. Therefore, one may compare it to the exact solutions of the harmonic oscillator, which are defined as $(2n+l-1/2)\hbar\omega/2$.^[2]

In addition, the matrix elements of r^μ ($\langle n'l'|r^\mu|nl\rangle$) have been calculated from the derived relation presented in Eq. (19) and the integration form shown in Eq. (8) where it is assumed that $m = 0$ and $f(yb) = b^\mu y^\mu$. The results of the calculated matrix elements are provided as supplementary material in Table 1.

The phenomenological nuclear interaction requires to determine the matrix elements of $f(r)$ ($\langle n'l'|f(r)|nl\rangle$) with $f(r)$ being the respective Gaussian and Yukawa potentials represented in Eq. (20). In the calculations of matrix elements, we have considered that the parameters $t_1 = -35.7$ MeV and $t_2 = 0.265$ fm⁻² for the Gaussian potential and $t_1 = -2777$ MeV and $t_2 = 0.4$ fm for the Yukawa potential. The non-degenerate eigenstates of the harmonic oscillator due to the main and orbital quantum numbers (n and l), some of the consequent states of harmonic oscillator repeat themselves with the same quantum numbers. Therefore, both matrix elements of Gaussian and Yukawa potentials remain unchanged.

It is interesting to see that the equation of matrix elements of Yukawa potential displayed in Eq. (22) is perfectly correct using Wolfram Mathematica. This is due to taking into account good expansion of the hypergeometric functions. In addition, if one interests other programming languages such as Fortran we suggest to use a numerical method to find $I^G(\rho, 1, b/t_2)$ in the first row of Eq. (22). That is equivalent to the numerical method for obtaining the incomplete gamma function $\Gamma((k+1)/2, \beta^2)$ in the second row of Eq. (22).

Table 1 The matrix elements of $r^\mu d_r^{(m)}$ with $m = 0, 1, 2$ for $b = 1.761$ fm and $0 \leq \mu \leq 2$ for the derived equations and integral form in Eq. (8).

i	μ	n'	l'	n	l	$\langle n'l' r^\mu nl\rangle$		$\langle n'l' r^\mu d_r^{(1)} nl\rangle$		$\langle n'l' r^\mu d_r^{(2)} nl\rangle$	
						Eq. (8)	Eq. (19)	Eq. (8)	Eq. (25)	Eq. (8)	Eq. (26)
1	0	1	0	1	0	1.000 000	1.000 000	-0.640 640	-0.640 640	0.161 172	0.161 172
2	1	1	0	1	0	1.987 450	1.987 450	-1.500 000	-1.500 000	0.640 641	0.640 641
3	2	1	0	1	0	4.653 410	4.653 410	-3.974 890	-3.974 890	2.250 000	2.250 000
4	0	1	1	1	0	0.921 318	0.921 318	-0.695 350	-0.695 350	0.296 981	0.296 981
5	1	1	1	1	0	2.157 180	2.157 180	-1.842 640	-1.842 640	1.043 030	1.043 030
6	2	1	1	1	0	5.716 360	5.716 360	-5.392 940	-5.392 940	3.685 270	3.685 270
7	0	1	1	1	0	0.921 318	0.921 318	-0.695 350	-0.695 350	0.296 981	0.296 981
8	1	1	1	1	0	2.157 180	2.157 180	-1.842 640	-1.842 640	1.043 030	1.043 030
9	2	1	1	1	0	5.716 360	5.716 360	-5.392 940	-5.392 940	3.685 270	3.685 270
10	0	1	2	1	0	0.774 597	0.774 597	-0.661 650	-0.661 650	0.374 530	0.374 530
11	1	1	2	1	0	2.052 630	2.052 630	-1.936 490	-1.936 490	1.323 300	1.323 300
12	2	1	2	1	0	6.007 530	6.007 530	-6.157 880	-6.157 880	4.841 230	4.841 230
13	0	1	2	1	0	0.774 597	0.774 597	-0.661 650	-0.661 650	0.374 530	0.374 530
14	1	1	2	1	0	2.052 630	2.052 630	-1.936 490	-1.936 490	1.323 300	1.323 300
15	2	1	2	1	0	6.007 530	6.007 530	-6.157 880	-6.157 880	4.841 230	4.841 230
16	0	2	0	1	0	0.000 000	0.000 000	0.261 541	0.261 541	-0.394 790	-0.394 790
17	1	2	0	1	0	-0.811 370	-0.811 370	1.224 740	1.224 740	-1.307 700	-1.307 700
18	2	2	0	1	0	-3.799 500	-3.799 500	4.868 230	4.868 230	-4.898 980	-4.898 980
19	0	1	3	1	0	0.622 924	0.622 924	-0.587 680	-0.587 680	0.401 592	0.401 592
20	1	1	3	1	0	1.823 150	1.823 150	-1.868 770	-1.868 770	1.469 200	1.469 200
21	2	1	3	1	0	5.797 450	5.797 450	-6.381 020	-6.381 020	5.606 320	5.606 320
22	0	1	3	1	0	0.622 924	0.622 924	-0.587 680	-0.587 680	0.401 592	0.401 592
23	1	1	3	1	0	1.823 150	1.823 150	-1.868 770	-1.868 770	1.469 200	1.469 200
24	2	1	3	1	0	5.797 450	5.797 450	-6.381 020	-6.381 020	5.606 320	5.606 320
25	0	2	1	1	0	0.291 346	0.291 346	0.000 000	0.000 000	-0.281 740	-0.281 740

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26	1	2	1	1	0	0.000 000	0.000 000	0.582 692	0.582 692	-1.099 450	-1.099 450
27	2	2	1	1	0	-1.807 670	-1.807 670	3.410 800	3.410 800	-4.661 540	-4.661 540
28	0	2	1	1	0	0.291 346	0.291 346	0.000 000	0.000 000	-0.281 740	-0.281 740
29	1	2	1	1	0	0.000 000	0.000 000	0.582 692	0.582 692	-1.099 450	-1.099 450
30	2	2	1	1	0	-1.807 670	-1.807 670	3.410 800	3.410 800	-4.661 540	-4.661 540
31	0	1	4	1	0	0.487 950	0.487 950	-0.500 160	-0.500 160	0.393 219	0.393 219
32	1	1	4	1	0	1.551 640	1.551 640	-1.707 830	-1.707 830	1.500 480	1.500 480
33	2	1	4	1	0	5.298 150	5.298 150	-6.206 560	-6.206 560	5.977 390	5.977 390
34	0	1	4	1	0	0.487 950	0.487 950	-0.500 160	-0.500 160	0.393 219	0.393 219
35	1	1	4	1	0	1.551 640	1.551 640	-1.707 830	-1.707 830	1.500 480	1.500 480
36	2	1	4	1	0	5.298 150	5.298 150	-6.206 560	-6.206 560	5.977 390	5.977 390
37	0	1	0	1	1	0.921 318	0.921 318	-0.231 780	-0.231 780	-0.296 980	-0.296 980
38	1	1	0	1	1	2.157 180	2.157 180	-0.921 320	-0.921 320	-0.347 680	-0.347 680
39	2	1	0	1	1	5.716 360	5.716 360	-3.235 770	-3.235 770	0.000 000	0.000 000
40	0	1	1	1	1	1.000 000	1.000 000	-0.427 090	-0.427 090	-0.161 170	-0.161 170
41	1	1	1	1	1	2.649 930	2.649 930	-1.500 000	-1.500 000	0.000 000	0.000 000
42	2	1	1	1	1	7.755 690	7.755 690	-5.299 860	-5.299 860	1.250 000	1.250 000
43	0	1	1	1	1	1.000 000	1.000 000	-0.427 090	-0.427 090	-0.161 170	-0.161 170
44	1	1	1	1	1	2.649 930	2.649 930	-1.500 000	-1.500 000	0.000 000	0.000 000
45	2	1	1	1	1	7.755 690	7.755 690	-5.299 860	-5.299 860	1.250 000	1.250 000
46	0	1	2	1	1	0.951 533	0.951 533	-0.538 620	-0.538 620	0.000 000	0.000 000
47	1	1	2	1	1	2.784 900	2.784 900	-1.903 070	-1.903 070	0.448 848	0.448 848
48	2	1	2	1	1	8.855 750	8.855 750	-6.962 260	-6.962 260	2.854 600	2.854 600
49	0	1	2	1	1	0.951 533	0.951 533	-0.538 620	-0.538 620	0.000 000	0.000 000
50	1	1	2	1	1	2.784 900	2.784 900	-1.903 070	-1.903 070	0.448 848	0.448 848
51	2	1	2	1	1	8.855 750	8.855 750	-6.962 260	-6.962 260	2.854 600	2.854 600
52	0	2	0	1	1	-0.376 130	-0.376 130	0.567 753	0.567 753	-0.363 730	-0.363 730
53	1	2	0	1	1	-1.761 330	-1.761 330	1.880 630	1.880 630	-1.135 510	-1.135 510
54	2	2	0	1	1	-7.001 090	-7.001 090	7.045 310	7.045 310	-4.513 520	-4.513 520
55	0	1	3	1	1	0.845 154	0.845 154	-0.577 540	-0.577 540	0.136 215	0.136 215
56	1	1	3	1	1	2.687 520	2.687 520	-2.112 890	-2.112 890	0.866 305	0.866 305
57	2	1	3	1	1	9.176 660	9.176 660	-8.062 550	-8.062 550	4.437 060	4.437 060
58	0	1	3	1	1	0.845 154	0.845 154	-0.577 540	-0.577 540	0.136 215	0.136 215
59	1	1	3	1	1	2.687 520	2.687 520	-2.112 890	-2.112 890	0.866 305	0.866 305
60	2	1	3	1	1	9.176 660	9.176 660	-8.062 550	-8.062 550	4.437 060	4.437 060
61	0	2	1	1	1	0.000 000	0.000 000	0.405 177	0.405 177	-0.509 670	-0.509 670
62	1	2	1	1	1	-0.837 980	-0.837 980	1.581 140	1.581 140	-1.620 710	-1.620 710
63	2	2	1	1	1	-4.905 130	-4.905 130	6.703 850	6.703 850	-6.324 560	-6.324 560
64	0	2	1	1	1	0.000 000	0.000 000	0.405 177	0.405 177	-0.509 670	-0.509 670
65	1	2	1	1	1	-0.837 980	-0.837 980	1.581 140	1.581 140	-1.620 710	-1.620 710
66	2	2	1	1	1	-4.905 130	-4.905 130	6.703 850	6.703 850	-6.324 560	-6.324 560
67	0	1	4	1	1	0.719 291	0.719 291	-0.565 500	-0.565 500	0.231 859	0.231 859
68	1	1	4	1	1	2.456 050	2.456 050	-2.157 870	-2.157 870	1.187 540	1.187 540
69	2	1	4	1	1	8.925 760	8.925 760	-8.596 190	-8.596 190	5.754 330	5.754 330
70	0	1	4	1	1	0.719 291	0.719 291	-0.565 500	-0.565 500	0.231 859	0.231 859
71	1	1	4	1	1	2.456 050	2.456 050	-2.157 870	-2.157 870	1.187 540	1.187 540
72	2	1	4	1	1	8.925 760	8.925 760	-8.596 190	-8.596 190	5.754 330	5.754 330
73	0	1	0	1	1	0.921 318	0.921 318	-0.231 780	-0.231 780	-0.296 980	-0.296 980
74	1	1	0	1	1	2.157 180	2.157 180	-0.921 320	-0.921 320	-0.347 680	-0.347 680
75	2	1	0	1	1	5.716 360	5.716 360	-3.235 770	-3.235 770	0.000 000	0.000 000
76	0	1	1	1	1	1.000 000	1.000 000	-0.427 090	-0.427 090	-0.161 170	-0.161 170
77	1	1	1	1	1	2.649 930	2.649 930	-1.500 000	-1.500 000	0.000 000	0.000 000
78	2	1	1	1	1	7.755 690	7.755 690	-5.299 860	-5.299 860	1.250 000	1.250 000
79	0	1	1	1	1	1.000 000	1.000 000	-0.427 090	-0.427 090	-0.161 170	-0.161 170
80	1	1	1	1	1	2.649 930	2.649 930	-1.500 000	-1.500 000	0.000 000	0.000 000
81	2	1	1	1	1	7.755 690	7.755 690	-5.299 860	-5.299 860	1.250 000	1.250 000
82	0	1	2	1	1	0.951 533	0.951 533	-0.538 620	-0.538 620	0.000 000	0.000 000

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83	1	1	2	1	1	2.784 900	2.784 900	-1.903 070	-1.903 070	0.448 848	0.448 848
84	2	1	2	1	1	8.855 750	8.855 750	-6.962 260	-6.962 260	2.854 600	2.854 600
85	0	1	2	1	1	0.951 533	0.951 533	-0.538 620	-0.538 620	0.000 000	0.000 000
86	1	1	2	1	1	2.784 900	2.784 900	-1.903 070	-1.903 070	0.448 848	0.448 848
87	2	1	2	1	1	8.855 750	8.855 750	-6.962 260	-6.962 260	2.854 600	2.854 600
88	0	2	0	1	1	-0.376 130	-0.376 130	0.567 753	0.567 753	-0.363 730	-0.363 730
89	1	2	0	1	1	-1.761 330	-1.761 330	1.880 630	1.880 630	-1.135 510	-1.135 510
90	2	2	0	1	1	-7.001 090	-7.001 090	7.045 310	7.045 310	-4.513 520	-4.513 520
91	0	1	3	1	1	0.845 154	0.845 154	-0.577 540	-0.577 540	0.136 215	0.136 215
92	1	1	3	1	1	2.687 520	2.687 520	-2.112 890	-2.112 890	0.866 305	0.866 305
93	2	1	3	1	1	9.176 660	9.176 660	-8.062 550	-8.062 550	4.437 060	4.437 060
94	0	1	3	1	1	0.845 154	0.845 154	-0.577 540	-0.577 540	0.136 215	0.136 215
95	1	1	3	1	1	2.687 520	2.687 520	-2.112 890	-2.112 890	0.866 305	0.866 305
96	2	1	3	1	1	9.176 660	9.176 660	-8.062 550	-8.062 550	4.437 060	4.437 060
97	0	2	1	1	1	0.000 000	0.000 000	0.405 177	0.405 177	-0.509 670	-0.509 670
98	1	2	1	1	1	-0.837 980	-0.837 980	1.581 140	1.581 140	-1.620 710	-1.620 710
99	2	2	1	1	1	-4.905 130	-4.905 130	6.703 850	6.703 850	-6.324 560	-6.324 560
100	0	2	1	1	1	0.000 000	0.000 000	0.405 177	0.405 177	-0.509 670	-0.509 670

Table 2 represents the results of the matrix elements of $\langle n'l'|V^G(r, t_1, t_2)|nl\rangle$ and $\langle n'l'|V^Y(r, t_1, t_2)|nl\rangle$ in the supplementary material. The results represent their calculated values from the respective derived relations. Then we compare to their corresponding values obtained from the integration methods shown in Eq. (8) where $m = 0$, $f(yb) = V^G(yb, t_1, t_2)$ for $\langle n'l'|V^G(r, t_1, t_2)|nl\rangle$ and $f(yb) = V^Y(yb, t_1, t_2)$ for $\langle n'l'|V^Y(r, t_1, t_2)|nl\rangle$.

The important roles of the single-particle form factors come from the generalized matrix element $\langle n'l'|f(r)|nl\rangle$ where $f(r) = r^\mu j_J(qr)$, J is the total spin of the nucleus (spin-multipolarity), and q indicates the transferred momentum. To distinguish between the Coulomb and the transverse form factors we suggest using $m = 0, J \geq 0$ and $\mu = 0$ for Coulomb form factor, and $m = 1, J \geq 1$, and $\mu = 0$ for transverse form factor.^[12]

In Eq. (23), the range of μ is $0 \leq \mu \leq 2$ for computing the matrix elements of the Coulomb form factors which are $\langle n'l'|j_J(qr) r^\mu|nl\rangle$. We provide results of this matrix elements in Table 3 in the supplementary materials. In addition, $1 \leq \mu \leq 2$ is used for the matrix elements of the transverse form factors in Eq. (24) defined as $\langle n'l'|j_J(qr) r^\mu d_r^{(1)}|nl\rangle$. Furthermore, the Eq. (8) has been introduced to calculate the matrix elements using integration methods with $f(yb) = b^\mu y^\mu j_J(qby)$. We thus get $\langle n'l'|j_J(qr) r^\mu|nl\rangle$ for $m = 0$ and $\langle n'l'|j_J(qr) r^\mu d_r^{(1)}|nl\rangle$ for $m = 1$.

Finally, the matrix elements shown in Eq. (25) and Eq. (26) play an important role in calculating the matrix elements of the kinetic and the potential energies. Using $\langle n'l'|r^\mu d_r^{(1)}|nl\rangle$ with $\mu = 0$ and $\langle n'l'|r^\mu d_r^{(2)}|nl\rangle$ with $\mu = -1$, one may compute the matrix element of the kinetic energy from the following relation $-\hbar\omega b^2 \langle n'l'|d_r^{(2)} + 2r^{-1} d_r^{(1)}|nl\rangle/2$.^[2] As a result, the matrix elements of the total Hamiltonian including the kinetic and potential energies are calculated using $-\hbar\omega b^2 \langle n'l'|d_r^{(2)} + 2r^{-1} d_r^{(1)}|nl\rangle/2 + \langle n'l'|r^2|nl\rangle \hbar\omega/2b^2$.^[2,14] We should note that the expectation value of the total Hamiltonian is $\langle nl|H|nl\rangle = (2n + l - 1/2) \hbar\omega$.

The calculated matrix elements of $\langle n'l'|r^\mu d_r^{(1)}|nl\rangle$ and $\langle n'l'|r^\mu d_r^{(2)}|nl\rangle$ presented in Eq. (25) and Eq. (26) and their integration forms demonstrated in Eq. (8) are listed in Table 1 for the range $0 \leq \mu \leq 2$.

Table 2 The calculated matrix elements of $V^G(r, t_1, t_2)$ and $V^Y(r, t_1, t_2)$ with $b = 1.761$ fm from the derived equations and integral forms.

i	n'	l'	n	l	$\langle n'l' V^G(r, t_1, t_2,) nl\rangle$		$\langle n'l' V^Y(r, t_1, t_2,) nl\rangle$	
					Eq. (8)	Eq. (21)	Eq. (8)	Eq. (22)
1	1	0	1	0	-14.514 700	-14.514 700	-57.805 302	-57.805 302
2	1	1	1	0	-9.906 760	-9.906 760	-17.322 399	-17.322 399
3	1	1	1	0	-9.906 760	-9.906 760	-17.322 399	-17.322 399
4	1	2	1	0	-6.170 370	-6.170 370	-5.729 880	-5.729 880
5	1	2	1	0	-6.170 370	-6.170 370	-5.729 880	-5.729 880
6	2	0	1	0	-8.020 620	-8.020 620	-61.737 000	-61.737 000
7	1	3	1	0	-3.676 080	-3.676 080	-2.040 940	-2.040 940
8	1	3	1	0	-3.676 080	-3.676 080	-2.040 940	-2.040 940
9	2	1	1	0	-8.786 650	-8.786 650	-23.570 900	-23.570 900
10	2	1	1	0	-8.786 650	-8.786 650	-23.570 900	-23.570 900

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11	1	4	1	0	-2.133 230	-2.133 230	-0.769 360	-0.769 360
12	1	4	1	0	-2.133 230	-2.133 230	-0.769 360	-0.769 360
13	2	2	1	0	-7.018 440	-7.018 440	-9.087 570	-9.087 570
14	2	2	1	0	-7.018 440	-7.018 440	-9.087 570	-9.087 570
15	3	0	1	0	-4.045 910	-4.045 910	-61.054 001	-61.054 001
16	1	5	1	0	-1.216 590	-1.216 590	-0.303 340	-0.303 340
17	1	5	1	0	-1.216 590	-1.216 590	-0.303 340	-0.303 340
18	2	3	1	0	-4.944 970	-4.944 970	-3.618 080	-3.618 080
19	2	3	1	0	-4.944 970	-4.944 970	-3.618 080	-3.618 080
20	3	1	1	0	-6.805 570	-6.805 570	-27.197 399	-27.197 399
21	1	0	1	1	-9.906 760	-9.906 760	-17.322 399	-17.322 399
22	1	1	1	1	-7.965 920	-7.965 920	-7.397 250	-7.397 250
23	1	1	1	1	-7.965 920	-7.965 920	-7.397 250	-7.397 250
24	1	2	1	1	-5.615 300	-5.615 300	-3.117 580	-3.117 580
25	1	2	1	1	-5.615 300	-5.615 300	-3.117 580	-3.117 580
26	2	0	1	1	-3.254 680	-3.254 680	-16.286 200	-16.286 200
27	1	3	1	1	-3.694 870	-3.694 870	-1.332 570	-1.332 570
28	1	3	1	1	-3.694 870	-3.694 870	-1.332 570	-1.332 570
29	2	1	1	1	-5.682 760	-5.682 760	-9.203 070	-9.203 070
30	2	1	1	1	-5.682 760	-5.682 760	-9.203 070	-9.203 070
31	1	4	1	1	-2.329 600	-2.329 600	-0.580 860	-0.580 860
32	1	4	1	1	-2.329 600	-2.329 600	-0.580 860	-0.580 860
33	2	2	1	1	-5.563 440	-5.563 440	-4.600 280	-4.600 280
34	2	2	1	1	-5.563 440	-5.563 440	-4.600 280	-4.600 280
35	3	0	1	1	-0.249 720	-0.249 720	-14.327 400	-14.327 400
36	1	5	1	1	-1.426 610	-1.426 610	-0.258 260	-0.258 260
37	1	5	1	1	-1.426 610	-1.426 610	-0.258 260	-0.258 260
38	2	3	1	1	-4.492 290	-4.492 290	-2.221 120	-2.221 120
39	2	3	1	1	-4.492 290	-4.492 290	-2.221 120	-2.221 120
40	3	1	1	1	-3.391 810	-3.391 810	-9.785 110	-9.785 110
41	1	0	1	1	-9.906 760	-9.906 760	-17.322 399	-17.322 399
42	1	1	1	1	-7.965 920	-7.965 920	-7.397 250	-7.397 250
43	1	1	1	1	-7.965 920	-7.965 920	-7.397 250	-7.397 250
44	1	2	1	1	-5.615 300	-5.615 300	-3.117 580	-3.117 580
45	1	2	1	1	-5.615 300	-5.615 300	-3.117 580	-3.117 580
46	2	0	1	1	-3.254 680	-3.254 680	-16.286 200	-16.286 200
47	1	3	1	1	-3.694 870	-3.694 870	-1.332 570	-1.332 570
48	1	3	1	1	-3.694 870	-3.694 870	-1.332 570	-1.332 570
49	2	1	1	1	-5.682 760	-5.682 760	-9.203 070	-9.203 070
50	2	1	1	1	-5.682 760	-5.682 760	-9.203 070	-9.203 070
51	1	4	1	1	-2.329 600	-2.329 600	-0.580 860	-0.580 860
52	1	4	1	1	-2.329 600	-2.329 600	-0.580 860	-0.580 860
53	2	2	1	1	-5.563 440	-5.563 440	-4.600 280	-4.600 280
54	2	2	1	1	-5.563 440	-5.563 440	-4.600 280	-4.600 280
55	3	0	1	1	-0.249 720	-0.249 720	-14.327 400	-14.327 400
56	1	5	1	1	-1.426 610	-1.426 610	-0.258 260	-0.258 260
57	1	5	1	1	-1.426 610	-1.426 610	-0.258 260	-0.258 260
58	2	3	1	1	-4.492 290	-4.492 290	-2.221 120	-2.221 120
59	2	3	1	1	-4.492 290	-4.492 290	-2.221 120	-2.221 120
60	3	1	1	1	-3.391 810	-3.391 810	-9.785 110	-9.785 110
61	1	0	1	2	-6.170 370	-6.170 370	-5.729 880	-5.729 880
62	1	1	1	2	-5.615 300	-5.615 300	-3.117 580	-3.117 580
63	1	1	1	2	-5.615 300	-5.615 300	-3.117 580	-3.117 580
64	1	2	1	2	-4.371 830	-4.371 830	-1.576 710	-1.576 710
65	1	2	1	2	-4.371 830	-4.371 830	-1.576 710	-1.576 710
66	2	0	1	2	-0.644 670	-0.644 670	-4.524 640	-4.524 640
67	1	3	1	2	-3.125 480	-3.125 480	-0.779 300	-0.779 300

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68	1	3	1	2	-3.125 480	-3.125 480	-0.779 300	-0.779 300
69	2	1	1	2	-3.031 330	-3.031 330	-3.471 390	-3.471 390
70	2	1	1	2	-3.031 330	-3.031 330	-3.471 390	-3.471 390
71	1	4	1	2	-2.116 010	-2.116 010	-0.383 070	-0.383 070
72	1	4	1	2	-2.116 010	-2.116 010	-0.383 070	-0.383 070
73	2	2	1	2	-3.690 200	-3.690 200	-2.137 150	-2.137 150
74	2	2	1	2	-3.690 200	-3.690 200	-2.137 150	-2.137 150
75	3	0	1	2	1.069 570	1.069 570	-3.346 420	-3.346 420
76	1	5	1	2	-1.379 170	-1.379 170	-0.188 640	-0.188 640
77	1	5	1	2	-1.379 170	-1.379 170	-0.188 640	-0.188 640
78	2	3	1	2	-3.395 720	-3.395 720	-1.210 750	-1.210 750
79	2	3	1	2	-3.395 720	-3.395 720	-1.210 750	-1.210 750
80	3	1	1	2	-1.126 540	-1.126 540	-3.327 140	-3.327 140
81	1	0	1	2	-6.170 370	-6.170 370	-5.729 880	-5.729 880
82	1	1	1	2	-5.615 300	-5.615 300	-3.117 580	-3.117 580
83	1	1	1	2	-5.615 300	-5.615 300	-3.117 580	-3.117 580
84	1	2	1	2	-4.371 830	-4.371 830	-1.576 710	-1.576 710
85	1	2	1	2	-4.371 830	-4.371 830	-1.576 710	-1.576 710
86	2	0	1	2	-0.644 670	-0.644 670	-4.524 640	-4.524 640
87	1	3	1	2	-3.125 480	-3.125 480	-0.779 300	-0.779 300
88	1	3	1	2	-3.125 480	-3.125 480	-0.779 300	-0.779 300
89	2	1	1	2	-3.031 330	-3.031 330	-3.471 390	-3.471 390
90	2	1	1	2	-3.031 330	-3.031 330	-3.471 390	-3.471 390
91	1	4	1	2	-2.116 010	-2.116 010	-0.383 070	-0.383 070
92	1	4	1	2	-2.116 010	-2.116 010	-0.383 070	-0.383 070
93	2	2	1	2	-3.690 200	-3.690 200	-2.137 150	-2.137 150
94	2	2	1	2	-3.690 200	-3.690 200	-2.137 150	-2.137 150
95	3	0	1	2	1.069 570	1.069 570	-3.346 420	-3.346 420
96	1	5	1	2	-1.379 170	-1.379 170	-0.188 640	-0.188 640
97	1	5	1	2	-1.379 170	-1.379 170	-0.188 640	-0.188 640
98	2	3	1	2	-3.395 720	-3.395 720	-1.210 750	-1.210 750
99	2	3	1	2	-3.395 720	-3.395 720	-1.210 750	-1.210 750
100	3	1	1	2	-1.126 540	-1.126 540	-3.327 140	-3.327 140

Table 3 The matrix elements of $r^\mu j_J(qr) d_r^{(m)}$ for $m = 1$, $b = 1.761$ fm, $0 \leq \mu \leq 2$, $0 \leq J \leq 3$ and $q = 1.5\text{--}3.5$ fm $^{-1}$ for the derived equations and integral form in Eq. (8).

i	μ	q	n'	l'	n	l	J	$\langle n'l' r^\mu j_J(qr) nl \rangle$		J	$\langle n'l' r^\mu j_J(qr) d_r^{(1)} nl \rangle$	
								Eq. (8)	Eq. (23)		Eq. (8)	Eq. (24)
1	0	1.5	1	0	1	0	0	0.174 640	0.174 640	1	-0.130 980	-0.130 980
2	0	3.0	1	0	1	0	0	0.000 930	0.000 930	1	-0.001 400	-0.001 400
3	0	1.5	1	0	1	0	1	0.266 730	0.266 730	2	-0.140 220	-0.140 220
4	0	3.0	1	0	1	0	1	0.037 879	0.037 879	2	-0.018 610	-0.018 610
5	0	1.5	1	0	1	0	2	0.216 316	0.216 316	3	-0.101 450	-0.101 450
6	0	3.0	1	0	1	0	2	0.070 941	0.070 941	3	-0.036 720	-0.036 720
7	1	1.5	1	0	1	0	0	0.098 464	0.098 464	1	-0.207 210	-0.207 210
8	1	3.0	1	0	1	0	0	-0.019 850	-0.019 850	1	0.010 840	0.010 840
9	1	1.5	1	0	1	0	1	0.406 335	0.406 335	2	-0.304 750	-0.304 750
10	1	3.0	1	0	1	0	1	0.004 329	0.004 329	2	-0.006 490	-0.006 490
11	1	1.5	1	0	1	0	2	0.434 996	0.434 996	3	-0.260 180	-0.260 180
12	1	3.0	1	0	1	0	2	0.057 732	0.057 732	3	-0.041 860	-0.041 860
13	2	1.5	1	0	1	0	0	-0.132 750	-0.132 750	1	-0.306 770	-0.306 770
14	2	3.0	1	0	1	0	0	-0.015 810	-0.015 810	1	0.019 393	0.019 393
15	2	1.5	1	0	1	0	1	0.642 831	0.642 831	2	-0.699 620	-0.699 620
16	2	3.0	1	0	1	0	1	-0.033 630	-0.033 630	2	0.021 578	0.021 578
17	2	1.5	1	0	1	0	2	0.945 424	0.945 424	3	-0.709 070	-0.709 070

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18	2	3.0	1	0	1	0	2	0.020 143	0.020 143	3	-0.030 210	-0.030 210
19	0	1.5	1	1	1	0	0	0.045 645	0.045 645	1	-0.096 060	-0.096 060
20	0	3.0	1	1	1	0	0	-0.009 200	-0.009 200	1	0.005 025	0.005 025
21	0	1.5	1	1	1	0	1	0.188 364	0.188 364	2	-0.141 270	-0.141 270
22	0	3.0	1	1	1	0	1	0.002 007	0.002 007	2	-0.003 010	-0.003 010
23	0	1.5	1	1	1	0	2	0.201 651	0.201 651	3	-0.120 610	-0.120 610
24	0	3.0	1	1	1	0	2	0.026 763	0.026 763	3	-0.019 400	-0.019 400
25	1	1.5	1	1	1	0	0	-0.061 540	-0.061 540	1	-0.142 210	-0.142 210
26	1	3.0	1	1	1	0	0	-0.007 330	-0.007 330	1	0.008 990	0.008 990
27	1	1.5	1	1	1	0	1	0.297 996	0.297 996	2	-0.324 320	-0.324 320
28	1	3.0	1	1	1	0	1	-0.015 590	-0.015 590	2	0.010 003	0.010 003
29	1	1.5	1	1	1	0	2	0.438 269	0.438 269	3	-0.328 700	-0.328 700
30	1	3.0	1	1	1	0	2	0.009 337	0.009 337	3	-0.014 010	-0.014 010
31	2	1.5	1	1	1	0	0	-0.410 150	-0.410 150	1	-0.139 390	-0.139 390
32	2	3.0	1	1	1	0	0	0.015 442	0.015 442	1	0.000 221	0.000 221
33	2	1.5	1	1	1	0	1	0.441 173	0.441 173	2	-0.769 150	-0.769 150
34	2	3.0	1	1	1	0	1	-0.027 890	-0.027 890	2	0.032 496	0.032 496
35	2	1.5	1	1	1	0	2	1.006 140	1.006 140	3	-0.941 690	-0.941 690
36	2	3.0	1	1	1	0	2	-0.031 030	-0.031 030	3	0.016 451	0.016 451
37	0	1.5	1	1	1	0	0	0.045 645	0.045 645	1	-0.096 060	-0.096 060
38	0	3.0	1	1	1	0	0	-0.009 200	-0.009 200	1	0.005 025	0.005 025
39	0	1.5	1	1	1	0	1	0.188 364	0.188 364	2	-0.141 270	-0.141 270
40	0	3.0	1	1	1	0	1	0.002 007	0.002 007	2	-0.003 010	-0.003 010
41	0	1.5	1	1	1	0	2	0.201 651	0.201 651	3	-0.120 610	-0.120 610
42	0	3.0	1	1	1	0	2	0.026 763	0.026 763	3	-0.019 400	-0.019 400
43	1	1.5	1	1	1	0	0	-0.061 540	-0.061 540	1	-0.142 210	-0.142 210
44	1	3.0	1	1	1	0	0	-0.007 330	-0.007 330	1	0.008 990	0.008 990
45	1	1.5	1	1	1	0	1	0.297 996	0.297 996	2	-0.324 320	-0.324 320
46	1	3.0	1	1	1	0	1	-0.015 590	-0.015 590	2	0.010 003	0.010 003
47	1	1.5	1	1	1	0	2	0.438 269	0.438 269	3	-0.328 700	-0.328 700
48	1	3.0	1	1	1	0	2	0.009 337	0.009 337	3	-0.014 010	-0.014 010
49	2	1.5	1	1	1	0	0	-0.410 150	-0.410 150	1	-0.139 390	-0.139 390
50	2	3.0	1	1	1	0	0	0.015 442	0.015 442	1	0.000 221	0.000 221
51	2	1.5	1	1	1	0	1	0.441 173	0.441 173	2	-0.769 150	-0.769 150
52	2	3.0	1	1	1	0	1	-0.027 890	-0.027 890	2	0.032 496	0.032 496
53	2	1.5	1	1	1	0	2	1.006 140	1.006 140	3	-0.941 690	-0.941 690
54	2	3.0	1	1	1	0	2	-0.031 030	-0.031 030	3	0.016 451	0.016 451
55	0	1.5	1	2	1	0	0	-0.022 100	-0.022 100	1	-0.051 060	-0.051 060
56	0	3.0	1	2	1	0	0	-0.002 630	-0.002 630	1	0.003 228	0.003 228
57	0	1.5	1	2	1	0	1	0.107 004	0.107 004	2	-0.116 460	-0.116 460
58	0	3.0	1	2	1	0	1	-0.005 600	-0.005 600	2	0.003 592	0.003 592
59	0	1.5	1	2	1	0	2	0.157 373	0.157 373	3	-0.118 030	-0.118 030
60	0	3.0	1	2	1	0	2	0.003 353	0.003 353	3	-0.005 030	-0.005 030
61	1	1.5	1	2	1	0	0	-0.147 280	-0.147 280	1	-0.050 050	-0.050 050
62	1	3.0	1	2	1	0	0	0.005 545	0.005 545	1	0.000 079	0.000 079
63	1	1.5	1	2	1	0	1	0.158 416	0.158 416	2	-0.276 190	-0.276 190
64	1	3.0	1	2	1	0	1	-0.010 010	-0.010 010	2	0.011 669	0.011 669
65	1	1.5	1	2	1	0	2	0.361 283	0.361 283	3	-0.338 140	-0.338 140
66	1	3.0	1	2	1	0	2	-0.011 140	-0.011 140	3	0.005 907	0.005 907
67	2	1.5	1	2	1	0	0	-0.539 970	-0.539 970	1	0.088 146	0.088 146
68	2	3.0	1	2	1	0	0	0.026 185	0.026 185	1	-0.019 250	-0.019 250
69	2	1.5	1	2	1	0	1	0.155 270	0.155 270	2	-0.658 380	-0.658 380
70	2	3.0	1	2	1	0	1	-0.000 250	-0.000 250	2	0.017 083	0.017 083
71	2	1.5	1	2	1	0	2	0.856 802	0.856 802	3	-1.008 760	-1.008 760
72	2	3.0	1	2	1	0	2	-0.036 200	-0.036 200	3	0.038 696	0.038 696
73	0	1.5	1	2	1	0	0	-0.022 100	-0.022 100	1	-0.051 060	-0.051 060
74	0	3.0	1	2	1	0	0	-0.002 630	-0.002 630	1	0.003 228	0.003 228

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75	0	1.5	1	2	1	0	1	0.107 004	0.107 004	2	-0.116 460	-0.116 460
76	0	3.0	1	2	1	0	1	-0.005 600	-0.005 600	2	0.003 592	0.003 592
77	0	1.5	1	2	1	0	2	0.157 373	0.157 373	3	-0.118 030	-0.118 030
78	0	3.0	1	2	1	0	2	0.003 353	0.003 353	3	-0.005 030	-0.005 030
79	1	1.5	1	2	1	0	0	-0.147 280	-0.147 280	1	-0.050 050	-0.050 050
80	1	3.0	1	2	1	0	0	0.005 545	0.005 545	1	0.000 079	0.000 079
81	1	1.5	1	2	1	0	1	0.158 416	0.158 416	2	-0.276 190	-0.276 190
82	1	3.0	1	2	1	0	1	-0.010 010	-0.010 010	2	0.011 669	0.011 669
83	1	1.5	1	2	1	0	2	0.361 283	0.361 283	3	-0.338 140	-0.338 140
84	1	3.0	1	2	1	0	2	-0.011 140	-0.011 140	3	0.005 907	0.005 907
85	2	1.5	1	2	1	0	0	-0.539 970	-0.539 970	1	0.088 146	0.088 146
86	2	3.0	1	2	1	0	0	0.026 185	0.026 185	1	-0.019 250	-0.019 250
87	2	1.5	1	2	1	0	1	0.155 270	0.155 270	2	-0.658 380	-0.658 380
88	2	3.0	1	2	1	0	1	-0.000 250	-0.000 250	2	0.017 083	0.017 083
89	2	1.5	1	2	1	0	2	0.856 802	0.856 802	3	-1.008 760	-1.008 760
90	2	3.0	1	2	1	0	2	-0.036 200	-0.036 200	3	0.038 696	0.038 696
91	0	1.5	1	0	1	1	0	0.045 645	0.045 645	1	0.027 590	0.027 590
92	0	3.0	1	0	1	1	0	-0.009 200	-0.009 200	1	0.022 585	0.022 585
93	0	1.5	1	0	1	1	1	0.188 364	0.188 364	2	-0.041 000	-0.041 000
94	0	3.0	1	0	1	1	1	0.002 007	0.002 007	2	0.029 876	0.029 876
95	0	1.5	1	0	1	1	2	0.201 651	0.201 651	3	-0.059 360	-0.059 360
96	0	3.0	1	0	1	1	2	0.026 763	0.026 763	3	0.022 079	0.022 079
97	1	1.5	1	0	1	1	0	-0.061 540	-0.061 540	1	0.046 155	0.046 155
98	1	3.0	1	0	1	1	0	-0.007 330	-0.007 330	1	0.010 996	0.010 996
99	1	1.5	1	0	1	1	1	0.297 996	0.297 996	2	-0.122 670	-0.122 670
100	1	3.0	1	0	1	1	1	-0.015 590	-0.015 590	2	0.036 766	0.036 766

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