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# The First Integral Method to Study a Class of Reaction-Diffusion Equations*

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***Abstract*** *In this letter, a class of reaction-diffusion equations, which arise in chemical reaction or ecology and other fields of physics, are investigated. A more general analytical solution of the equation is obtained by using the first integral method.*

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## 1 Introduction

It is well known that many important dynamics processes can be described by specific nonlinear partial differential equations. When a nonlinear partial differential equation is used to characterize a physical parameter indicating some kinds of reaction-diffusion, propagation, or aggregation properties, it is of fundamental physical interest to solve the partial differential equation with a certain type of traveling wave solutions. Although in the past several decades much progress in finding exact solutions of these nonlinear evolution equations had been achieved, there are only limited approaches[1−6] available presently due to the complexity of mathematics.

In the present paper we shall investigate the explicit analytical solutions for the two-dimensional reaction-diffusion equation by using the first integral method. This kind of method has been successfully used to deal with the Burgers-KdV equation and compound Burgers-KdV equation, see Refs. [7] ∼ [9].

Let us consider the following reaction-diffusion model in two dimensions,

$$(u_t - Du_{xx} + \alpha u^3 + \beta u^2 + \gamma u)_x + \lambda u_{yy} = 0\,, \qquad (1)$$

where $D$, $\alpha$, $\beta$, $\gamma$, and $\lambda$ are real constants, and $D$ is also called diffusion coefficient.

For $\lambda = 0$, equation (1) is a reaction-diffusion equation arising in chemical reaction or ecology and other fields of physics. The choice $D = 1$, $\beta = 0$, and $\gamma = -\alpha$ leads Eq. (1) to the two-dimensional Chaffee–Infante equation,

$$[u_t - u_{xx} + \alpha(u^3 - u)]_x + \lambda u_{yy} = 0\,, \qquad (2)$$

the choice $D = 1$, $\gamma = 0$, and $\alpha = -\beta = 1$ leads Eq. (1) to the two-dimensional Huxley equation,

$$[u_t - u_{xx} + u^3 - u^2]_x + \lambda u_{yy} = 0\,, \qquad (3)$$

and if taking $D = 1$, $\alpha = 1$, $\beta = -(a+1)$, and $\gamma = a$ in Eq. (1), we can obtain the following two-dimensional Fitzhugh–Nagumo equation:

$$[u_t - u_{xx} + u^3 - (a+1)u^2 + au]_x + \lambda u_{yy} = 0\,. \qquad (4)$$

## 2 Exact Solutions to Two-Dimensional Reaction-Diffusion Equation

Without loss of generality, we may assume that equation (1) has traveling wave solution in the form

$$u(x,y,t) = u(\xi)\,, \qquad \xi = hx + ly - \omega t\,, \qquad (5)$$

where $h$, $l$, $\omega$ are real constants to be determined.

Substituting Eq. (5) into Eq. (1) then integrating it once yields

$$h[-\omega u'(\xi) - Dh^2 u''(\xi) + \alpha u^3(\xi) + \beta u^2(\xi) + \gamma u(\xi)] + \lambda l^2 u'(\xi) + I_0 = 0\,, \qquad (6)$$

where $I_0$ is an arbitrary integration constant.

Equation (6) is a nonlinear ordinary differential equation, and we can rewrite it as

$$u_{\xi\xi} - au_\xi - bu^3 - cu^2 - du - e = 0\,, \qquad (7)$$

where

$$a = \frac{l^2\lambda - h\omega}{Dh^3}\,, \quad b = \frac{\alpha}{Dh^2}\,, \quad c = \frac{\beta}{Dh^2}\,, \quad d = \frac{\gamma}{Dh^2}\,, \quad e = \frac{I_0}{Dh^3}\,.$$

Let $x = u$, $y = u_\xi$, then equation (7) is equivalent to

$$\dot{x} = y\,, \qquad \dot{y} = ay + bx^3 + cx^2 + dx + e\,. \qquad (8)$$

According to the first integral method in Ref. [7], we first introduce the Division Theorem.

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***Division Theorem*** Suppose that $P(\omega, z)$ and $Q(\omega, z)$ are polynomials in $\mathbb{C}[\omega, z]$, $P(\omega, z)$ is irreducible in $\mathbb{C}[\omega, z]$. If $Q(\xi, z)$ vanishes at all zero points of $P(\omega, z)$, then there exists a polynomial $G(\omega, z)$ in $\mathbb{C}[\omega, z]$, such that

$$Q(\omega, z) = P(\omega, z)G(\omega, z)\,.$$

Applying the above Division Theorem, we can seek the first integral to Eq. (8).

Suppose that $x = x(\xi)$ and $y = y(\xi)$ are the nontrivial solutions to Eq. (8), and $P[x(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(x)y^i$ is an irreducible polynomial in $\mathbb{C}[\omega, z]$, such that

$$P[x(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(x)y^i = 0\,, \tag{9}$$

where $a_i(x)$ $(i = 0, 1, 2, \ldots, m)$ are polynomials of $x$ and all relatively prime in $\mathbb{C}[\omega, z]$, $a_m(x) \neq 0$. Equation (9) is also called the first integral to Eq. (8).

Here we take $m = 2$ in Eq. (9). Note that $\mathrm{d}P/\mathrm{d}\xi$ is a polynomial in $x$ and $y$, and $P[x(\xi), y(\xi)] = 0$ implies $\mathrm{d}P/\mathrm{d}\xi|_{(8)} = 0$. By the Division Theorem, there exists a polynomial $H(x, y) = \alpha(x) + \beta(x)y$ in $\mathbb{C}[\omega, z]$, then

$$\frac{\mathrm{d}P}{\mathrm{d}\xi}\Big|_{(8)} = \sum_{i=0}^{2} a_i'(x)y^{i+1} + \sum_{i=0}^{2} i a_i(x)y^{i-1}(ay + bx^3 + cx^2 + \mathrm{d}x + e) = [\alpha(x) + \beta(x)y]\sum_{i=0}^{2} a_i(x)y^i\,, \tag{10}$$

where the prime denotes derivative.

Comparing the coefficients of $y^i$ $(i = 3, 2, 1, 0)$ on both sides of Eq. (10) yields

$$(y^3)\quad a_2'(x) = \beta(x)a_2(x)\,, \tag{11}$$

$$(y^2)\quad a_1'(x) = (\alpha(x) - 2a)a_2(x) + \beta(x)a_1(x)\,, \tag{12}$$

$$(y^1)\quad a_0'(x) = -2(bx^3 + cx^2 + dx + e)a_2(x) + (\alpha(x) - a)a_1(x) + \beta(x)a_0(x)\,, \tag{13}$$

$$(y^0)\quad a_1(x)(bx^3 + cx^2 + dx + e) = \alpha(x)a_0(x)\,. \tag{14}$$

From Eq. (11) we deduce that $a_2(x)$ is a constant and $\beta(x) = 0$ for $a_i(x)$ $(i = 0, 1, 2)$ being polynomials.

To simplify, taking $a_2(x) = 1$, we have

$$a_1(x) = \int [\alpha(x) - 2a]\,\mathrm{d}x\,, \tag{15}$$

$$a_0(x) = \int [-2(bx^3 + cx^2 + dx + e) + (\alpha(x) - a)a_1(x)]\,\mathrm{d}x. \tag{16}$$

By Eqs. (14), (15), and (16) we can conclude that $\deg\alpha(x) = 0$ or $\deg\alpha(x) = 1$ corresponding to $\deg a_1(x) = 1$ or $\deg a_1(x) = 2$, respectively. Otherwise, if $\deg\alpha(x) = k > 1$, then we deduce $\deg a_1(x) = k + 1$ and $\deg a_0(x) = 2k + 2$. This yields a contradiction with Eq. (14).

In the case of $\deg\alpha(x) = 0$, we assume $\alpha(x) = \alpha$ $(\alpha \in \mathbb{C}\ )$ and

$$a_1(x) = A_1x + A_0\,, \quad A_1, A_0 \in \mathbb{C} \quad \text{with} \quad A_1 \neq 0\,. \tag{17}$$

From Eq. (15) we get $A_1 = \alpha - 2a$, or $\alpha = A_1 + 2a$. And from Eq. (16) we have

$$a_0(x) = -\frac{b}{2}x^4 - \frac{2}{3}cx^3 + \Big(\frac{A_1(A_1 + a)}{2} - d\Big)x^2 + [A_0(A_1 + a) - 2e]x + I_1\,, \tag{18}$$

where $I_1$ is an arbitrary integration constant.

Substituting Eqs. (17) and (18) into Eq. (14) and setting all coefficients of $x^i$ $(i = 4, 3, 2, 1, 0)$ to zero yields

$$\begin{gathered} A_1b = -\frac{b}{2}(A_1 + 2a)\,, \quad A_1c + A_0b = -\frac{2c}{3}(A_1 + 2a)\,, \quad A_1d + A_0c = -(A_1 + 2a)d + \frac{A_1(A_1 + a)(A_1 + 2a)}{2}\,, \\ A_1e + A_0d = -2(A_1 + 2a)e + A_0(A_1 + a)(A_1 + 2a)\,, \qquad A_0e = I_1(A_1 + 2a)\,. \end{gathered} \tag{19}$$

By solving Eq. (19) we have

$$A_1 = -\frac{2}{3}a\,, \quad A_0 = -\frac{2}{9b}ac\,, \quad I_1 = \frac{c^2}{162b^2}\Big(2a^2 - \frac{c^2}{b}\Big)\,, \quad 2a^2 - 3c^2 + 9d = 0\,, \quad 9dc - 4a^2c - 81be = 0\,. \tag{20}$$

From Eqs. (18) and (20) we have

$$\begin{aligned} a_1(x) &= A_1x + A_0 = -\frac{2}{3}ax - \frac{2}{9b}ac\,, \\ a_0(x) &= -\frac{b}{2}x^4 - \frac{2}{3}cx^3 + \Big(\frac{a^2}{9} - \frac{c^2}{3b}\Big)x^2 + \frac{2c}{27b}\Big(a^2 - \frac{c^2}{b}\Big)x + \frac{c^2}{162b^2}\Big(2a^2 - \frac{c^2}{b}\Big)\,. \end{aligned} \tag{21}$$

When

$$P[x(\xi), y(\xi)] = \sum_{i=0}^{m} a_i(x) y^i = 0\,,$$

we have

$$y^2 - \Big(\frac{2a}{3}x + \frac{2ac}{9b}\Big)y - \frac{b}{2}x^4 - \frac{2}{3}cx^3 + \Big(\frac{a^2}{9} - \frac{c^2}{3b}\Big)x^2 + \frac{2c}{27b}\Big(a^2 - \frac{c^2}{b}\Big)x + \frac{c^2}{162b^2}\Big(2a^2 - \frac{c^2}{b}\Big) = 0\,. \tag{22}$$

Then $y$ can be expressed in terms of $x$ by Eq. (22), i.e.

$$y = \Big(\frac{a}{3}x + \frac{ac}{9b}\Big) \pm \frac{1}{9b^2}\sqrt{\frac{b}{2}}\,(3bx + c)^2\,. \tag{23}$$

Combining Eqs. (8) and (23), we have

$$\frac{9b^2\,\mathrm{d}x}{ab(3bx + c) \pm \sqrt{b/2}\,(3bx + c)^2} = \mathrm{d}\xi\,. \tag{24}$$

Thus we can obtain an exact solution to Eq. (7) as follows by solving Eq. (24) directly,

$$u(\xi) = \frac{a}{3}I_2\frac{\mathrm{e}^{a\xi/3}}{1 \mp I_2\sqrt{b/2}\ \mathrm{e}^{a\xi/3}} - \frac{c}{3b}\,, \tag{25}$$

where $I_2$ is an arbitrary integration constant.

From Eq. (20) we can obtain

$$a = \pm\frac{3}{h}\sqrt{\frac{\beta^2 - 3\alpha\gamma}{6D\alpha}}\,, \quad \omega = \frac{l^2\lambda}{h} \mp 3Dh\sqrt{\frac{\beta^2 - 3\alpha\gamma}{6D\alpha}}\,.$$

Thus a more general analytical solution to Eq. (1) can be represented as

$$u(x, y, t) = \pm qI_2\frac{\mathrm{e}^{q\xi}}{1 \mp I_2\sqrt{p}\ \mathrm{e}^{q\xi}} - \frac{\beta}{3\alpha}\,, \tag{26}$$

where

$$p = \frac{\alpha}{2Dh^2}\,, \quad q = \frac{1}{h}\sqrt{\frac{\beta^2 - 3\alpha\gamma}{6D\alpha}}\,,$$

$$\xi = hx + ly - \Big(\frac{\lambda}{h}l^2 \mp Dh^2 q\Big)t\,.$$

Now let us discuss the result Eq. (26).

***Case 1*** If $p > 0$, i.e. $D\alpha > 0$, let $I_2 = \sqrt{1/p}$ ,

(i) If $q > 0$, i.e. $\beta^2 - 3\alpha\gamma > 0$, from Eq. (26) we have

$$u(x, y, t) = \pm q\sqrt{\frac{1}{p}}\,\frac{\mathrm{e}^{q\xi}}{1 \mp \mathrm{e}^{q\xi}} - \frac{\beta}{3\alpha}\,. \tag{27}$$

Making use of the equalities

$$\frac{\mathrm{e}^{\eta}}{1 + \mathrm{e}^{\eta}} = \frac{1}{2}\Big[\tanh\Big(\frac{\eta}{2}\Big) + 1\Big]\,,$$

$$\frac{\mathrm{e}^{\eta}}{1 - \mathrm{e}^{\eta}} = \frac{-1}{2}\Big[\coth\Big(\frac{\eta}{2}\Big) + 1\Big]\,,$$

we get the following solitary wave solution of Eq. (1),

$$u_1(x, y, t) = \pm\frac{q}{2}\sqrt{\frac{1}{p}}\,\Big[\tanh\Big(\frac{q}{2}\xi\Big) + 1\Big] - \frac{\beta}{3\alpha}\,, \tag{28}$$

$$u_2(x, y, t) = \mp\frac{q}{2}\sqrt{\frac{1}{p}}\,\Big[\coth\Big(\frac{q}{2}\xi\Big) + 1\Big] - \frac{\beta}{3\alpha}\,, \tag{29}$$

where $p$, $q$, and $\xi$ are given in Eq. (26).

(ii) If $q < 0$, i.e. $\beta^2 - 3\alpha\gamma < 0$, from Eq. (26) we have

$$u(x, y, t) = \pm\mathrm{i}q^*\sqrt{\frac{1}{p}}\,\frac{\mathrm{e}^{\mathrm{i}q^*\xi}}{1 \mp \mathrm{e}^{\mathrm{i}q^*\xi}} - \frac{\beta}{3\alpha}\,. \tag{30}$$

Similarly, using relationship

$$\frac{\mathrm{e}^{\mathrm{i}\eta}}{1 + \mathrm{e}^{\mathrm{i}\eta}} = \frac{1}{2} - \mathrm{i}\Big(\sin\eta - \frac{1}{2}\tanh\frac{\eta}{2}\Big)\,,$$

$$\frac{\mathrm{e}^{\mathrm{i}\eta}}{1 - \mathrm{e}^{\mathrm{i}\eta}} = -\frac{1}{2} + \frac{\mathrm{i}}{2}\coth\frac{\eta}{2}\,,$$

we obtain that equation (1) has the following formal solitary wave solution:

$$u_3(x, y, t) = \pm\frac{q^*}{2}\sqrt{\frac{1}{p}}\,\Big[2\sin(q^*\xi) - \tanh\Big(\frac{q^*}{2}\xi\Big) + \mathrm{i}\Big] - \frac{\beta}{3\alpha}\,, \tag{31}$$

$$u_4(x, y, t) = \mp\frac{q^*}{2}\sqrt{\frac{1}{p}}\,\Big[\coth\Big(\frac{q^*}{2}\xi\Big) + \mathrm{i}\Big] - \frac{\beta}{3\alpha}\,, \tag{32}$$

where $q^* = \sqrt{(3\alpha\gamma - \beta^2)/(6D\alpha)}$, and $p$ and $\xi$ are given by Eq. (26)

***Case 2*** If $p < 0$, i.e. $\alpha D < 0$, let $I_2 = \sqrt{-1/p}$.

(i) If $q > 0$, i.e. $\beta^2 - 3\alpha\gamma > 0$, from Eq. (26) we have

$$u(x, y, t) = \pm q\sqrt{\frac{-1}{p}}\,\frac{\mathrm{e}^{q\xi}}{1 \mp \mathrm{i}\,\mathrm{e}^{q\xi}} - \frac{\beta}{3\alpha}\,. \tag{33}$$

Using relationship

$$\frac{\mathrm{e}^{\eta}}{1 \pm \mathrm{i}\,\mathrm{e}^{\eta}} = \frac{1}{2}[\mathrm{sech}\ \eta \mp \mathrm{i}(\tanh\eta + 1)]\,,$$

we also have the following formal solitary wave solution to Eq. (1),

$$u_5(x, y, t) = \pm\frac{q}{2}\sqrt{\frac{-1}{p}}\,[\mathrm{sech}\,(q\xi) - \mathrm{i}\tanh(q\xi) - \mathrm{i}] - \frac{\beta}{3\alpha}\,, \tag{34}$$

$$u_6(x, y, t) = \pm\frac{q}{2}\sqrt{\frac{-1}{p}}\,[\mathrm{sech}\,(q\xi) + \mathrm{i}\tanh(q\xi) + \mathrm{i}] - \frac{\beta}{3\alpha}\,, \tag{35}$$

where $p$, $q$, and $\xi$ are given by Eq. (26).

(ii) If $q < 0$, i.e. $\beta^2 - 3\alpha\gamma < 0$, from Eq. (26) we have

$$u(x,y,t) = \pm \mathrm{i}q^* \sqrt{\frac{-1}{p}}\,\frac{\mathrm{e}^{\mathrm{i}q^*\xi}}{1 \mp \mathrm{i}\,\mathrm{e}^{\mathrm{i}q^*\xi}} - \frac{\beta}{3\alpha}\,. \tag{36}$$

Using relationship

$$\frac{\mathrm{e}^{\mathrm{i}\eta}}{1+\mathrm{i}\,\mathrm{e}^{\mathrm{i}\eta}} = -\frac{1}{2} - \frac{\mathrm{i}}{2} - \Big(\tanh\frac{\eta}{2} - 1\Big)^{-1}, \qquad \frac{\mathrm{e}^{\mathrm{i}\eta}}{1-\mathrm{i}\,\mathrm{e}^{\mathrm{i}\eta}} = -\frac{1}{2} + \frac{\mathrm{i}}{2} + \Big(\tanh\frac{\eta}{2} + 1\Big)^{-1},$$

equation (1) has the following formal solitary wave solution

$$u_7(x,y,t) = \pm\frac{q^*}{2}\sqrt{\frac{-1}{p}}\,\Big[1 - \mathrm{i} - 2\mathrm{i}\Big(\tanh\Big(\frac{q^*}{2}\xi\Big) - 1\Big)^{-1}\Big] - \frac{\beta}{3\alpha}\,, \tag{37}$$

$$u_8(x,y,t) = \pm\frac{q^*}{2}\sqrt{\frac{-1}{p}}\,\Big[-1 - \mathrm{i} + 2\mathrm{i}\Big(\tanh\Big(\frac{q^*}{2}\xi\Big) + 1\Big)^{-1}\Big] - \frac{\beta}{3\alpha}\,, \tag{38}$$

where $q^* = \sqrt{(3\alpha\gamma - \beta^2)/(6D\alpha)}$, $p$ and $\xi$ are given in Eq. (26).

In the case of $\deg\alpha(x) = 1$, the argument is identical, so we omit it.

If assuming $m = 3, 4$ in Eq. (9), respectively, using the similar arguments as earlier we obtain that equation (8) does not have any first integral in the form (9). We have no need of discussion for the cases $m \geq 5$ due to the fact that in general the polynomial equation with the degree greater than or equal to 5 is not solvable.

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