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Energy Conditions and Conservation Laws in LRS Bianchi Type I Spacetimes via Noether Symmetries*

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Abstract *In this paper, we have completely classified the locally rotationally symmetric (LRS) Bianchi type I spacetimes via Noether symmetries (NS). The usual Lagrangian corresponding to LRS Bianchi type I metric is used to find the set of determining equations. To achieve a complete classification, these determining equations are generally integrated to find the components of NS vector field and the metric coefficients. During this procedure, several cases arise which give different Noether algebras of dimension* $5, \ldots, 9$, *11, and 17. A comparison is established between the obtained NS and the Killing and homothetic vectors. Corresponding to all NS generators, the conservation laws are stated by using Noether's theorem. The metrics which we have obtained as a result of our classification are shown to be anisotropic or perfect fluids which satisfy certain energy conditions.*

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1 Introduction

The Einstein's field equations (EFEs), $G_{ab} = kT_{ab}$, are ten tensor equations in the Einstein's theory of general relativity, which relate the spacetime curvature with the energy and momentum within spacetime. The term G_{ab} appearing in these equations expresses the curvature of spacetime and is known as the Einstein tensor. Moreover, k signifies the gravitational constant and T_{ab} denotes the stress-energy tensor, which gives the description of density and flux of energy and momentum in the spacetime.

An exact solution of the EFEs is a Lorentz metric g_{ab}, which is obtained by solving these equations in closed form and is conformable to a physically realistic T_{ab}. The study of the exact solutions of these equations is proved to be one of the important activities in different branches of physics. They describe the structure of spacetime including the inertial motion of objects in the spacetime. Moreover, these solutions lead to the prediction of black holes and different models of evolution of universe. The problem which one faces in finding the exact solutions of these equations is their highly nonlinear nature. These equations cannot be solved without some simplifying assumption, such as symmetry restriction on g_{ab}. Using such restrictions, there are numerous cases where the EFEs are solved completely.[1]

The most basic symmetry is expressed in terms of a Killing vector (KV) X satisfying the relation $\mathcal{L}_X g_{ab} = 0$, where $\mathcal{L}$ denotes the Lie derivative operator and g_{ab} is the metric tensor. The KVs are closely related to the conservation laws in a spacetime. For a detailed study of exact solutions of EFEs with the help of symmetry restrictions on g_{ab} and the corresponding conservation laws, we refer to Refs. [1–3].

Some other conventional symmetries, which have been studied in the literature include homothetic vectors ($\mathcal{L}_X g_{ab} = 2\psi g_{ab}$); ψ being a constant, curvature collineations ($\mathcal{L}_X R^a_{bcd} = 0$), Ricci collineations ($\mathcal{L}_X R_{ab} = 0$) and matter collineations ($\mathcal{L}_X T_{ab} = 0$). Recently, these collineations have been investigated for some physically important spacetimes.[4−9]

In 1918, Emmy Noether[10] proposed her work in terms of Noether theorem. As a result of this theorem, one can find the expression for conserved quantity for each continuous symmetry transformation that leaves the action invariant. NS are also called the variational symmetries and they are associated with mechanical systems possessing a Lagrangian. In particular, for a metric $\mathrm{d}s^2 = g_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$, the associated Lagrangian is given by $L = g_{ab}\dot{x}^a\dot{x}^b$. In this expression, a dot denotes differentiation with respect to the geodesics parameter s of the world line of a point particle moving in a spacetime. It is well known that every KV is an NS but there may exist some NS, which are not KVs. Thus the additional NS may yield some extra conservation laws. Homothetic vectors (HVs) are also closely related with NS. Corresponding to every homothetic vector X, we have an NS, $X+2\psi s\partial_s$. Conversely, if the vector field $X + 2\psi s\partial_s$ is an NS, then X is an HV provided that X does not depend on s.[11]

In literature, NS, their relation will Killing and homothetic vectors and the corresponding conservation laws have been studied by many researchers, for details we refer to Refs. [11–19].

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The term T_{ab} appearing in EFEs is crucial as it describes the physics of a spacetime. The exact solutions of these equations may not give physically interesting results unless the source of T_{ab} is specified. For different sources, T_{ab} has some particular form. For example, for an anisotropic fluid, $T_{ab} = (\rho+p_\perp)u_a u_b + (p_{||}-p_\perp)n_a n_b + p_\perp g_{ab}$; ρ, u_a and n_a being energy density, four-velocity and spacelike unit vector respectively. The quantities $p_\perp$ and $p_{||}$ respectively represent the perpendicular and parallel pressures to n_a. Moreover, $u_a u^a = -1$, $n_a n^a = 1$ and $u_a n^a = 0$.[20] Similarly, for a perfect fluid we have the same form of T_{ab} with $p_{||} = p_\perp = p$.

The positive energy condition is a relation satisfied by the component T_{00} of stress-energy tensor, which ensures that the energy density is non negative. The physical importance of this condition is evident from the fact that the empty vacuum may become unstable if both positive and negative energy regions are allowed.

There are some other energy conditions including weak, strong, null and dominant energy conditions, which generalize the condition $T_{00} \geq 0$ to the whole tensor T_{ab}. The weak energy condition states that $T_{ab}v^a v^b \geq 0$, for any timelike vector v^a at a point of the spacetime manifold. For an anisotropic source, all the energy conditions take the form:

$$\begin{aligned}
&\text{Dominant energy condition :} && \rho \geq 0\,, \quad \rho \geq |p_{||}|\,, \quad \rho \geq |p_\perp|\,,\\
&\text{Strong energy condition :} && \rho + p_{||} \geq 0\,, \quad \rho + p_\perp \geq 0\,, \quad \rho + p_{||} + 2p_\perp \geq 0\,,\\
&\text{Weak energy condition :} && \rho \geq 0\,, \quad \rho + p_{||} \geq 0\,, \quad \rho + p_\perp \geq 0,\\
&\text{Null energy condition :} && \rho + p_{||} \geq 0\,, \quad \rho + p_\perp \geq 0\,.
\end{aligned} \tag{1}$$

In particular, if $p_{||} = p_\perp$, then these conditions reduce to the energy conditions for a perfect fluid.

According to the Bianchi classification of all the 3-dimensional real Lie algebras, there are nine types of Bianchi spatially homogeneous but not necessarily isotropic spacetimes. As a subclass, these models contain the isotropic Friedmann-Robertson-Walker (FRW) universes. The Bianchi type models are of vital importance because the physical variables in these models are dependent on time only. Consequently, the EFEs and other governing equations reduce to ordinary differential equations.

Among the Bianchi type models, the Bianchi type I spacetimes are those models for which the group G_3 of translations of the 3-dimensional Euclidian space is Abelian. In the literature, Bianchi type I spacetimes have been thoroughly studied from the symmetry point of view. Paliathanasis *et al.*[21] presented the symmetry classification of the Klein-Gordon equation in Bianchi I spacetimes, which in turn related the Lie symmetries of this equation with the conformal Killing vectors (CKVs) of the underlying geometry. In the same analysis, it was also shown that the resulting Lie symmetries of the conformal algebra are also NS. Tsamparlis *et al.*[22] studied the CKVs of Bianchi type I spacetimes and conjectured that there are only two conformally flat and two non conformally flat families of these spacetimes admitting CKVs. The same authors stated that for dynamical system whose equations of motion are of the form $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c + f(x^a)$; $f(x^a)$ being an arbitrary function of its argument, the computation of Lie and NS reduce to the probelm of finding the special projective collieations.[23] These general results are then applied to the analytic computation of the Bianchi I metric.

In this paper, we present a complete classification of LRS Bianchi type I spacetimes via NS and the corresponding conservation laws. The bounds for energy conditions are also calculated for all the obtained models. In next section, we derive the list of determining equations for NS. In Secs. 3–9, we present different metrics, their Noether generators and corresponding conservation laws. For each of the obtained model in these sections, a brief discussion on the energy conditions is provided. A conclusion of the present work is appended at the end of the paper.

2 Determining Equations

The metric of the LRS Bianchi type I spacetimes is given by:[1]

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)\left[dy^2 + dz^2\right]\,, \tag{2}$$

such that $A(t) \neq 0$ and $B(t) \neq 0$. For this metric, the EFEs with $k = 1$ give:

$$\begin{aligned}
&T_{00} = \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\,, \quad T_{11} = -\frac{A^2}{B^2}(2B\ddot{B} + \dot{B}^2)\,,\\
&T_{22} = T_{33} = -\frac{B}{A}(A\ddot{B} + B\ddot{A} + \dot{A}\dot{B})\,.
\end{aligned} \tag{3}$$

Here a dot on A and B denotes differentiation with respect to t. For an anisotropic fluid, these components become:

$$T_{00} = \rho\,, \quad T_{11} = A^2 p_{||}\,, \quad T_{22} = T_{33} = B^2 p_\perp\,, \tag{4}$$

while for a perfect fluid, we have the same values of T_{ab} with $p_{||} = p_\perp = p$. Following is the Lagrangian corresponding to the metric (2):

$$L = -\dot{t}^2 + A^2(t)\dot{x}^2 + B^2(t)\left[\dot{y}^2 + \dot{z}^2\right]\,. \tag{5}$$

An NS vector field X is a vector field of the form $X = \xi(\partial/\partial s) + X^j(\partial/\partial a_j)$, satisfying the following condition:

$$X^{(1)}L + L(D\xi) = DF\,, \tag{6}$$

where $X^{(1)} = X + X_s^j(\partial/\partial\dot{a}_j)$ is the first prolongation of X and $X_s^j = DX^j - \dot{a}_j D\xi$ with $D = (\partial/\partial s) + \dot{a}_j(\partial/\partial a_j)$. Moreover, ξ, X^j and the Gauge function F all depend on s and a_j, where $a_j = (t, x, y, z)$ are depending variables of s such that $\dot{a}_j = \partial a_j/\partial s$.

Using the Noether's theorem, the corresponding conservation law for each NS can be found with the help of the expression:

$$\Upsilon = \xi L + (X^j - \dot{a}_j\xi)\frac{\partial L}{\partial \dot{a}_j} - F\,. \tag{7}$$

We may simplify Eq. (6) by using the Lagrangian (5) to get the following set of determining equations:

$$F_{,s} = \xi_{,t} = \xi_{,x} = \xi_{,y} = \xi_{,z} = 0\,, \tag{8}$$
$$2X^0_{,t} = \xi_{,s}\,, \tag{9}$$
$$2\dot{A}X^0 + 2AX^1_{,x} = A\xi_{,s}\,, \tag{10}$$
$$2\dot{B}X^0 + 2BX^2_{,y} = B\xi_{,s}\,, \tag{11}$$
$$2\dot{B}X^0 + 2BX^3_{,z} = B\xi_{,s}\,, \tag{12}$$
$$X^0_{,x} - A^2X^1_{,t} = 0\,, \tag{13}$$
$$X^0_{,y} - B^2X^2_{,t} = 0\,, \tag{14}$$
$$X^0_{,z} - B^2X^3_{,t} = 0\,, \tag{15}$$
$$A^2X^1_{,y} + B^2X^2_{,x} = 0\,, \tag{16}$$
$$A^2X^1_{,z} + B^2X^3_{,x} = 0\,, \tag{17}$$
$$X^2_{,z} + X^3_{,y} = 0\,, \tag{18}$$
$$2X^0_{,s} = -F_{,t}\,, \tag{19}$$
$$2A^2X^1_{,s} = F_{,x}\,, \tag{20}$$
$$2B^2X^2_{,s} = F_{,y}\,, \tag{21}$$
$$2B^2X^3_{,s} = F_{,z}\,. \tag{22}$$

The components X^a of the NS vector field, the Gauge function F and the metric functions A and B appearing in the above system can be found by decoupling and then integrating these equations systematically. In this way, we may get the exact form of LRS Bianchi type I metrics along with their NS. During this procedure, several cases arise which restrict A and B to satisfy certain conditions and give the exact form of LRS Bianchi type I metric admitting NS having dimension $5, \ldots, 9$, 11, and 17. To avoid the repetition, we exclude to write the basic calculations and present the metrics along with their NS, conservation laws, Lie algebra and some physical implications in the upcoming sections.

3 Minimal Set of NS

The minimal set of NS admitted by LRS Bianchi Type I metric is found to be:

$$X_0 = \partial_s\,, \quad X_1 = \partial_x\,, \quad X_2 = \partial_y\,,$$
$$X_3 = \partial_z\,, \quad X_4 = z\partial_y - y\partial_z\,, \tag{23}$$

where X_0 is the symmetry corresponding to the Lagrangian and $X_1, \ldots, X_4$ are the minimum KVs of the metric (2). The above minimal set of NS is obtained under the following restrictions on metric functions.

Table 1 Metrics admitting 5 NS

No.	$A(t)$	$B(t)$
5a	$A \neq B^\alpha$ for all $\alpha \in \Re$	$B = a_1t + a_2, \quad a_1 \neq 0$
5b	$A \neq \mathrm{e}^{\alpha t}$ for all $\alpha \in \Re$	$B = \mathrm{e}^{\beta t}, \quad \beta \neq 0$
5c	$\dot{A} \neq 0, \quad A \neq B$ and $A \neq (at + 2b)^{(a-2d)/a}$	$\ddot{B} \neq 0, \quad B \neq \mathrm{e}^{\beta t}$

Using Eq. (7), the conservation laws for the above set of minimal NS are obtained as:

$$\Upsilon_0 = -L\,, \quad \Upsilon_1 = 2A^2\dot{x}\,, \quad \Upsilon_2 = 2B^2\dot{y}\,,$$
$$\Upsilon_3 = 2B^2\dot{z}\,, \quad \Upsilon_4 = 2B^2(z\dot{y} - y\dot{z})\,. \tag{24}$$

The corresponding Lie algebra for the generators given in Eq. (23) is:

$$[X_2, X_4] = -X_3\,, \quad [X_3, X_4] = X_2\,. \tag{25}$$

The metrics **5a**–**5c** are anisotropic fluids for which:

$$\rho = \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\,,$$
$$p_{||} = -\Big(\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2}\Big)\,,$$
$$p_\perp = -\Big(\frac{\ddot{B}}{B} + \frac{\ddot{A}}{A} + \frac{\dot{A}\dot{B}}{AB}\Big)\,. \tag{26}$$

One may use these values in Eq. (1) to find the energy bounds for the metrics **5a-5c**. For example, for the metric **5b**, the energy conditions restrict the metric function A as follows:

$$\text{Dominant energy condition}: \quad \frac{\beta\dot{A}}{A} \geq 0\,, \quad -2\frac{\beta\dot{A}}{A} - 2\beta^2 \leq \frac{\ddot{A}}{A} \leq \frac{\beta\dot{A}}{A}\,,$$
$$\text{Strong energy condition}: \quad \frac{\beta\dot{A}}{A} \geq 0\,, \quad \beta^2 + \frac{\ddot{A}}{A} \leq 0\,,$$
$$\text{Weak energy condition}: \quad \frac{\beta\dot{A}}{A} \geq 0\,, \quad \frac{\beta\dot{A}}{A} - \frac{\ddot{A}}{A} \geq 0\,. \tag{27}$$

4 Six NS

If $A = \alpha$, where α is a non zero constant and B satisfies the conditions $\ddot{B} \neq 0$ and $B \neq \mathrm{e}^{\beta t}$, then the metric (2) becomes:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \alpha^2\mathrm{d}x^2 + B^2(t)[\mathrm{d}y^2 + \mathrm{d}z^2]\,. \tag{28}$$

For this metric, we obtain six NS, out of which five are same as given in Eq. (23) and the sixth one is a proper

NS, $X_5 = (s/2\alpha^2)\partial_x$, with the Gauge function $F = x$. The corresponding invariant for this Noether generator is $\Upsilon_5 = s\dot{x} - x$ and the Lie algebra of these six NS generators is given by:

$$[X_2, X_4] = -X_3,\ \ [X_3, X_4] = X_2,\ \ [X_0, X_5] = \frac{1}{2\alpha^2}X_1. \quad (29)$$

For the metric (28), being an anisotropic fluid, the physical terms are found to be:

$$\rho = \frac{\dot{B}^2}{B^2}\,, \quad p_{||} = -\Big(\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2}\Big)\,, \quad p_\perp = -\frac{\ddot{B}}{B}\,. \quad (30)$$

Here the dominant energy condition holds if $\ddot{B}/B + \dot{B}^2/B^2 \geq 0$ and $\ddot{B}/B \leq 0$ and the remaining energy conditions are satisfied provided that $\ddot{B}/B \leq 0$.

5 Seven NS

In Table 2, we present some LRS Bianchi type I metrics each of which admits a 7-dimensional Lie algebra of NS. For each of these metrics, five NS are same as given in Eq. (23), while the extra two NS along with their conservation laws and Lie algebra are listed with each metric.

Table 2 Metrics admitting 7 NS.

No.	$A(t)$	$B(t)$	NS	Invariants	Lie algebra
7a	$B^{(b-2c)/b}$,	$(a_1t + a_2)$,	$X_5 = x\partial_x$,	$\Upsilon_5 = 2A^2x\dot{x}$,	$[X_0, X_6] = X_0$,
	$b \neq 2c \neq 0$	$a_1 \neq 0$	$X_6 = s\partial_s + [(a_1t + a_2)/2a_1]\partial_t$	$\Upsilon_6 = -sL - [(a_1t + a_2)/a_1]\dot{t}$	$[X_1, X_5] = X_1$
7b	$\ddot{A} \neq 0$ such that	$B = \beta$	$X_5 = (s/\beta)\partial_y$, $F = 2\beta y$	$\Upsilon_5 = 2\beta(s\dot{y} - y)$	$[X_0, X_5] = X_2/\beta$,
	$A \neq (at + 2b)^{(a-2c)/a}$,				$[X_0, X_6] = X_3/\beta$,
	$\cosh(kt)$, $\cos(kt)$,		$X_6 = (s/\beta)\partial_z$, $F = 2\beta z$	$\Upsilon_6 = 2\beta(s\dot{z} - z)$	$[X_4, X_5] = X_6$,
	$\sinh(kt)$, $\sin(kt)$ and e^{kt}				$[X_6, X_4] = X_5$
7c	$e^{-at/b}$,	$e^{\beta t}$,	$X_5 = \partial_t - \beta(y\partial_y + z\partial_z)$	$\Upsilon_5 = -2\dot{t} - 2\beta^3(y\dot{y} + z\dot{z})$,	$[X_1, X_6] = X_1$,
					$[X_5, X_2] = \beta X_2$
	$-a/b \neq \beta$	$\beta \neq 0$	$X_6 = x\partial_x$	$\Upsilon_6 = 2A^2x\dot{x}$,	$[X_5, X_3] = \beta X_3$
7d	$A = B$	$\ddot{B} \neq 0$	$X_5 = -y\partial_x + x\partial_y$	$\Upsilon_5 = 2B^2(x\dot{y} - y\dot{x})$	$[X_1, X_5] = X_2$,
					$[X_1, X_6] = X_3$
	$A \neq (at + 2b)^{(a-2c)/a}$	$B \neq e^{\beta t}$	$X_6 = -z\partial_x + x\partial_z$	$\Upsilon_6 = 2B^2(x\dot{z} - z\dot{x})$	$[X_5, X_2] = [X_6, X_3] = X_1$,
					$[X_4, X_5] = X_6$,
					$[X_6, X_4] = X_5$,
					$[X_5, X_6] = X_4$

For the metric **7a**, X_5 is a KV and X_6 corresponds to a homothetic vector $[(a_1t + a_2)/2a_1]\partial_t$ with the homothetic constant 1/2. In case **7b**, both X_5 and X_6 represent proper NS. Finally, both X_5 and X_6 are KVs for the metrics **7c** and **7d**.

The metric in case **7a** is an anisotropic fluid with:

$$\rho = \frac{3a_1^2}{(a_1t + a_2)^2} - \frac{4ca_1^2}{b(a_1t + a_2)^2}\,,$$
$$p_{||} = \frac{a_1^2}{(a_1t + a_2)^2}\,, \quad p_\perp = -\frac{a_1^2(b - 2c)^2}{(a_1t + a_2)^2}\,. \quad (31)$$

The above expressions satisfy the dominant energy conditions if $c/b \leq 1/2$ and $3 - 4c/b \geq (b - 2c)^2$, while the weak energy conditions hold if $c/b \leq 0.75$ and $3 - 4c/b \geq -(b - 2c)^2$. Moreover, the strong energy conditions are satisfied when $c/b \leq 1$, $3 - 4c/b \geq -(b - 2c)^2$, and $2(1 - c/b) \geq -(b - 2c)^2$.

Similarly, the metric in case **7b** is an anisotropic fluid whose energy density and parallel pressure vanish and $p_\perp = -\ddot{A}/A$. Here the dominant energy condition is clearly failed, while the remaining energy conditions are satisfied provided that $\ddot{A}/A < 0$.

The energy momentum tensor components for the model **7c**, being an anisotropic fluid, produces the following expressions:

$$\rho = -\frac{2\beta a}{b} + \beta^2\,, \quad p_{||} = -3\beta^2\,,$$
$$p_\perp = -\Big(\beta^2 + \frac{a^2}{b^2} - \frac{\beta a}{b}\Big)\,. \quad (32)$$

For the above values, the strong and dominant energy conditions are failed, while the weak energy conditions hold when $2\beta a/b \leq \beta^2 \leq -\beta a/b$ and $a^2/b^2 \leq -\beta a/b$.

Finally, for the metric **7d** we have:

$$\rho = \frac{3\dot{B}^2}{B^2}\,, \quad p_{||} = p_\perp = -\Big(\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2}\Big)\,. \quad (33)$$

Here we have obtained a perfect fluid matter such that the dominant energy conditions hold if $\ddot{B}/B + 2\dot{B}^2/B^2 \geq 0$ and $\dot{B}^2/B^2 - \ddot{B}/B \geq 0$, while strong and weak energy conditions respectively require $\ddot{B}/B \leq 0$ and $\dot{B}^2/B^2 \geq \ddot{B}/B$.

6 Eight NS

In Table 3, we give all the LRS Bianchi type I metrics admitting eight NS, out of which five are same as given in Eq. (23).

For metric **8a**, X_5 and X_6 are proper NS, while X_7 corresponds to an HV $[(a_1t + a_2)/2a_1]\partial_t + x/2\partial_x$. In case of metric **8b**, X_5 corresponds to an HV $(B/2\dot{B})\partial_t$; X_6 is a proper NS while X_7 is a KV. Finally for case **8c**, X_5 is an NS corresponding to the HV $(A/2\dot{A})\partial_t$ and X_6, X_7 are KVs.

The metric **8a** represents an anisotropic fluid with zero perpendicular pressure and $\rho = -p_{||} = a_1^2/(a_1 t + a_2)^2$. All the energy conditions are satisfied here. Similarly, The metric **8b** is also an anisotropic fluid for which we have:

$$\rho = \frac{(a-2c)^2}{(at+2b)^2}\,, \quad p_{||} = \frac{(a-2c)(6c-a)}{(at+2b)^2}\,,$$
$$p_\perp = \frac{2c(a-2c)}{(at+2b)^2}\,. \tag{34}$$

such that the strong and weak energy conditions hold if either $a \geq 2c \geq 0$ or $a \leq 2c \leq 0$, while for dominant energy condition we must have $(a-2c)^2 \geq |(a-2c)(a-6c)|$ and $(a-2c)^2 \geq |2c(a-2c)|$. The physical terms for case **8c** are given by:

$$\rho = \frac{3a^2 - 4a(d+2c) + 4c(c+2d)}{(at+2b)^2}\,,$$
$$p_{||} = \frac{(a-2c)(6c-a)}{(at+2b)^2}\,,$$
$$p_\perp = \frac{4(a-c)(d+c) - a^2 - 4d^2}{(at+2b)^2}\,. \tag{35}$$

One may simplify the energy conditions using the above values, like the previous cases.

Table 3 Metrics admitting 8 NS.

No.	$A(t)$	$B(t)$	NS	Invariants	Lie algebra
8a	$\alpha \neq 0$	$a_1 t + a_2$, $a_1 \neq 0$	$X_5 = \frac{s^2}{2}\partial_s + \frac{s(a_1t+a_2)}{2a_1}\partial_t + \frac{sx}{2}\partial_x$, $F = -\frac{t^2}{2} + \frac{\alpha^2 x^2}{2} - \frac{a_2 t}{a_1}$, $X_6 = \frac{s}{2\alpha^2}\partial_x$, $F = x$, $X_7 = s\partial_s + \frac{a_1t+a_2}{2a_1}\partial_t + \frac{x}{2}\partial_x$	$\Upsilon_5 = -\frac{s^2}{2}L - \frac{s(a_1t+a_2)}{a_1}\dot{t} + s\alpha^2 x\dot{x} + \frac{t^2}{2} - \frac{\alpha^2x^2}{2} + \frac{a_2 t}{a_1}$, $\Upsilon_6 = s\dot{x} - x$, $\Upsilon_7 = -sL - \frac{a_1t+a_2}{a_1}\dot{t} + \alpha^2 x\dot{x}$	$[X_0, X_5] = X_6$, $[X_0, X_6] = X_0$, $[X_1, X_6] = \frac{X_1}{2}$, $[X_1, X_5] = \alpha^2 X_7$, $[X_6, X_5] = X_5$
8b	$\alpha \neq 0$	$(at+2b)^{(a-2c)/a}$, $a \neq 2c \neq 0$	$X_5 = s\partial_s + \frac{B}{2\dot{B}}\partial_t$, $X_6 = \frac{s\partial_x}{2\alpha^2}$, $F = x$, $X_7 = y\partial_y + z\partial_z$	$\Upsilon_5 = -sL - \frac{B\dot{t}}{\dot{B}}$, $\Upsilon_6 = s\dot{x} - x$, $\Upsilon_7 = 2B^2(y\dot{y} + z\dot{z})$	$[X_0, X_5] = X_0$, $[X_2, X_7] = X_2$, $[X_5, X_6] = X_6$, $[X_0, X_6] = \frac{X_1}{2\alpha^2}$, $[X_3, X_7] = X_3$
8c	$(at+2b)^{[(a-2d)/a]}$,	$(at+2b)^{(a-2c)/a}$ $a \neq 2d \neq 2c \neq 0$	$X_5 = s\partial_s + \frac{A}{2\dot{A}}\partial_t$, $X_6 = x\partial_x - \frac{A}{\dot{A}}\partial_t$, $X_7 = y\partial_y + z\partial_z$	$\Upsilon_5 = -sL - \frac{A\dot{t}}{\dot{A}}$, $\Upsilon_6 = 2A^2 x\dot{x} + \frac{2A\dot{t}}{\dot{A}}$, $\Upsilon_7 = 2B^2(y\dot{y} + z\dot{z})$	$[X_0, X_5] = X_0$, $[X_2, X_7] = X_2$, $[X_1, X_6] = X_1$, $[X_3, X_7] = X_3$

7 Nine NS

There are nine metrics each of which possesses 9-dimensional algebra of NS. All such metrics and the four additional NS different from those given in Eq. (23) for each of these metrics along with their conservation laws and Lie algebra are presented in Table 4.

For the metric **9a**, X_5 and X_6 represent KVs, X_7 is a proper NS while X_8 is an NS corresponding to an HV $(A/2a_1)\partial_t$. In cases **9b–9f**, X_5 and X_6 are KVs, while X_7 and X_8 are proper NS. The metric given in case **9g** admits three additional KVs X_5, X_6, and X_7 along with a proper Noether symmetry X_8. For the metric **9h**, X_5 is an NS which corresponds to an HV $(A/2\dot{A})\partial_t + (y/2)\partial_y + (z/2)\partial_z$, X_6 is a KV while X_7 and X_8 are proper NS. Finally, in case **9i**, we have three additional KVs X_6, X_7, X_8, and one NS X_5 corresponding to the HV $A\partial_t/2\dot{A}$.

The metric **9a** represents a perfect fluid, while all the remaining cases give anisotropic fluids. For the metric **9a**, we find $p_{||} = p_\perp = -\rho/3 = -a_1^2/(a_1t + a_2)^2$, which satisfy all the energy conditions. For the models in cases **9b–9d**, we get $\rho = p_{||} = 0$ and $p_\perp = -k^2$, which do not satisfy any energy condition except the positive energy condition, $\rho \geq 0$. Similarly, for models **9e** and **9f**, we have $\rho = p_{||} = 0$ and $p_\perp = k^2$. Here the dominant energy condition fails, while all the remaining energy conditions are trivially satisfied. For the model **9g**, we obtain $\rho = -p_\perp = \beta^2$ and $p_{||} = -3\beta^2$, which do not satisfy any energy condition except $\rho \geq 0$. The metric given in **9h** is an anisotropic fluid for which $\rho = p_{||} = 0$ and $p_\perp = 2c(a-2c)/(at+2b)^2$. The dominant energy condition is clearly failed, while the remaining energy conditions are satisfied provided that $c(a-2c) \geq 0$. The following physical terms for the metric **9i** reveal that it represents a perfect fluid model:

$$\rho = \frac{3(a-2c)^2}{(at+2b)^2}\,, \quad p_{||} = p_\perp = \frac{(a-2c)(6c-a)}{(at+2b)^2}\,. \tag{36}$$

The corresponding weak energy conditions hold for $a(a-2c) \geq 0$, whereas the strong energy conditions require $a \geq 2c \geq 0$ or $a \leq 2c \leq 0$. Moreover, the dominant energy conditions are satisfied if $a(a-2c) \geq 0$, and $(a-2c)(a-3c) \leq 0$.

Table 4 Metrics admitting 9 NS.

No.	$A(t)$	$B(t)$	NS	Invariants	Lie algebra
9a	a_1t+a_2, $a_1\neq 0$	$B=A$	$X_5=-y\partial_x+x\partial_y$, $X_6=-z\partial_x+x\partial_z$, $X_7=\frac{s^2\partial_s}{2}+\frac{sA\partial_t}{2a_1}, \quad F=-\frac{tA}{2a_1}$, $X_8=s\partial_s+\frac{A}{2a_1}\partial_t$	$\Upsilon_5=2B^2(x\dot{y}-y\dot{x})$, $\Upsilon_6=2B^2(x\dot{z}-z\dot{x})$, $\Upsilon_7=-\frac{s^2}{2}L-\frac{sA}{a_1}\dot{t}+\frac{tA}{2a_1}$, $\Upsilon_8=-sL-\frac{A}{a_1}\dot{t}$	$[X_1,X_5]=X_2, \quad [X_1,X_6]=X_3$, $[X_0,X_8]=X_0, \quad [X_8,X_7]=X_7$, $[X_6,X_4]=X_5, \quad [X_5,X_6]=X_4$, $[X_0,X_7]=X_8, \quad [X_4,X_5]=X_6$, $[X_5,X_2]=[X_6,X_3]=X_1$
9b	e^{kt}, $k\neq 0$	β	$X_5=x\partial_t-(\frac{\mathrm{e}^{-2kt}}{2k}+\frac{kx^2}{2})\partial_x$, $X_6=\partial_t-kx\partial_x$, $X_7=\frac{s}{\beta}\partial_y, \quad F=2\beta y$, $X_8=\frac{s}{\beta}\partial_z, \quad F=2\beta z$	$\Upsilon_5=-2x\dot{t}-(k^3x^2+\frac{k}{\mathrm{e}^{2kt}})\dot{x}$, $\Upsilon_6=-2(\dot{t}+k^3x\dot{x})$, $\Upsilon_7=2\beta(s\dot{y}-y)$, $\Upsilon_8=2\beta(s\dot{z}-z)$	$[X_0,X_7]=\frac{X_2}{\beta}, \quad [X_0,X_8]=\frac{X_3}{\beta}$, $[X_1,X_5]=X_6, \quad [X_6,X_1]=kX_1$, $[X_5,X_6]=kX_5, \quad [X_4,X_7]=X_8$, $[X_8,X_4]=X_7$
9c	$\cosh(kt)$	β	$X_5=\frac{\sin(kx)}{k}\partial_t+\frac{\dot{A}\cos(kx)}{k^2A}\partial_x$, $X_6=\frac{\dot{A}\sin(kx)}{k^2A}\partial_x-\frac{\cos(kx)}{k}\partial_t$, $X_7=\frac{s}{\beta}\partial_y, \quad F=2\beta y$, $X_8=\frac{s}{\beta}\partial_z, \quad F=2\beta z$	$\Upsilon_5=\frac{2A\dot{A}}{k^2}\cos(kx)\dot{x}$ $-\frac{2\sin(kx)}{k}\dot{t}$, $\Upsilon_6=\frac{2\cos(kx)}{k}\dot{t}$ $+\frac{2A\dot{A}}{k^2}\sin(kx)\dot{x}$, $\Upsilon_7=2\beta(s\dot{y}-y)$, $\Upsilon_8=2\beta(s\dot{z}-z)$	$[X_0,X_7]=\frac{X_2}{\beta}, \quad [X_0,X_8]=\frac{X_3}{\beta}$, $[X_5,X_6]=\frac{X_1}{k}, \quad [X_4,X_7]=X_8$, $[X_5,X_1]=kX_6, \quad [X_1,X_6]=kX_5$, $[X_8,X_4]=X_7$
9d	$\sinh(kt)$	β	$X_5=\frac{\sinh(kx)}{k}\partial_t-\frac{\dot{A}\cosh(kx)}{k^2A}\partial_x$, $X_6=\frac{\cosh(kx)}{k}\partial_t-\frac{\dot{A}\sinh(kx)}{k^2A}\partial_x$, $X_7=\frac{s}{\beta}\partial_y, \quad F=2\beta y$, $X_8=\frac{s}{\beta}\partial_z, \quad F=2\beta z$	$\Upsilon_5=-\frac{2\sinh(kx)}{k}\dot{t}$ $-\frac{2A\dot{A}}{k^2}\cosh(kx)\dot{x}$, $\Upsilon_6=-\frac{2\cosh(kx)}{k}\dot{t}$ $-\frac{2A\dot{A}}{k^2}\sinh(kx)\dot{x}$, $\Upsilon_7=2\beta(s\dot{y}-y)$, $\Upsilon_8=2\beta(s\dot{z}-z)$	Same as case **9c** except $[X_1,X_5]=kX_6$
9e	$\cos(kt)$	β	Similar to **9d**	Similar to **9d**	Same as case **9c**
9f	$\sin(kt)$	β	Similar to **9d**	Similar to **9d**	Same as case **9d**
9g	α	$\mathrm{e}^{\beta t}$, $\beta\neq 0$	$X_5=\frac{y}{\beta}\partial_t-yz\partial_z$ $+(\frac{z^2-y^2}{2}-\frac{1}{2\beta^2B^2})\partial_y$, $X_6=yz\partial_y-\frac{z}{\beta}\partial_t$ $+(\frac{z^2-y^2}{2}+\frac{1}{2\beta^2B^2})\partial_z$, $X_7=\partial_t-\beta(y\partial_y+z\partial_z)$, $X_8=\frac{s}{2\alpha^2}\partial_x, \quad F=x$	$\Upsilon_5=-2B^2yz\dot{z}-\frac{2y}{\beta}\dot{t}$ $+(B^2(z^2-y^2)-\frac{1}{\beta^2})\dot{y}$, $\Upsilon_6=\frac{2z}{\beta}\dot{t}+2B^2yz\dot{y}$ $+(B^2(z^2-y^2)+\frac{1}{\beta^2})\dot{z}$, $\Upsilon_7=-2\dot{t}-2\beta B^2(y\dot{y}+z\dot{z})$, $\Upsilon_8=s\dot{x}-x$	$[X_0,X_8]=\frac{X_1}{2\alpha^2}, \quad [X_5,X_4]=X_6$, $[X_2,X_5]=[X_6,X_3]=\frac{X_7}{\beta}$, $[X_2,X_6]=[X_3,X_5]=X_4$, $[X_7,X_3]=\beta X_3, \quad [X_7,X_2]=\beta X_2$, $[X_5,X_7]=\frac{X_5}{\beta}, \quad [X_4,X_6]=X_5$, $[X_6,X_7]=\frac{X_6}{\beta}$
9h	$(at+2b)^{\chi}$, $\chi=\frac{a-2c}{a}$, $a\neq 2c\neq 0$	β	$X_5=s\partial_s+\frac{A}{2\dot{A}}\partial_t+\frac{y}{2}\partial_y+\frac{z}{2}\partial_z$, $X_6=x\partial_x-\frac{A}{\dot{A}}\partial_t$, $X_7=\frac{s}{\beta}\partial_y, \quad F=2\beta y,$, $X_8=\frac{s}{\beta}\partial_z, \quad F=2\beta z$	$\Upsilon_5=-sL-\frac{A\dot{t}}{\dot{A}}+\beta^2(y\dot{y}+z\dot{z})$, $\Upsilon_6=2A^2x\dot{x}+\frac{2A\dot{t}}{\dot{A}}$, $\Upsilon_7=2\beta(s\dot{y}-y)$, $\Upsilon_8=2\beta(s\dot{z}-z)$	$[X_0,X_5]=X_0, \quad [X_1,X_6]=X_1$, $2[X_5,X_8]=[X_4,X_7]=X_8$, $2[X_3,X_5]=\beta[X_0,X_8]=X_3$, $2[X_2,X_5]=\beta[X_0,X_7]=X_2$, $2[X_5,X_7]=[X_8,X_4]=X_7$
9i	as in **9h**	$B=A$,	$X_5=s\partial_s+\frac{A}{2\dot{A}}\partial_t$, $X_6=-y\partial_x+x\partial_y$, $X_7=-z\partial_x+x\partial_z$, $X_8=x\partial_x+y\partial_y+z\partial_z-\frac{A}{\dot{A}}\partial_t$	$\Upsilon_5=-sL-\frac{A\dot{t}}{\dot{A}}$, $\Upsilon_6=2A^2(x\dot{y}-y\dot{x})$, $\Upsilon_7=2A^2(x\dot{z}-z\dot{x})$, $\Upsilon_8=2A^2(x\dot{x}+y\dot{y}+z\dot{z})-\frac{2A\dot{t}}{\dot{A}}$	$[X_0,X_5]=X_0, \quad [X_4,X_6]=X_7$, $[X_1,X_6]=[X_2,X_8]=X_2$, $[X_1,X_7]=[X_3,X_8]=X_3$, $[X_1,X_8]=[X_6,X_2]=X_1$, $[X_7,X_3]=X_1, \quad [X_7,X_4]=X_6$, $[X_6,X_7]=X_4$

8 Eleven NS

Following is the only one metric which admits eleven NS:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{e}^{2\beta t}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\,, \qquad (37)$$

where $\beta \neq 0$. The set of eleven NS for the above metric contains the minimal set of NS and the extra six NS (KVs) are obtained as:

$$\begin{aligned}
&X_5 = -z\partial_x + x\partial_z\,, \quad X_6 = -y\partial_x + x\partial_y\,,\\
&X_7 = \frac{y}{\beta}\partial_t - xy\partial_x - yz\partial_z\\
&\qquad + \Big(\frac{x^2-y^2+z^2}{2} - \frac{1}{2\beta^2\,\mathrm{e}^{2\beta t}}\Big)\partial_y\,,\\
&X_8 = -\frac{z}{\beta}\partial_t + xz\partial_x + yz\partial_y\\
&\qquad + \Big(\frac{z^2-y^2-x^2}{2} + \frac{1}{2\beta^2\,\mathrm{e}^{2\beta t}}\Big)\partial_z\,,\\
&X_9 = x\partial_t - \beta xy\partial_y - \beta xz\partial_z\\
&\qquad + \Big(\frac{\beta(y^2-x^2+z^2)}{2} - \frac{1}{2\beta\,\mathrm{e}^{2\beta t}}\Big)\partial_x\,,\\
&X_{10} = \partial_t - \beta x\partial_x - \beta y\partial_y - \beta z\partial_z\,. \qquad (38)
\end{aligned}$$

The Lie algebra for the above set of generators is found to be:

$$\begin{aligned}
&\beta[X_1,X_5] = [X_{10},X_3] = \beta X_3\,,\\
&\beta[X_1,X_7] = \beta[X_5,X_4] = [X_9,X_2] = \beta X_6\,,\\
&\beta[X_1,X_6] = [X_{10},X_2] = \beta X_2\,,\\
&\beta[X_8,X_1] = \beta[X_4,X_6] = [X_9,X_3] = \beta X_5\,,\\
&\beta[X_2,X_7] = \beta[X_8,X_3] = [X_1,X_9] = X_{10}\,,\\
&\beta[X_6,X_2] = \beta[X_5,X_3] = [X_{10},X_1] = \beta X_1\,,\\
&[X_2,X_8] = [X_3,X_7] = [X_6,X_5] = X_4\,,\\
&\beta[X_7,X_4] = [X_5,X_9] = [X_8,X_{10}] = \beta X_8\,,\\
&\beta[X_4,X_8] = [X_9,X_6] = [X_7,X_{10}] = \beta X_7\,,\\
&\beta^2[X_6,X_7] = \beta^2[X_8,X_5] = [X_9,X_{10}] = \beta X_9\,,
\end{aligned}$$

and the corresponding conservation laws are:

$$\begin{aligned}
&\Upsilon_5 = 2\,\mathrm{e}^{2\beta t}(x\dot z - z\dot x)\,, \quad \Upsilon_6 = 2\,\mathrm{e}^{2\beta t}(x\dot y - y\dot x)\,,\\
&\Upsilon_7 = -\frac{2y}{\beta}\dot t - 2\,\mathrm{e}^{2\beta t}y(x\dot x + z\dot z)\\
&\qquad - \frac{\dot y}{\beta^2} + \mathrm{e}^{2\beta t}\dot y(x^2-y^2+z^2)\,,\\
&\Upsilon_8 = \frac{2z}{\beta}\dot t + 2\,\mathrm{e}^{2\beta t}z(x\dot x + y\dot y)\\
&\qquad + \frac{\dot z}{\beta^2} + \mathrm{e}^{2\beta t}\dot z(z^2-y^2-x^2)\,,\\
&\Upsilon_9 = -2x\dot t - 2\beta\,\mathrm{e}^{2\beta t}x(y\dot y + z\dot z)\\
&\qquad - \frac{\dot x}{\beta} + \beta\,\mathrm{e}^{2\beta t}\dot x(y^2-x^2+z^2)\,,\\
&\Upsilon_{10} = -2\dot t + 2\beta\,\mathrm{e}^{2\beta t}(x\dot x - y\dot y - z\dot z)\,. \qquad (39)
\end{aligned}$$

For the metric (37), we have $\rho = 3\beta^2$ and $p = p_{||} = p_\perp = -3\beta^2$. Thus it gives a perfect fluid. Here the strong energy condition is violated while the remaining energy conditions are satisfied.

9 Maximal Set of NS

It is well known that the the dimension of Noether algebra for flat Minkowski metric is 17. Following is an another metric admitting 17 NS.

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (a_1t + a_2)^2\,\mathrm{d}x^2 + \beta^2(\mathrm{d}y^2 + \mathrm{d}z^2)\,, \qquad (40)$$

where $a_1 \neq 0$ and $\beta \neq 0$. Five NS of the above metric are same as given in Eq. (23), while the remaining twelve are given as follows:

$$\begin{aligned}
&X_5 = s\partial_s + \frac{A}{2a_1}\partial_t + \frac{y}{2}\partial_y + \frac{z}{2}\partial_z\,,\\
&X_6 = \frac{s^2}{2}\partial_s + \frac{sA}{2a_1}\partial_t + \frac{sy}{2}\partial_y + \frac{sz}{2}\partial_z\,,\\
&F = -\frac{t^2}{2} + \beta^2\frac{y^2+z^2}{2} - \frac{a_2A}{a_1^2}\,,\\
&X_7 = \frac{s}{\beta}\partial_y\,, \quad F = 2\beta y\,,\\
&X_8 = \frac{s}{\beta}\partial_z\,, \quad F = 2\beta z\,,\\
&X_9 = -\frac{s}{2}\cosh(a_1x)\partial_t + \frac{s}{2A}\sinh(a_1x)\partial_x\,,\\
&F = \frac{A}{a_1}\cosh(a_1x)\,,\\
&X_{10} = -\frac{s}{2}\sinh(a_1x)\partial_t + \frac{s}{2A}\cosh(a_1x)\partial_x\,,\\
&F = \frac{A}{a_1}\sinh(a_1x),\,,\\
&X_{11} = \beta^2y\cosh(a_1x)\partial_t - \frac{\beta^2y}{A}\sinh(a_1x)\partial_x\\
&\qquad + \frac{A}{a_1}\cosh(a_1x)\partial_y\,,\\
&X_{12} = \beta^2y\sinh(a_1x)\partial_t - \frac{\beta^2y}{A}\cosh(a_1x)\partial_x\\
&\qquad + \frac{A}{a_1}\sinh(a_1x)\partial_y\,,\\
&X_{13} = \beta^2z\cosh(a_1x)\partial_t - \frac{\beta^2z}{A}\sinh(a_1x)\partial_x\\
&\qquad + \frac{A}{a_1}\cosh(a_1x)\partial_z\,,\\
&X_{14} = \beta^2z\sinh(a_1x)\partial_t - \frac{\beta^2z}{A}\cosh(a_1x)\partial_x\\
&\qquad + \frac{A}{a_1}\sinh(a_1x)\partial_z\,,\\
&X_{15} = \cosh(a_1x)\partial_t - \frac{1}{A}\sinh(a_1x)\partial_x\,,\\
&X_{16} = \sinh(a_1x)\partial_t - \frac{1}{A}\cosh(a_1x)\partial_x\,.
\end{aligned}$$

In the above set, X_5 is an NS and its corresponding HV is $(A/2a_1)\partial_t + (y/2)\partial_y + (z/2)\partial_z$. Moreover, $X_6, \ldots, X_{10}$ are proper NS and $X_{11}, \ldots, X_{16}$ are KVs. The Lie algebra for these generators is given by:

$$
\begin{aligned}
&[X_0, X_6] = X_5\,, \quad [X_0, X_5] = X_0\,,\\
&[X_{15}, X_{11}] = \beta[X_0, X_7] = X_2\,,\\
&[X_{12}, X_{16}] = 2[X_2, X_5] = X_2\,,\\
&[X_{15}, X_{13}] = \beta[X_0, X_8] = [X_{14}, X_{16}] = X_3\,,\\
&[X_9, X_0] = [X_{15}, X_5] = \frac{X_{15}}{2}\,,\\
&[X_{10}, X_0] = [X_{16}, X_5] = \frac{X_{16}}{2}\,,\\
&[X_1, X_9] = a_1X_{10}\,, \quad [X_1, X_{10}] = a_1X_9\,,\\
&[X_1, X_{11}] = a_1X_{12}\,, \quad [X_1, X_{12}] = a_1X_{11}\,,\\
&[X_1, X_{13}] = a_1X_{14}\,, \quad [X_1, X_{14}] = a_1X_{13}\,,\\
&[X_1, X_{15}] = a_1X_{16}\,, \quad [X_1, X_{16}] = a_1X_{15}\,,\\
&[X_2, X_6] = [X_{11}, X_9] = [X_{10}, X_{12}] = \frac{\beta X_7}{2}\,,\\
&[X_4, X_7] = 2[X_5, X_8] = X_8\,,\\
&[X_2, X_{11}] = [X_3, X_{13}] = \beta^2 X_{15}\,,\\
&[X_2, X_{12}] = [X_3, X_{14}] = \beta^2 X_{16}\,,\\
&[X_3, X_6] = [X_{13}, X_9] = [X_{10}, X_{14}] = \frac{\beta X_8}{2}\,,\\
&[X_8, X_4] = 2[X_5, X_7] = X_7\,, \quad [X_5, X_6] = X_6\,,\\
&[X_6, X_{15}] = 2[X_5, X_9] = X_9\,,\\
&[X_6, X_{16}] = 2[X_5, X_{10}] = X_{10}\,,\\
&[X_{11}, X_7] = [X_{13}, X_8] = 2\beta X_9\,,\\
&[X_{12}, X_7] = [X_{14}, X_8] = 2\beta X_{10}\,,\\
&a_1[X_{12}, X_{11}] = a_1[X_{14}, X_{13}] = \beta^2 X_1\,,\\
&[X_{12}, X_{14}] = [X_{13}, X_{11}] = \beta^2 X_4\,.
\end{aligned}
$$

In this case, the conservation laws are obtained as:

$$
\begin{aligned}
\Upsilon_5 &= -sL - \frac{A}{a_1}\dot{t} + \beta^2(y\dot{y} + z\dot{z})\,,\\
\Upsilon_6 &= -\frac{s^2}{2}L - \frac{sA}{a_1}\dot{t} + s\beta^2(y\dot{y} + z\dot{z}) + \frac{t^2}{2}\\
&\quad - \frac{\beta^2(y^2+z^2)}{2} + \frac{a_2A}{a_1^2}\,,\\
\Upsilon_7 &= 2\beta(s\dot{y} - y)\,, \quad \Upsilon_8 = 2\beta(s\dot{z} - z)\,,\\
\Upsilon_9 &= s\cosh(a_1x)\dot{t} + sA\sinh(a_1x)\dot{x} - \frac{A}{a_1}\cosh(a_1x)\,,\\
\Upsilon_{10} &= s\sinh(a_1x)\dot{t} + sA\cosh(a_1x)\dot{x} - \frac{A}{a_1}\sinh(a_1x)\,,\\
\Upsilon_{11} &= -2\beta^2y\cosh(a_1x)\dot{t} - 2\beta^2yA\sinh(a_1x)\dot{x}\\
&\quad + \frac{2\beta^2A}{a_1}\sinh(a_1x)\dot{y}\,,\\
\Upsilon_{12} &= -2\beta^2y\sinh(a_1x)\dot{t} - 2\beta^2yA\cosh(a_1x)\dot{x}\\
&\quad + \frac{2\beta^2A}{a_1}\cosh(a_1x)\dot{y}\,,\\
\Upsilon_{13} &= -2\beta^2z\cosh(a_1x)\dot{t} - 2\beta^2zA\sinh(a_1x)\dot{x}\\
&\quad + \frac{2\beta^2A}{a_1}\sinh(a_1x)\dot{z}\,,\\
\Upsilon_{14} &= -2\beta^2z\sinh(a_1x)\dot{t} - 2\beta^2zA\cosh(a_1x)\dot{x}\\
&\quad + \frac{2\beta^2A}{a_1}\cosh(a_1x)\dot{z}\,,\\
\Upsilon_{15} &= -2(\cosh(a_1x)\dot{t} + A\sinh(a_1x)\dot{x})\,,\\
\Upsilon_{16} &= -2(\sinh(a_1x)\dot{t} + A\cosh(a_1x)\dot{x})\,. \qquad (41)
\end{aligned}
$$

For the metric (40), we have $T_{ab} = 0$. Thus it represents a vacuum solution.

10 Conclusion

In this paper, we have studied the NS of LRS Bianchi type I spacetimes. For a complete classification, the Noether determining equations are generally solved, which in result categorized the mentioned spacetimes metric into seven different classes according to the dimension of Noether algebra. The possible dimension of Lie algebra of Noether symmetry turned out to be 5, 6, 7, 8, 9, 11, and 17. These NS are compared with Killing and homothetic vectors and it is shown that the possible dimension of Killing algebra for LRS Bianchi type I spacetime is 4, 5, 6, 7 or 10. Besides this, the conservations laws are presented for all the Noether symmetry generators by using the well known Noether's theorem. Finally, it is observed that most of the obtained metrics are anisotropic or perfect fluids satisfying different energy conditions.

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