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Entropy, Fisher Information and Variance with Frost-Musulin Potenial

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Abstract *This study presents the Shannon and Renyi information entropy for both position and momentum space and the Fisher information for the position-dependent mass Schrödinger equation with the Frost-Musulin potential. The analysis of the quantum mechanical probability has been obtained via the Fisher information. The variance information of this potential is equally computed. This controls both the chemical properties and physical properties of some of the molecular systems. We have observed the behaviour of the Shannon entropy. Renyi entropy, Fisher information and variance with the quantum number n respectively.*

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Key words: Fisher information, Shannon entropy, position-dependent mass

1 Introduction

There has been a great interest in studying information theoretic measures for different quantum systems because the information theory of quantum-mechanical systems are related to the modern quantum communication and density functional methods.[1−2] The quantum information plays a crucial role in the measurement of uncertainty and other quantum parameters of the system.[2] The information entropy is a superior measure of spreading and then of quantum uncertainity, a property of fundamental relevance for the adequate characterization of the position and momentum densities.[3−5] The entropies have been used for numerous practical purposes e.g. measure the squeezing of quantum fluctuation[6] and to reconstruct the charge and momentum densities of atomic and molecular systems[7−8] by means of maximum entropy procedures. The Fisher information is the basic variable of the principle of extreme physical information which has been used to obtain various fundamental equations of motion in physics.[9] It is equally being used to rederive classical thermodynamics without the usual concept of Boltzmann's entropy. It is understood that the position-space Shannon entropy $S(\rho)$ measures the uncertainity in the localization of a particle in space while the momentum-space n the Shannon entropy measures the uncertainity in predicting the momentum of the particle.[10] These entropies are closely related to fundamental and experimentally measurable quantities such as the kinetic energy and magnetic susceptibility, which make them useful in the study of the structure and dynamics of atomic and molecular systems.[8] Thus in this paper, we calculate both the position and momentum of the Shannon entropy and Renyi entropy, the Fisher information and Variance of a system and the observed behavior of these quantities with the quantum number n. We compute the wave function in Sec. 2. In Sec. 3, we show the basic equation in this work. Section 4 presents the Shannon, Renyi entropies, Fisher information and variance. In the final section, we give the conclusion.

2 Wave Function of Radial Schrödinger Equation with Frost-Musulin Potential

Given the Radial Schrödinger Equation as[11]

$$U(r) = \frac{2\mu}{\hbar^2}\Big[E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\Big]U(r) = 0\,, \quad (1)$$

and the Frost–Musulin potential as

$$V(r) = \Big[1 - \frac{(r + \delta r_e[r - r_e])\,\mathrm{e}^{-\delta(r-r_e)}}{r}\Big]\,, \quad (2)$$

where

$$\frac{1}{r^2} = \frac{\delta^2}{(1-\mathrm{e}^{-\delta r})^2}\,. \quad (3)$$

Substitute Eqs. (2) and (3) into Eq. (1) and by defining a variable of the form $y = \mathrm{e}^{-\delta r}$, we have a differential equation of the form

$$\frac{\mathrm{d}^2U_{n,\ell}(y)}{\mathrm{d}y^2} + \frac{1-y}{y(1-y)}\frac{\mathrm{d}U_{n,\ell}}{\mathrm{d}y} + \frac{Ay^2 + By + C}{y^2(1-y)^2}U_{n,\ell}(y) = 0\,, \quad (4)$$

where

$$A = \frac{2\mu E_{n,\ell}}{\delta^2\hbar} + \frac{2\mu D_e}{\delta^2\hbar^2}[b\delta r_e(1+r_e) + b - 1]\,,$$

$$B = -\frac{2\mu}{\delta^2\hbar^2}[2E_{n,\ell} - 2D_e(1-b) + D_e r_e\delta b(2+r_e)]\,,$$

$$C = \frac{2\mu En,l}{\delta^2\hbar^2} + \frac{2\mu D_e}{\delta^2\hbar^2}[b + b\delta r_e - 1] - \ell(\ell+1)\,.$$

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The wavefunction is obtained as

$$U_{n,\ell}(y) = N_{n,\ell} y^{\eta}(1-y)^{\lambda}[\mathrm{P}_n^{(2\eta,2\lambda-1)}(1-2y)]\,, \quad (5)$$

where

$$\eta = \sqrt{\ell(\ell+1) - 2\mu D_e(b + b\delta r_e - 1) - \frac{2\mu E_{n,\ell}}{\delta^2\hbar^2}}\,, \quad (6)$$

$$\lambda = \frac{1}{2}\Big[1 + \sqrt{4\ell(\ell+1) - 8\mu b D_e(\delta r_e + 1) + \frac{8\mu b D_e r_e}{\delta^2\hbar^2}}\Big]\,. \quad (7)$$

3 Basic Equations

In this section, we write some equations that are useful in this work. The Shannon information entropy $S(\rho)$ and $S(\gamma)$ of the electron density $\rho(r)$ in the coordinate space for position and momentum space respectively are defined as[12−14]

$$S(\rho) = -\int_0^{\infty} \rho(\boldsymbol{r}) \ln \rho(\boldsymbol{r})\,\mathrm{d}x\,, \quad (8)$$

$$S(\gamma) = -\int_0^{\infty} \gamma(\boldsymbol{p}) \ln \gamma(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p}\,, \quad (9)$$

where $\gamma(\boldsymbol{p})$ denotes the momentum density. The Renyi position entropy is defined as[15−18]

$$R_q(\rho) = \frac{1}{1-q} \ln\Big[\int \rho(r)^q\,\mathrm{d}r\Big]\,, \quad 0 < q < \infty\,, \quad q \neq 1\,. \quad (10)$$

It is a generalization of the Shannon entropy. The corresponding momentum space entropy is given as[14]

$$R_q(\gamma) = \frac{1}{1-q} \ln\Big[\int \gamma(\boldsymbol{p})^q\,\mathrm{d}p\Big], \quad 0 < q < \infty, \quad q \neq 1\,. \quad (11)$$

The Fisher information entropy of the radial probability distribution $\rho(r)$ function is calculated by using[19]

$$I_\rho = \int_0^{\infty} \frac{1}{\rho(r)}\Big[\frac{\mathrm{d}\rho(r)}{\mathrm{d}r}\Big]^2 \mathrm{d}r\,. \quad (12)$$

The position Variance is given as[19]

$$V_\rho = \int (r - \langle r\rangle_\rho)^2 \rho(r)\,\mathrm{d}r\,. \quad (13)$$

4 Frost-Musulim Potential with Shannon Entropy, Renyi Entropy, Fisher Information and Variance

To obtain the Shannon and Renyi entropies in the position space, we define the radial probability distribution function $\rho(r)$ for this problem as the square of the radial wave function and obtain the following

$$\rho(r) = \mathbb{C}\,\mathrm{e}^{-2\eta\alpha r}(1 - \mathrm{e}^{-\alpha r})^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(1 - 2\,\mathrm{e}^{-\alpha r})], \quad (14)$$

$$\gamma(p) = \mathbb{C}\,\mathrm{e}^{-2\eta\alpha p}(1 - \mathrm{e}^{-\alpha p})^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(1 - 2\,\mathrm{e}^{-\alpha p})]^2, \quad (15)$$

where,

$$[P_n^{(2\eta,2\lambda-1)}(1 - 2e^{-\alpha p})] \equiv {}_2F_1(-n, n + 2(\eta+\lambda), 2\eta+1; e^{-\alpha r})$$

and $\mathbb{C} = N_{n\ell}^2$.

4.1 *Shannon Entropy for the Frost-Musulin Potential*

Entropy is a thermodynamic quantity representing the unavailability of a system's thermal energy for conversion into mechanical work. It is often interpreted as the degree of disorder or randomness in the system. Shannon entropy is the expected value or average of the information contained in each message. It measures the information in a message as opposed to the portion of the message that is determined. Shannon entropy can determine the minimum channel capacity required to reliably transmit the source as encoded binary digit.

To obtain the position space for the Shannon entropy, we substitute Eq. (14) into Eq. (8) and have

$$S(\rho) = -\frac{\mathbb{C}}{\alpha}\int_1^0 y^{\eta}(1-y)^{\lambda}[P_n^{(2\eta,2\lambda-1)}(1-2y)]^2 \ln \rho(y)\,\mathrm{d}y\,, \quad y = \mathrm{e}^{-\alpha r}\,, \quad (16)$$

$$S(\rho) = -\frac{\mathbb{C}}{\alpha}\beta_1 D_1 \int_0^1 z^{2\lambda}(1-z)^{2\eta} 2F_1[-n, n+2(\eta+\lambda), 2\eta+1; z]\,, \quad y = 1 - z\,, \quad (17)$$

where we have used the following for mathematical simplicity;

$$\beta_1 = [P_n^{(2\eta,2\lambda-1)}(1-2y)]\,, \quad D_1 = \ln[P_n^{(2\eta,2\lambda-1)}(1-2y)]\,.$$

By using the identity in Appendix B and standard integral in Appendix A, we obtain the Shannon entropy in the position space as

$$\begin{aligned} S(\rho) = &-\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}(z-1)^m \\ &\times \ln\Big[\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}(z-1)^m\Big]\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)} \\ &\times \sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}(z-1)^m\frac{\Gamma(2\lambda+1)\Gamma(\gamma)\Gamma(n+2\lambda-1)\Gamma(\gamma+n-2\eta-1)}{\Gamma(n+2\eta+2\lambda)\Gamma(\gamma+n)\Gamma(\gamma-2\eta-1)}\,, \end{aligned} \quad (18a)$$

when $n = 1$,

$$S(\rho) = -\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+2)}{\Gamma(2\eta+2\lambda+1)}\frac{\Gamma(2\eta+2\lambda+1)}{\Gamma(2\eta+1)}\ln\Big[\frac{\Gamma(2\eta+2)}{\Gamma(2\eta+2\lambda+1)}\frac{\Gamma(2\eta+2\lambda+1)}{\Gamma(2\eta+1)}\Big]$$

$$\times \frac{\Gamma(2\eta+2)}{\Gamma(2\eta+2\lambda+1)}\frac{\Gamma(2\eta+2\lambda+1)}{\Gamma(2\eta+1)}\frac{\Gamma(2\lambda+1)\Gamma(2\lambda)\Gamma(2\eta)}{\Gamma(2\eta+2\lambda+1)\Gamma(-2\eta-1)}\,, \tag{18b}$$

when $n=2$,

$$\begin{aligned}S(\rho) = &-\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+3)}{\Gamma(2\eta+2\lambda+2)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)}{\Gamma(2\eta+1)}\Big)\\ &\times \ln\Big[\frac{\Gamma(2\eta+3)}{\Gamma(2\eta+2\lambda+2)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)}{\Gamma(2\eta+1)}\Big)\Big]\\ &\times \frac{\Gamma(2\eta+3)}{\Gamma(2\eta+2\lambda+2)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)}{\Gamma(2\eta+1)}\Big)\frac{\Gamma(2\lambda+1)\Gamma(2\lambda+1)\Gamma(2\eta+1)}{\Gamma(2\eta+2\lambda+2)\Gamma(-2\eta-1)}\,,\end{aligned} \tag{18c}$$

when $n=3$

$$\begin{aligned}S(\rho) = &-\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+4)}{\Gamma(2\eta+2\lambda+3)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)+\Gamma(2\eta+2\lambda+3)}{\Gamma(2\eta+1)}\Big)\\ &\times \ln\Big[\frac{\Gamma(2\eta+4)}{\Gamma(2\eta+2\lambda+3)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)+\Gamma(2\eta+2\lambda+3)}{\Gamma(2\eta+1)}\Big)\Big]\frac{\Gamma(2\eta+4)}{\Gamma(2\eta+2\lambda+3)}\\ &\times \Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)+\Gamma(2\eta+2\lambda+3)}{\Gamma(2\eta+1)}\Big)\frac{\Gamma(2\lambda+1)\Gamma(2\lambda+2)\Gamma(2\eta+2)}{\Gamma(2\eta+2\lambda+3)\Gamma(-2\eta-1)}\,.\end{aligned} \tag{18d}$$

The momentum space of the Shannon entropy is obtained by substituting Eq. (15) into Eq. (9) to have

$$S(\gamma) = -4\pi\int_0^\infty p^2\gamma(p)\ln\gamma(p)\,\mathrm{d}p\,, \tag{19}$$

$$S(\gamma) = -4\mathbb{C}\pi\boldsymbol{C}^{2\lambda}\bar{w}[P_n^{(2\eta,2\lambda-1)}(\boldsymbol{C}-1)]^2\int_0^\infty p^2\,\mathrm{e}^{-2\eta\alpha p}\,\mathrm{d}p\,,\quad \boldsymbol{C}=1-\mathrm{e}^{-\alpha p}\,, \tag{20}$$

where $\bar{w}=\ln\gamma(p)$. Using the relation in Appendix B and integral in Appendix A, we obtain the Shannon entropy in the momentum space as

$$\begin{aligned}S(\gamma) = &\frac{3.142\mathbb{C}\boldsymbol{C}^{2\lambda}\ln\big[\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}(z-1)^m\big]}{\alpha^3[\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)-2\mu E_{n,\ell}/\delta^2\hbar^2}]^3}\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n}\\ &\times\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}\Big(n+2k\Big[\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)-\frac{2\mu E_{n\ell}}{\delta^2\hbar^2}}\Big]\Big)\\ &\times\Big(n+k\sqrt{4\ell(\ell+1)-8\mu D_e(b+b\delta r_e-1)-\frac{8\mu E_{n\ell}}{\delta^2\hbar^2}}\Big)\\ &\times\Big(\frac{\boldsymbol{C}-2}{2}\Big)^m\Big(n+n-k\sqrt{4\ell(\ell+1)-8\mu D_e(b+b\delta r_e-1)-\frac{8\mu E_{n\ell}}{\delta^2\hbar^2}}\Big)(\boldsymbol{C}-2)^{n-k}(\boldsymbol{C})^k\,.\end{aligned} \tag{21}$$

4.2 *Renyi Entropy for the Frost-Musulin Potential*

To obtain the position-space for the Renyi-entropy, we substitute Eq. (14) into Eq. (10) to have

$$R_q(\rho) = \frac{1}{1-q}\ln\Big[\Big(\frac{\mathbb{C}}{\alpha}\Big)^q\int_1^0(y^{2\eta-1}(1-y)^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(1-2y)]^2)^q\,\mathrm{d}y\Big]\,,\quad y=\mathrm{e}^{-\alpha r}\,, \tag{22}$$

$$R_q(\rho) = \frac{1}{1-q}\ln\Big[\Big(\frac{\mathbb{C}}{\alpha}\Big)^q\int_1^0(t^{2\lambda}(1-t)^{2\eta-1}[P_n^{(2\eta,2\lambda-1)}(t)][P_n^{(2\eta,2\lambda-1)}(t)])^q\,\mathrm{d}t\Big]\,,\quad t=1-y\,, \tag{23}$$

$$\begin{aligned}R_q(\rho) = &\frac{1}{1-q}\ln\Big[\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+n+1)}{n!\Gamma(2\eta+2\lambda+n)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}\Big(\frac{t-1}{2}\Big)^m\\ &\times\frac{\Gamma(2\eta)\Gamma(r)\Gamma(2\lambda+n)\Gamma(2\lambda+n-2\eta}{\Gamma(2\eta+2\lambda+n)\Gamma(2\lambda+n)\Gamma(2\lambda-2\eta)}\Big]^q\end{aligned} \tag{24a}$$

when $n=1$,

$$R_q(\rho) = \frac{1}{1-q}\ln\Big[\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+2)}{\Gamma(2\eta+2\lambda+1)}\frac{\Gamma(2\eta+2\lambda+1)}{\Gamma(2\eta+1)}\frac{\Gamma(2\eta)\Gamma(2\lambda+1)\Gamma(2\lambda)}{\Gamma(2\eta+2\lambda+1)\Gamma(-2\lambda-1)}\Big]^q\,, \tag{24b}$$

when $n=2$

$$R_q(\rho) = \frac{1}{1-q}\ln\Big[\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+3)}{2\Gamma(2\eta+2\lambda+2)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)}{\Gamma(2\eta+1)}\Big)\frac{\Gamma(2\eta+1)\Gamma(2\lambda+1)\Gamma(2\lambda+1)}{\Gamma(2\eta+2\lambda+2)\Gamma(-2\lambda-1)}\Big]^q\,, \tag{24c}$$

when $n = 3$,

$$R_q(\rho) = \frac{1}{1-q}\ln\Big[\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+4)}{2\Gamma(2\eta+2\lambda+3)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)+\Gamma(2\eta+2\lambda+3)}{\Gamma(2\eta+1)}\Big) \times \frac{\Gamma(2\eta+2)\Gamma(2\lambda+1)\Gamma(2\lambda+2)}{\Gamma(2\eta+2\lambda+3)\Gamma(-2\lambda-1)}\Big]^q . \tag{24d}$$

The Renyi entropy in the momentum space is obtained by substituting Eq. (15) into Eq. (11) to have

$$R_q(\gamma) = \frac{1}{1-q}\ln\Big[4\pi \boldsymbol{C}^{2\lambda q}\int_0^\infty (p^2\,\mathrm{e}^{-2\eta\alpha p}[P_n^{(2\eta,2\lambda-1)}(\boldsymbol{C}-1)]^2)^q\,\mathrm{d}p\Big] , \tag{25}$$

which gives

$$R_q(\gamma) = \frac{1}{1-q}\ln\Big[12.568\Big(\frac{\mathbb{C}\boldsymbol{C}^{2\lambda}[\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)-2\mu E_{n\ell}/\delta^2\hbar^2}]^{-3}}{4\alpha^3}\Big)^q\Big] . \tag{26}$$

4.3 *Fisher Information Entropy for the Frost-Musulin Potential*

To obtain the Fisher information entropy, we substitute Eq. (14) into Eq. (13) and then solve it bit by bit as follows

$$\frac{\mathrm{d}\rho(y)}{\mathrm{d}y} = 2\alpha\mathbb{C}[-\eta y^{2\eta}(1-y)^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(1-2y)]^2 - \lambda y^{2\eta}(1-y)^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(1-2y)]^2] + \eta y^{2\eta}(1-y)^{2\lambda}[P_n^{(2\eta,2\lambda-1)}]^2 , \quad y = \mathrm{e}^{-\alpha r} , \tag{27}$$

$$I(\rho) = 4\alpha\mathbb{C}\beta_2[\eta I_1 + \lambda^2 I_2 + 4I_3 + 2\eta\lambda I_4 - 4(\eta I_5 + \lambda I_6)] , \quad y = \frac{1-z}{2} , \tag{28}$$

where

$$\beta_2 = \frac{2}{1-z} , \quad I_1 = \int_{-1}^{1}\Big(\frac{1-z}{2}\Big)^{2\eta}\Big(\frac{1+z}{2}\Big)^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(z)]^2\,\mathrm{d}z ,$$

$$I_2 = \int_{-1}^{1}\Big(\frac{1-z}{2}\Big)^{2\eta}\Big(\frac{1+z}{2}\Big)^{2(\lambda-1)}[P_n^{(2\eta,2\lambda-1)}(z)]^2\,\mathrm{d}z ,$$

$$I_3 = \int_{-1}^{1}\Big(\frac{1-z}{2}\Big)^{2\eta}\Big(\frac{1+z}{2}\Big)^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(z)]^2\,\mathrm{d}z , \quad I_4 = \int_{-1}^{1}\Big(\frac{1-z}{2}\Big)^{2\eta}\Big(\frac{1+z}{2}\Big)^{2\lambda-1}[P_n^{(2\eta,2\lambda-1)}(z)]^2\,\mathrm{d}z ,$$

$$I_5 = \int_{-1}^{1}\frac{1}{z}\Big(\frac{1-z}{2}\Big)^{2\eta}\Big(\frac{1+z}{2}\Big)^{2\lambda}[P_n^{(2\eta,2\lambda-1)}(z)]^2\,\mathrm{d}z , \quad I_6 = \int_{-1}^{1}\frac{1}{z}\Big(\frac{1-z}{2}\Big)^{2\eta}\Big(\frac{1+z}{2}\Big)^{2\lambda-1}[P_n^{(2\eta,2\lambda-1)}(z)]^2\,\mathrm{d}z .$$

Thus, we obtain the Fisher information entropy as

$$I(\rho) = 4\alpha\mathbb{C}\beta_2\Big[\frac{\Gamma(2\eta+n+1)\Gamma(2\lambda+n+1)}{n!\Gamma(2\eta+2\lambda+2n+1)\Gamma(2\eta+2\lambda+n+1)} + \frac{\lambda^2\Gamma(2\eta+n+1)\Gamma(2\lambda+n-1)}{n!\eta\Gamma(2\eta+2\lambda+2n+1)\Gamma(2\eta+2\lambda+n-1)} + \frac{4\Gamma(2\eta+n+1)\Gamma(2\lambda+n+1)}{n!\eta\Gamma(2\eta+2\lambda+2n+1)\Gamma(2\eta+2\lambda+n+1)} + \frac{2\lambda\Gamma(2\eta+n+1)\Gamma(2\lambda+n)}{n!\Gamma(2\eta+2\lambda+2n)\Gamma(2\eta+2\lambda+n)} - \frac{1}{z}\frac{4\lambda\Gamma(2\eta+n+1)\Gamma(2\lambda+n)}{n!\eta\Gamma(2\eta+2\lambda+2n)\Gamma(2\eta+2\lambda+n)} - \frac{4\Gamma(2\eta+n+1)\Gamma(2\lambda+n+1)}{n!\Gamma(2\eta+2\lambda+2n+1)\Gamma(2\eta+2\lambda+n+1)}\Big] , \tag{29a}$$

when $n = 0$,

$$I(\rho) = 4\alpha\mathbb{C}\beta_2\Big[\frac{\Gamma(2\eta+1)\Gamma(2\lambda+1)}{\Gamma(2\eta+2\lambda+1)\Gamma(2\eta+2\lambda+1)} + \frac{\lambda^2\Gamma(2\eta+1)\Gamma(2\lambda-1)}{\eta\Gamma(2\eta+2\lambda+1)\Gamma(2\eta+2\lambda-1)} + \frac{4\Gamma(2\eta+1)\Gamma(2\lambda+1)}{\eta\Gamma(2\eta+2\lambda+1)\Gamma(2\eta+2\lambda+1)} + \frac{2\lambda\Gamma(2\eta+n+1)\Gamma(2\lambda)}{\Gamma(2\eta+2\lambda)\Gamma(2\eta+2\lambda)} - \frac{1}{z}\frac{4\lambda\Gamma(2\eta+1)\Gamma(2\lambda)}{\eta\Gamma(2\eta+2\lambda)\Gamma(2\eta+2\lambda)} - \frac{4\Gamma(2\eta+1)\Gamma(2\lambda+1)}{\Gamma(2\eta+2\lambda+1)\Gamma(2\eta+2\lambda+1)}\Big] , \tag{29b}$$

when $n = 1$,

$$I(\rho) = 4\alpha\mathbb{C}\beta_2\Big[\frac{\Gamma(2\eta+2)\Gamma(2\lambda+2)}{\Gamma(2\eta+2\lambda+3)\Gamma(2\eta+2\lambda+2)} + \frac{\lambda^2\Gamma(2\eta+2)\Gamma(2\lambda)}{\eta\Gamma(2\eta+2\lambda+3)\Gamma(2\eta+2\lambda)} + \frac{4\Gamma(2\eta+2)\Gamma(2\lambda+2)}{\eta\Gamma(2\eta+2\lambda+3)\Gamma(2\eta+2\lambda+2)} + \frac{2\lambda\Gamma(2\eta+2)\Gamma(2\lambda+1)}{\Gamma(2\eta+2\lambda+2)\Gamma(2\eta+2\lambda+1)} - \frac{1}{z}\frac{4\lambda\Gamma(2\eta+2)\Gamma(2\lambda+1)}{\eta\Gamma(2\eta+2\lambda+2)\Gamma(2\eta+2\lambda+1)} - \frac{4\Gamma(2\eta+2)\Gamma(2\lambda+2)}{\Gamma(2\eta+2\lambda+3)\Gamma(2\eta+2\lambda+2)}\Big] , \tag{29c}$$

when $n = 2$,

$$I(\rho) = 4\alpha\mathbb{C}\beta_2\Big[\frac{\Gamma(2\eta+3)\Gamma(2\lambda+3)}{2\Gamma(2\eta+2\lambda+5)\Gamma(2\eta+2\lambda+3)} + \frac{\lambda^2\Gamma(2\eta+3)\Gamma(2\lambda+1)}{\eta\Gamma(2\eta+2\lambda+5)\Gamma(2\eta+2\lambda+1)} + \frac{2\Gamma(2\eta+3)\Gamma(2\lambda+3)}{\eta\Gamma(2\eta+2\lambda+5)\Gamma(2\eta+2\lambda+3)}$$

$$+\frac{2\lambda\Gamma(2\eta+3)\Gamma(2\lambda+2)}{\Gamma(2\eta+2\lambda+4)\Gamma(2\eta+2\lambda+2)}-\frac{1}{z}\frac{2\lambda\Gamma(2\eta+3)\Gamma(2\lambda+2)}{\eta\Gamma(2\eta+2\lambda+4)\Gamma(2\eta+2\lambda+2)}-\frac{2\Gamma(2\eta+3)\Gamma(2\lambda+3)}{\Gamma(2\eta+2\lambda+5)\Gamma(2\eta+2\lambda+3)}\Big], \quad (29d)$$

when $n=3$,

$$I(\rho)=4\alpha\mathbb{C}\beta_2\Big[\frac{\Gamma(2\eta+4)\Gamma(2\lambda+4)}{6\Gamma(2\eta+2\lambda+7)\Gamma(2\eta+2\lambda+4)}+\frac{\lambda^2\Gamma(2\eta+4)\Gamma(2\lambda+2)}{6\eta\Gamma(2\eta+2\lambda+7)\Gamma(2\eta+2\lambda+2)}+\frac{2\Gamma(2\eta+4)\Gamma(2\lambda+4)}{3\eta\Gamma(2\eta+2\lambda+7)\Gamma(2\eta+2\lambda+4)}$$
$$+\frac{\lambda\Gamma(2\eta+4)\Gamma(2\lambda+3)}{3\Gamma(2\eta+2\lambda+6)\Gamma(2\eta+2\lambda+3)}-\frac{1}{z}\frac{2\lambda\Gamma(2\eta+4)\Gamma(2\lambda+3)}{3\eta\Gamma(2\eta+2\lambda+6)\Gamma(2\eta+2\lambda+3)}-\frac{2\Gamma(2\eta+4)\Gamma(2\lambda+4)}{3\Gamma(2\eta+2\lambda+7)\Gamma(2\eta+2\lambda+4)}\Big].\quad (29e)$$

4.4 *Variance with Frost-Musulin Potential*

Using Eq. (13), we obtain the following

$$V_\rho=\int(r-\langle r\rangle_\rho)^2\rho(r)\,\mathrm{d}r=\langle r^2\rangle_\rho-\langle r\rangle_\rho^2=\int_0^\infty r^2\rho(r)\,\mathrm{d}r-\int_0^\infty r\rho(r)\,\mathrm{d}r\,, \quad (30)$$

$$V_\rho=\frac{X\times(Y-X)^2}{Z}\,, \quad (31a)$$

where,

$$X=\mathbb{C}\boldsymbol{C}^{2\lambda}\Big(\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}(\boldsymbol{C}-2)^m\Big)^2$$

$$Y=4\alpha\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)-\frac{-2\mu E_{n,\ell}}{\delta^2\hbar^2}}\,,\quad Z=\Big(2\alpha\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)-\frac{-2\mu E_{n,\ell}}{\delta^2\hbar^2}}\Big)^4,$$

when $n=1$,

$$V_\rho=\frac{\big[\mathbb{C}\boldsymbol{C}^{2\lambda}\big(\frac{\Gamma(2\eta+2)}{\Gamma(2\eta+1)}\big)^2\big]\big((\lambda-\frac{1}{2})-\mathbb{C}\boldsymbol{C}^{2\lambda}\big(\frac{\Gamma(2\eta+2)}{\Gamma(2\eta+1)}\big)^2\big)^2}{(2\alpha\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)--2\mu E_{n,\ell}/\delta^2\hbar^2})^4}\,, \quad (31b)$$

when $n=2$,

$$V_\rho=\frac{\mathbb{C}\boldsymbol{C}^{2\lambda}\big(\frac{\Gamma(2\eta+3)}{\Gamma(2\eta+2\lambda+2)}\big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)}{\Gamma(2\eta+1)}\big)\big)^2\big((\lambda-\frac{1}{2})-\mathbb{C}\boldsymbol{C}^{2\lambda}\big(\frac{\Gamma(2\eta+3)}{\Gamma(2\eta+2\lambda+2)}\big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)}{\Gamma(2\eta+1)}\big)\big)^2\big)^2}{(2\alpha\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)--2\mu E_{n,\ell}/\delta^2\hbar^2})^4}\,, \quad (31c)$$

when $n=3$,

$$V_\rho=\frac{U\times(V-U)^2}{W}\,, \quad (31d)$$

where,

$$U=\mathbb{C}\boldsymbol{C}^{2\lambda}\Big(\frac{\Gamma(2\eta+4)}{\Gamma(2\eta+2\lambda+3)}\Big(\frac{\Gamma(2\eta+2\lambda+1)+\Gamma(2\eta+2\lambda+2)+\Gamma(2\eta+2\lambda+3)}{\Gamma(2\eta+1)}\Big)\Big)^2,$$

$$V=\Big(\lambda-\frac{1}{2}\Big),\quad W=\Big(2\alpha\sqrt{\ell(\ell+1)-2\mu D_e(b+b\delta r_e-1)-\frac{-2\mu E_{n,\ell}}{\delta^2\hbar^2}}\Big)^4.$$

Here, we obtain the information content in terms of the Shannon entropy and Renyi entropy. The information content in terms of Shannon entropy is given as

$$S_T=S(\rho)+S(\gamma)\geq 3+3\ln 3.142\,, \quad (32)$$

$$S_T=\Big[\ln\Big[\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}(z-1)^m\Big]\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}\Big]$$
$$\times\Big[3.142\mathbb{C}\boldsymbol{C}^{2\lambda}\binom{n+2\eta}{k}^2\Big(\frac{\boldsymbol{C}-2}{2}\Big)^m\binom{n+2\eta}{n-k}(\boldsymbol{C}-2)^{n-k}(\boldsymbol{C})^k-\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+n+1)}{\Gamma(2\eta+2\lambda+n)}$$
$$\times\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m}{\Gamma(2\eta+m+1)}\Big[\frac{\Gamma(2\lambda+1)\Gamma(2\eta)\Gamma(2\lambda+n-1)\Gamma(2\lambda+n-2\eta-1)}{\Gamma(2\eta+2\lambda+n)\Gamma(2\eta+n)\Gamma(2\lambda-2\eta-1)}(z-1)^{2m}\Big]\Big]\,, \quad (33)$$

Equation (32) thus gives the uncertainty relation. The information content in terms of Renyi entropy is defined as

$$R_{(q)T}=R_q(\rho)+R_q(\gamma)\,, \quad (34)$$

$$R_{(q)T}=\frac{1}{1-q}\ln\Big[12.568\Big(\frac{\mathbb{C}\boldsymbol{C}^{2\lambda}}{4\lambda^3\alpha^3}\Big)^q+\Big[\frac{\mathbb{C}}{\alpha}\frac{\Gamma(2\eta+n+1)}{n!\Gamma(2\eta+2\lambda+n}$$
$$\times\sum_{m=0}^{n}\binom{n}{m}\frac{\Gamma(2\eta+2\lambda+n+m)}{\Gamma(2\eta+m+1)}\Big(\frac{t-1}{2}\Big)^m$$

$$\times \frac{\Gamma(2\eta)\Gamma(r)\Gamma(2\lambda+n)\Gamma(r+n-2\eta)}{\Gamma(2\eta+2\lambda+n)\Gamma(r+n)\Gamma(r-2\eta)}\Big]^q\Big]. \quad (35)$$

5 Conclusion

In this paper, we have calculated the position-space and momentum-space of the Shannon entropy and Renyi entropy, the Fisher information and the variance of the Frost-Musulin potential. The information-theoretic measures of the localization of the quantum-mechanical distribution density of the system are obtained in a closed and compact form. The Shannon entropy gives the average of information contained in each message in any quantum mechanical system. The Fisher information describes the concentration of the density around its nodes, providing a measure of the oscillatory character of the corresponding wave function. This describes the chemical and physical properties of the systems. The computed variance gives the spreading around the centroid. However, we have computed the Shannon entropy, the Renyi entropy, the Fisher information and variance for various values of n. It is observed that the Shannon entropy, the Renyi entropy, the Fisher information and variance all decrease with increasing quantum number n respectively.

Appendix A

$$\int_0^w z^y(1-pz)^t \mathrm{d}z = \frac{w^{1+y}}{1+y}\,{}_2F_1(-t,1+y,2+y;pw),$$

$$\int_0^\infty z^n \mathrm{e}^{-tz}\mathrm{d}z = \frac{n!}{P^{n+1}}, \quad p>0, \quad n=1,2,3,\ldots,$$

$$\int_{-1}^1 \Big(\frac{1-x}{2}\Big)^a\Big(\frac{1+x}{2}\Big)^b [P_n^{(a,b)}(x)]^2 = \frac{2\Gamma(a+n+1)\Gamma(b+n+1)}{n!a\Gamma(a+b+2n+1)\Gamma(a+b+n+1)}.$$

Appendix B

$$P_n^{(\alpha,\beta)}(s) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \times \sum_{m=0}^n \binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)}\Big(\frac{s-1}{2}\Big)^m,$$

$$P_n^{(\alpha,\beta)}(s) = \frac{1}{2^n}\sum_{k=0}^n \binom{n+\alpha}{k} \times \binom{n+\beta}{k}\binom{n+\beta}{n-k}(s-1)^{n-k}(s+1)^k,$$

$$\binom{n}{m} = \frac{n!}{(n-m)!m!} = \frac{\Gamma(n+1)}{\Gamma(n-m+1)\Gamma(m+1)}.$$

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