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Whittaker Order Reduction Method of Relativistic Birkhoffian Systems*

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Abstract *The order reduction method of the relativistic Birkhoffian equations is studied. For a relativistic autonomous Birkhoffian system, if the conservative law of the Birkhoffian holds, the conservative quantity can be called the generalized energy integral. Through the generalized energy integral, the order of the system can be reduced. If the relativistic Birkhoffian system has a generalized energy integral, then the Birkhoffian equations can be reduced by at least two degrees and the Birkhoffian form can be kept. The relations among the relativistic Birkhoffian mechanics, the relativistic Hamiltonian mechanics and the relativistic Lagrangian mechanics are discussed, and the Whittaker order reduction method of the relativistic Lagrangian system is obtained. And an example is given to illustrate the application of the result.*

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Key words: relativity, Birkhoffian system, Lagrangian system, energy integral, order reduction method

1 Introduction

Relativistic analytical mechanics is an important aspect with the modern development in the field of theoretical physics. Since 1987, we constructed the theory of analytical mechanics for the relativistic system, and gave its basic theoretical frame.[1−9] In 1927, G.D. Birkhoff made primary researches on Birkhoffian dynamics.[10] In 1983, R.M. Santilli studied the transformation theory of Birkhoffian equations and generalization of Galilei relativity, and summarized comprehensively the origin of Birkhoffian equations and the later studies on them.[11] Later, MEI Feng-Xiang gave the Birkhoffian equations of nonholonomic constrained systems, and studied the Noether theory, the Lie theory, stability of motion, and geometrical description of Birkhoffian systems; moreover, he constructed the theoretical frame of Birkhoffian dynamics.[12−22] The Birkhoffian dynamics is more general than the Hamiltonian dynamics. The Hamiltonian dynamics has been extensively applied in the field of modern physics, so the Birkhoffian dynamics should play an important role in the field of modern physics. Recently, we constructed the basic theory of relativistic Birkhoffian dynamics, and gave its basic theoretical frame.[23−27]

The integral method of dynamical equations of complicated system is an important aspect with the modern development of the field of mathematical mechanics and mathematical physics. In 1904, Whittaker initiated the order reduction method for a holonomic conservative system by using the energy integral.[28] After that, this method was always paid attention by mathematicians, mechanicians, and physicists. Since 1984, the Whittaker order reduction method has been generalized to classical mechanics system, Vacco dynamical system, the variable mass system, the noninertial reference frame, and the Birkhoffian system in Refs. [29] $\sim$ [36]. But, the studies were confined to the classical mechanics system.

This paper studies the order reduction method of the relativistic Birkhoffian system and the relativistic Lagrangian system. For a relativistic autonomous Birkhoffian system, if the conservative law of the Birkhoffian holds, the conservative quantity can be called the generalized energy integral. Through the generalized energy integral, the order of the system can be reduced. If the relativistic Birkhoffian system has a generalized energy integral, then the Birkhoffian equations can be reduced by at least two degrees and the Birkhoffian form can be kept. The relations among the relativistic Birkhoffian system, the relativistic Hamiltonian system, and the relativistic Lagrangian system are discussed, and the order reduction method of the relativistic Lagrangian system is obtained. Also, an example is given to illustrate the application of the results.

2 Generalized Energy Integral of Relativistic Birkhoffian Systems

Here we will consider a mechanical system of N particles. At time t, the i-th particle's velocity is $\dot{\boldsymbol{r}}_i$, the

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limiting velocity is c, the classical mass is m_{oi}, and its relativistic mass is

$$m_i = \frac{m_{oi}}{\sqrt{1 - \dot{\boldsymbol{r}}_i^2/c^2}} \qquad (i = 1, \ldots, N)\,. \tag{1}$$

We can construct the Birkhoffian B^* and the Birkhoff's functions R_ν^* ($\nu = 1, \ldots, 2n$) of a relativistic system as

$$\begin{aligned} &B^* = B^*(m_i(t, a^\mu), t, a^\mu)\,, \\ &R_\nu^* = R_\nu^*(m_i(t, a^\mu), t, a^\mu) \quad (\nu, \mu = 1, \ldots, 2n)\,, \end{aligned} \tag{2}$$

and let

$$\begin{aligned} &\tilde{B}^* = \tilde{B}^*(t, a^\mu) = B^*(m_i(t, a^\mu), t, a^\mu)\,, \\ &\tilde{R}_\nu^* = \tilde{R}_\nu^*(t, a^\nu) = R_\nu^*(m_i(t, a^\mu), t, a^\mu)\,. \end{aligned} \tag{3}$$

For an ideal holonomic or free relativistic system, the Birkhoffian equations of the system can be written in the form[26]

$$\Big(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu}\Big)\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial t} = 0 \quad (\nu, \mu = 1, \ldots, 2n)\,, \tag{4}$$

where the repeated subscripts represent the summation. In general, it is supposed that the system (4) is nonsingular, i.e.

$$\det(\omega_{\mu\nu}^*) \neq 0, \quad \omega_{\mu\nu}^* = \Big(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu}\Big)\,. \tag{5}$$

If $\tilde{R}_\mu^*$ and $\tilde{B}^*$ do not include time t explicitly, the system (4) is autonomous. If $\tilde{R}_\mu^*$ do not include time t explicitly, the system (4) is semi-autonomous.

Corollary 1 For a relativistic autonomous Birkhoffian system, the conservative law of the Birkhoffian holds, i.e.

$$\tilde{B}^*(a^\mu) = B^*(m_i(a^\mu), a^\mu) = h = \text{const.} \quad (\mu = 1, \cdots, 2n)\,. \tag{6}$$

We call Eq. (6) the generalized energy integral of the system.

In fact, for the relativistic autonomous Birkhoffian system, equations (4) have the simple form as

$$\Big(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu}\Big)\dot{a}^\nu - \frac{\partial \tilde{B}^*}{\partial a^\mu} = 0 \quad (\nu, \mu = 1, \ldots, 2n)\,. \tag{7}$$

Multiplying the k-th equation by $\dot{a}^k (k = 1, \ldots, 2n)$, and adding all these equations up, we can obtain

$$-\Big(\frac{\partial \tilde{B}^*}{\partial a^1}\dot{a}^1 + \frac{\partial \tilde{B}^*}{\partial a^2}\dot{a}^2 + \cdots + \frac{\partial \tilde{B}^*}{\partial a^{2n}}\dot{a}^{2n}\Big) = 0\,, \tag{8}$$

that is, $\mathrm{d}\tilde{B}^*/\mathrm{d}t = 0$ or $\tilde{B}^*(a^1, a^2, \ldots, a^{2n}) = \text{const.}$

3 Order Reduction Method of Relativistic Birkhoffian Systems

Using the generalized energy integral, we now reduce the order of Eq. (7).

Generally, we assume

$$\frac{\partial \tilde{B}^*}{\partial a^\mu} \neq 0\,. \tag{9}$$

From Eq. (6), we get

$$a^\mu = K(a^\nu, h) \quad (\nu = 1, \ldots, 2n; \quad \nu \neq \mu)\,, \tag{10}$$

$$\frac{\partial K}{\partial a^\nu} = -\frac{\partial \tilde{B}^*}{\partial a^\nu}\Big(\frac{\partial \tilde{B}^*}{\partial a^\mu}\Big)^{-1}\,. \tag{11}$$

Letting

$$\begin{aligned} &a'^2 = \frac{\mathrm{d}a^2}{\mathrm{d}a^1} = \frac{\dot{a}^2}{\dot{a}^1}, \ldots\,, \\ &a'^\nu = \frac{\mathrm{d}a^\nu}{\mathrm{d}a^1} = \frac{\dot{a}^\nu}{\dot{a}^1}, \ldots\,, \\ &a'^{2n} = \frac{\mathrm{d}a^{2n}}{\mathrm{d}a^1} = \frac{\dot{a}^{2n}}{\dot{a}^1}\,. \end{aligned} \tag{12}$$

From the μ-th equation of the relativistic Birkhoffian system (7), we obtain

$$\frac{1}{\dot{a}^1} = \Big[\Big(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu}\Big)\dot{a}'^\nu + \Big(\frac{\partial \tilde{R}_1^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^1}\Big)\Big]\Big(\frac{\partial \tilde{B}^*}{\partial a^\mu}\Big)^{-1} \quad (\nu = 2, \ldots, 2n)\,. \tag{13}$$

Substituting Eq. (13) into equations (7) (except the first and μ-th equations), we get

$$\begin{aligned} &\Big[\frac{\partial \tilde{R}_\nu^*}{\partial a^\rho} - \frac{\partial \tilde{R}_\rho^*}{\partial a^\nu} + \frac{\partial K}{\partial a^\rho}\Big(\frac{\partial \tilde{R}_\nu^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^\nu}\Big)\Big]a'^\nu \\ &+ \Big(\frac{\partial \tilde{R}_\mu^*}{\partial a^\rho} - \frac{\partial \tilde{R}_\rho^*}{\partial a^\nu}\Big)a'^\mu + \frac{\partial K}{\partial a^\rho}\Big(\frac{\partial \tilde{R}_1^*}{\partial a^\mu} - \frac{\partial \tilde{R}_\mu^*}{\partial a^1}\Big) \\ &+ \Big(\frac{\partial \tilde{R}_1^*}{\partial a^\rho} - \frac{\partial \tilde{R}_\rho^*}{\partial a^1}\Big) = 0 \\ &(\nu = 2, \ldots, 2n;\ \nu \neq \mu;\ \rho \neq 1, \mu)\,. \end{aligned} \tag{14}$$

Considering

$$a'^\mu = \frac{\partial K}{\partial a^\nu}a'^\nu + \frac{\partial K}{\partial a^1} \quad (\nu = 2, \cdots, 2n;\ \nu \neq \mu)\,, \tag{15}$$

and making

$$\begin{aligned} &\tilde{R}_\nu^{*\prime}(a^1, \ldots, a^{\mu-1}, a^{\mu+1}, \ldots, a^{2n}) \\ &= \tilde{R}_\nu^*(a^1, \ldots, a^{\mu-1}, K, a^{\mu+1}, \ldots, a^{2n}) \\ &(\nu = 1, \ldots, 2n)\,, \end{aligned} \tag{16}$$

we have

$$\begin{aligned} &\Big[\Big(\frac{\partial \tilde{R}_\nu^{*\prime}}{\partial a^\rho} + \frac{\partial \tilde{R}_\mu^{*\prime}}{\partial a^\rho}\frac{\partial K}{\partial a^\nu}\Big) - \Big(\frac{\partial \tilde{R}_\rho^{*\prime}}{\partial a^\nu} + \frac{\partial \tilde{R}_\mu^{*\prime}}{\partial a^\nu}\frac{\partial K}{\partial a^\rho}\Big)\Big]a'^\nu \\ &+\Big(\frac{\partial \tilde{R}_1^{*\prime}}{\partial a^\rho} + \frac{\partial \tilde{R}_\mu^{*\prime}}{\partial a^\rho}\frac{\partial K}{\partial a^1}\Big) - \Big(\frac{\partial \tilde{R}_\rho^{*\prime}}{\partial a^1} + \frac{\partial \tilde{R}_\mu^{*\prime}}{\partial a^1}\frac{\partial K}{\partial a^\rho}\Big) = 0\,. \end{aligned} \tag{17}$$

Let

$$\frac{\partial \tilde{\tilde{R}}_1^*}{\partial a^\rho} = -\Big(\frac{\partial \tilde{R}_\nu^{*\prime}}{\partial a^\rho} + \frac{\partial \tilde{R}_\mu^{*\prime}}{\partial a^\rho}\frac{\partial K}{\partial a^\nu}\Big) \quad (\nu = 2, \ldots, 2n; \rho = 1, \ldots, 2n; \nu \neq \mu)\,, \tag{18}$$

$$\frac{\partial \tilde{\tilde{R}}_1^{*\prime}}{\partial a^\rho} = -\Big(\frac{\partial \tilde{R}_1^{*\prime}}{\partial a^\rho} + \frac{\partial \tilde{R}_\mu^{*\prime}}{\partial a^\rho}\frac{\partial K}{\partial a^1}\Big)$$
$$(\rho = 2, \ldots, 2n; \nu, \rho \neq \mu)\,, \tag{19}$$

finally, we obtain

$$\Big(\frac{\partial \tilde{\tilde{R}}_\nu^*}{\partial a^\rho} - \frac{\partial \tilde{\tilde{R}}_\rho^*}{\partial a^\nu}\Big)a'^\nu - \Big(\frac{\partial \tilde{\tilde{R}}_1^*}{\partial a^\rho} + \frac{\partial \tilde{\tilde{R}}_\rho^*}{\partial a^1}\Big) = 0$$
$$(\nu, \rho = 2, \ldots, 2n; \quad \nu, \rho \neq \mu)\,. \tag{20}$$

Through the above procedure, equations (7) are reduced by two orders while the Birkhoffian form is still kept. After solving Eqs. (20), if equations (6) and (13) are supplanted, we can obtain the solution of the original relativistic Birkhoffian system (7). So we have

Corollary 2 If a relativistic Birkhoffian system has a generalized energy integral, then the Birkhoffian equations can be reduced by at least two degrees and keep Birkhoffian form.

4 Whittaker Order Reduction Method of Relativistic Lagrangian Systems

Now, we discuss the relations among the relativistic Birkhoffian system, the relativistic Hamiltonian system and the relativistic Lagrangian system, and obtain the order reduction method of the relativistic Lagrangian system.

If $\tilde{R}_\mu^*$ does not include time t explicitly, for the relativistic Birkhoffian system (4), we let

$$a^\mu = \begin{cases} q_\mu & (\mu = 1, \ldots, n)\,, \\ p_{\mu-n} & (\mu = n+1, \ldots, 2n)\,, \end{cases}$$
$$\tilde{R}_\mu^* = \begin{cases} p_\mu & (\mu = 1, \ldots, n)\,, \\ 0 & (\mu = n+1, \ldots, 2n)\,, \end{cases} \tag{21}$$
$$\tilde{B}^* = H^*\,,$$

then the Birkhoffian tensors of the relativistic system are

$$(\tilde{\omega}_{\mu\nu}^*) = \begin{bmatrix} 0_{n\times n} & +1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{bmatrix}. \tag{22}$$

We can obtain immediately the canonical equations of the relativistic Hamiltonian system

$$\dot{q}_s = \frac{\partial H^*}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H^*}{\partial q_s} \quad (s = 1, \ldots, n)\,, \tag{23}$$

and the relativistic Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L^*}{\partial \dot{q}_s} - \frac{\partial L^*}{\partial q_s} = 0 \quad (s = 1, \ldots, n)\,, \tag{24}$$

where

$$H^*(t, q_s, p_s) = p_s\dot{q}_s - L^*, \quad p_s = \frac{\partial L^*}{\partial \dot{q}_s}\,,$$
$$L^*(t, q_s, \dot{q}_s) = T^* - V\,,$$
$$T^* = \sum_{i=1}^{N} m_{oi}c^2(1 - \sqrt{1 - \dot{\boldsymbol{r}}_i^2/c^2}) \quad (s = 1, \ldots, n)\,. \tag{25}$$

Therefore, this paper gives the generalized energy integral and reduction of the relativistic Lagrangian system.

If L^* does not include time t, then the system (24) has the relativistic energy integral

$$\frac{\partial L^*}{\partial \dot{q}_s}\dot{q}_s - L^* = h = \text{const.} \tag{26}$$

We let

$$\dot{q}_r = \dot{q}_1 q_r', \quad q_r' = \frac{\mathrm{d}q_r}{\mathrm{d}q_1} \quad (r = 2, \ldots, n)\,, \tag{27}$$
$$\Omega(q\prime_r, \dot{q}_1, q_s) = L^*(\dot{q}_s, q_s)\,. \tag{28}$$

Substituting Eqs. (27) into the energy integral (26), we can obtain

$$\dot{q}_1 = \dot{q}_1(q_r', q_s) \quad (r = 2, \ldots, n; \quad s = 1, \ldots, n)\,. \tag{29}$$

By using Eqs. (28) and (29), we define the relativistic Whittaker function

$$W(q_r', q_s) \equiv \frac{\partial \Omega}{\partial \dot{q}_1}\,. \tag{30}$$

From Eqs. (26) $\sim$ (30), we can obtain

$$\frac{\partial W}{\partial q\prime_r} = \frac{\partial L^*}{\partial \dot{q}_r}, \quad \frac{\partial W}{\partial q_s} = \frac{1}{\dot{q}_1}\frac{\partial L^*}{\partial q_s}$$
$$(r = 2, \ldots, n; \quad s = 1, \cdots, n)\,. \tag{31}$$

Substituting Eqs. (31) into the last $(n-1)$ equations of Eqs. (24), we have

$$\frac{\mathrm{d}}{\mathrm{d}q_1}\frac{\partial W}{\partial q_r'} - \frac{\partial W}{\partial q_r} = 0 \quad (r = 2, \ldots, n)\,. \tag{32}$$

Corollary 3 If a relativistic Lagrangian system has a generalized energy integral, then the order of Lagrangian equaions can be reduced and keep Lagrangian form.

5 Examples

As an example, we study a relativistic system of three particles, in which the Birkhoffian is

$$\tilde{B}^* = [m_{o1}(a^1)^2 + m_{o2}(a^2)^2 + m_{o3}(a^3)^2$$
$$- (m_{o1} + m_{o2} + m_{o3})]c^2\,, \tag{33}$$

and Birkhoffian's functions are

$$\tilde{R}_1^* = \tilde{R}_2^* = 0, \quad \tilde{R}_3^* = 2m_{o1}a^1c^2\,,$$
$$\tilde{R}_4^* = 2m_{o2}a^2c^2\,, \tag{34}$$

where

$$a^1 = (1 - \dot{q}_1^2/c^2)^{-1/4}, \quad a^2 = (1 - \dot{q}_2^2/c^2)^{-1/4},$$
$$a^3 = (1 - \dot{q}_3^2/c^2)^{-1/4}\,. \tag{35}$$

Birkhoffian equations are

$$\dot{a}^3 - a^1 = 0, \qquad \dot{a}^4 - a^2 = 0\,,$$
$$m_{o1}\dot{a}^1 + m_{o3}a^3 = 0, \qquad \dot{a}^2 = 0\,. \tag{36}$$

For $\tilde{B}^*$, $\tilde{R}_1^*$, $\tilde{R}_2^*$, $\tilde{R}_3^*$, and $\tilde{R}_4^*$ do not include time t apparently, they satisfy corollary 1 and the system has a generalized energy integral

$$\tilde{B}^* = [m_{o1}(a^1)^2 + m_{o2}(a^2)^2 + m_{o3}(a^3)^2 - (m_{o1} + m_{o2} + m_{o3})]c^2 = h = \text{const.} \tag{37}$$

So according to corollary 2, the relativistic Birkhoffian system can be reduced in orders. From the reduced equations

$$\Big(\frac{\partial \tilde{\tilde{R}}_\nu^*}{\partial a^\rho} - \frac{\partial \tilde{\tilde{R}}_\rho^*}{\partial a^\nu}\Big) a'^\nu - \Big(\frac{\partial \tilde{\tilde{R}}_1^*}{\partial a^\rho} - \frac{\partial \tilde{\tilde{R}}_\rho^*}{\partial a^1}\Big) = 0 \quad (\nu, \rho = 2, 4)\,, \tag{38}$$

we obtain

$$a'^2 = 0\,,$$
$$a'^4 = m_{o1}c^2 a^2[(h - m_{o1}c^2(a^1)^2 - m_{o2}c^2(a^2)^2 + (m_{o1} + m_{o2} + m_{o3})c^2)m_{o3}c^2]^{-1/2}\,. \tag{39}$$

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