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Generating Lie Point Symmetry Groups of (2+1)-Dimensional Broer–Kaup Equation via a Simple Direct Method*

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Abstract *Using the (2+1)-dimensional Broer–Kaup equation as an simple example, a new direct method is developed to find symmetry groups and symmetry algebras and then exact solutions of nonlinear mathematical physical equations.*

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Key words: symmetry groups, CK direct method, exact solution, symmetries

Over the past decades, Lie group method has been used as a powerful tool in mathematical physics[1−3] since Sophus Lie[4−7] introduced it at the end of the 19th century. Sophus Lie's method, the classical Lie method, is an algorithmic procedure to find the symmetry group and reduction of a PDE. Bluman and Cole[8] introduced the nonclassical method, which allows us to find the conditional symmetries associated with a PDE. Any classical symmetry is a nonclassical symmetry but it is not true conversely. Another way to find symmetry reductions is the direct method introduced by Clarkson and Kruskal.[9] Many authors have applied it for kinds of models.[10,11] These two methods have been studied in detail by Olver.[12] In the last century, Noether[13] found that Lie's method could be generalized by adding the derivatives of the dependent variables as well as the independent and dependent variables to the transformations. All of these methods for finding symmetries and associated reductions of a given PDE is the use of group theory.

The direct method to find similarity reductions is a very simple method that does not use group theory. The main idea is to seek a reduction of a given PDE in the form

$$\begin{aligned} u(x_1,\ x_2,\ \ldots,\ x_n) \\ &= W(x_1,\ x_2,\ \ldots,\ x_n,\ U(z_1,\ z_2,\ \ldots,\ z_m))\,, \qquad (1)\\ &(z_i = z_i(x_1,\ x_2,\ \ldots,\ x_n),\ i=1,\ 2,\ \ldots,\ m<n)\,, \end{aligned}$$

which is the most general form for a similarity reduction.[14] For a given PDE,

$$F(x_i,\ u,\ u_{x_i},\ u_{x_ix_j},\ \ldots,\ i,\ j=1,\ 2,\ \ldots\ n)=0\,, \qquad (2)$$

substituting Eq. (1) into Eq. (2) and demanding that the result be a lower-dimensional partial (or ordinary) differential equation for $U(z_1,\ z_2,\ \ldots,\ z_m)$ imposes conditions upon W, z_i and their derivatives that enable one to solve for W and z_i.

For many real physical systems, it is enough to seek the symmetry reductions in a simple form

$$\begin{aligned} u(x_1,x_2,\ldots,x_n) &= \alpha(x_1,x_2,\ldots,x_n)+\beta(x_1,x_2,\ldots,x_n)\\ &\quad\times U(z_1,z_2,\ldots,z_m) \end{aligned} \qquad (3)$$

instead of the general form (1).

The traditional direct method is used to find group invariant solutions and then get the related similar reductions.[11] In this paper, we try to extend and modify the method to directly generate finite symmetry transformation group and symmetry algebra for the (2+1)-dimensional Broer–Kaup equation,

$$\begin{aligned} &u_t+2u_xv+2uv_x+u_{xx}=0\,,\\ &v_{yt}+2v_yv_x+2vv_{xy}+2u_{xx}-v_{xxy}=0\,, \end{aligned} \qquad (4)$$

which possesses symmetry group in the form (3), i.e.

$$\begin{aligned} u(x,\ y,\ t) &= \alpha_1+\beta_1 U(\xi,\ \eta,\ \tau)\,,\\ v(x,\ y,\ t) &= \alpha_2+\beta_2 V(\xi,\ \eta,\ \tau)\,, \end{aligned} \qquad (5)$$

where $\alpha_1 = \alpha_1(x,\ y,\ t)$, $\beta_1 = \beta_1(x,\ y,\ t)$, $\alpha_2 = \alpha_2(x,\ y,\ t)$, $\beta_2 = \beta_2(x,\ y,\ t)$, $\xi = \xi(x,\ y,\ t)$, $\eta = \eta(x,\ y,\ t)$, and $\tau = \tau(x,\ y,\ t)$ are functions of $\{x,\ y,\ t\}$ and should be determined by requiring $U(\xi,\ \eta,\ \tau)$ to satisfy the same (2+1)-dimensional system as $u = u(x,\ y,\ t)$ with the transformation $\{u,\ x,\ y,\ t\} \to \{U,\ \xi,\ \eta,\ \tau\}$.

The invariance properties of some of the integrable nonlinear evolution equations in (2+1)-dimensions, which are very important in mathematics and physics, have been studied through the classical Lie group approaches. For instance, the Kadomatsev–Petvishilli equation,[15] the Davey–Stewartson equation,[16] the three-wave interaction problem,[17] the cylindrical Kadomtsev–Petviashvilli equation,[18] the stimulated Raman scattering equation,[19] the Nizhnik–Novikov–Veselov equation, the breaking soliton equation, the sine-Gordon system,[20] and the long dispersive wave equation,[21,22] etc. The related symmetry groups can be obtained by means of

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the standard Lie group theory, however, the final known expressions of these types of symmetry groups are quite complicated and quite difficult for real applications especially for physicists and other non-mathematical scientists. In this short report, starting with the simple idea mentioned above, the transformation group and then the symmetry algebra of the Broer–Kaup equation are given in detail.

The Broer–Kaup (BK) equation (4) can be obtained from the symmetry reduction of KP equation.[23] It has rich local reaction structure and is widely used in statistic physics, plasma physics, and nonlinear fiber optic communication.

Now let u, v has the form (5). Substitute it into the first equation of Eq. (4) and restrict $U(\xi,\eta,\tau)$ and $V(\xi,\eta,\tau)$ to satisfy the same equation as Eq. (4),

$$\begin{aligned}V_{\eta\tau}(\xi,\eta,\tau) &= -2V_\xi(\xi,\eta,\tau)V_\eta(\xi,\eta,\tau)\\&\quad - 2V(\xi,\eta,\tau)V_{\xi\eta}(\xi,\eta,\tau)\\&\quad - 2U_{\xi\xi}(\xi,\eta,\tau) + V_{\xi\xi\eta}(\xi,\eta,\tau)\,,\\U_\tau(\xi,\eta,\tau) &= -2U_\xi(\xi,\eta,\tau)V(\xi,\eta,\tau)\\&\quad - 2U(\xi,\eta,\tau)V_\xi(\xi,\eta,\tau)\\&\quad - U_{\xi\xi}(\xi,\eta,\tau)\,.\end{aligned} \tag{6}$$

Ruling out U_τ and $V_{\eta\tau}$ we have

$$\begin{aligned}&\beta_1\tau_x^2U_{\xi^4} + \beta_1\eta_x^2U_{\eta^2} + \beta_1\eta_tU_\eta + 2\alpha_1\beta_2\xi_xV_\xi\\&\quad + F(x,\ y,\ t,\ U_\xi,\ldots) = 0\,,\end{aligned} \tag{7}$$

where $U_{\xi^n} = \partial_\xi^nU$, $U_{\eta^n} = \partial_\eta^nU$, and $F(x,\ y,\ t,\ U_\xi,\ldots)$ does not depend on U_{ξ^4}, U_{η^2}, U_η, V_ξ, which tells that

$$\tau_x = 0,\ \eta_x = 0,\ \eta_t = 0,\ \alpha_1 = 0\,. \tag{8}$$

And then equation (7) becomes

$$\begin{aligned}&((2\beta_{1x}\beta_2 + 2\beta_1\beta_{2x})V - 2\beta_1(\tau_t - \xi_x\beta_2)V_\xi\\&\quad + 2\beta_1\alpha_{2x} + \beta_{1t} + \beta_{1xx} + 2\beta_{1x}\alpha_2)U\\&\quad + (-2\beta_1(\tau_t - \xi_x\beta_2)V + 2\beta_{1x}\xi_x + \beta_1\xi_t + 2\beta_1\xi_x\alpha_2\\&\quad + \beta_1\xi_{xx})U - U_{\xi^2}\beta_1(\tau_t - \xi_x^2) = 0\,.\end{aligned} \tag{9}$$

In the same way, by substitution of Eq. (5) with Eqs. (6) and (8) one gets

$$\beta_2\tau_t\xi_yV_{\xi\tau} - \beta_2\tau_y\xi_x^2V_{\xi^2\tau} + F'(x,\ y,\ t,\ U_\xi,\ldots) = 0\,, \tag{10}$$

where $F'(x,\ y,\ t,\ U_\xi,\ldots)$ does not depend on $V_{\xi\tau}$ or $V_{\xi^2\tau}$, which means

$$\xi_y = 0, \qquad \tau_y = 0\,. \tag{11}$$

Comparing the coefficients of Eq. (9) and by using Eq. (8) we get

$$\begin{aligned}&\eta = \eta(y), \quad \tau = \tau(t), \quad \xi = \tau_t^{1/2}x + \xi_1(t)\,,\\&\beta_2 = \tau_t^{1/2}, \quad \beta_{1x} = 0\,.\end{aligned} \tag{12}$$

Under the help of Eq. (12), equation (9) turns to

$$\begin{aligned}&(\beta_{1t} + 2\beta_1\alpha_{2x})U\\&\quad + \beta_1\Big(2\tau_t^{1/2}\alpha_2 + \xi_{1t}(t) + \frac{1}{2}\frac{x\tau_{tt}}{\tau_t^{1/2}}\Big)U_\xi = 0\,,\end{aligned} \tag{13}$$

which tells

$$\begin{aligned}\alpha_2 &= -\frac{1}{4}\frac{x\tau_{tt} + 2\xi_{1t}(t)\tau_t^{1/2}}{\tau_t}\,,\\\beta_1 &= F_1(y)\tau_t^{1/2}\,.\end{aligned} \tag{14}$$

On the basis of the above, equation (10) becomes

$$-2\tau_t^{3/2}(\eta_y - F_1(y))U_{\xi^2} = 0\,,$$

which means

$$F_1(y) = \eta_y\,. \tag{15}$$

In summary, for the general Lie point symmetry group of the Broer–Kaup equation, we have the following theorem.

Theorem If $\{U = U(x,y,t),\ V = V(x,y,t)\}$ are solutions of the Broer–Kaup equation (4) then so is $\{u,\ v\}$, which is expressed by

$$\begin{aligned}u &= \eta_y\tau_t^{1/2}U(\xi,\eta,\tau)\,,\\v &= -\frac{1}{4}\frac{x\tau_{tt} + 2\xi_{1t}(t)\tau_t^{1/2}}{\tau_t}\tau_t^{1/2}V(\xi,\eta,\tau)\end{aligned}$$

with Eq. (12).

Whence the finite symmetry group is obtained, to find its related symmetry algebra is a straightforward quite work. For the Broer–Kaup equation, the Lie point symmetries have the forms:

$$\begin{aligned}\sigma &\equiv \sigma_1(f(t)) + \sigma_2(g(y)) + \sigma_3(h(t))\\&\equiv \begin{pmatrix} f(t)u_t + \frac{1}{2}\dot f(t)(xu_x + u)\\ f(t)v_t + \frac{1}{2}\dot f(t)(xv_x + v) - \frac{1}{4}x\ddot f(t)\end{pmatrix}\\&\quad + \begin{pmatrix} g(y)u_y + g_y(y)u\\ g(y)v_y\end{pmatrix} + \begin{pmatrix} h(t)u_x\\ h(t)v_x - \frac{1}{2}\dot h(t)\end{pmatrix}\,,\end{aligned}$$

which can be obtained from the Theorem by restricting

$$\begin{aligned}&\tau(t) = t + \epsilon f(t), \quad \xi_1(t) = \epsilon h(t)\,,\\&\eta(y) = \epsilon g(y)\,.\end{aligned}$$

It is very important and difficult work to find some types of localized excitations in high dimensions. By means of the theorem given in this paper, we can obtain many kinds of new exact solutions starting from some simple known solutions for the Broer–Kaup equation.

Obviously, many kinds of simple exact solutions of the (2+1)-dimensional Broer–Kaup equation (4) has been given by many authors, say, a variable separation solution,

$$\begin{aligned}u &= \frac{(c_3 - c_1c_2)p_xq_y}{(1 + c_1p + c_2q + c_3pq)^2}\,,\\v &= \frac{(c_1 - c_3q)p_x}{1 + c_1p + c_2q + c_3pq} - \frac{1}{2}\frac{p_t + p_{xx}}{p_x}\,,\end{aligned} \tag{16}$$

where $p = p(x,t)$, $q = q(y)$, and c_i, $(i = 1,2,3)$ are arbitrary constants, has been given in Ref. [24].

When we take a special form of $p(x,t)$ and $q(y)$ as $p(x,t) = \exp(2x+2t)$, $q(y) = \exp(2y)$, say, for the field u, we have a simple straight line soliton solution of Broer–Kaup equation,

$$u = \frac{4\exp(2x)\exp(2y)\exp(2t)(c_3 - c_1c_2)}{(1 + c_1\exp(2x)\exp(2t) + c_2\exp(2y) + c_3\exp(2x)\exp(2y)\exp(2t))^2}\,. \tag{17}$$

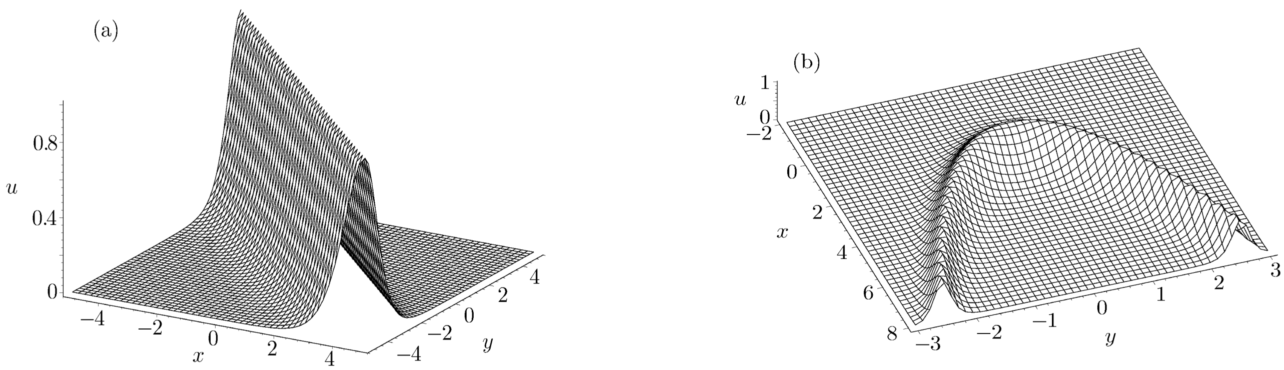

Fig. 1 A typical straight line solution to the (2+1)-dimensional Broer–Kaup equation at $t = 0$ (a) and its special curve shape group deformation at $t = -\pi/6$ (b).

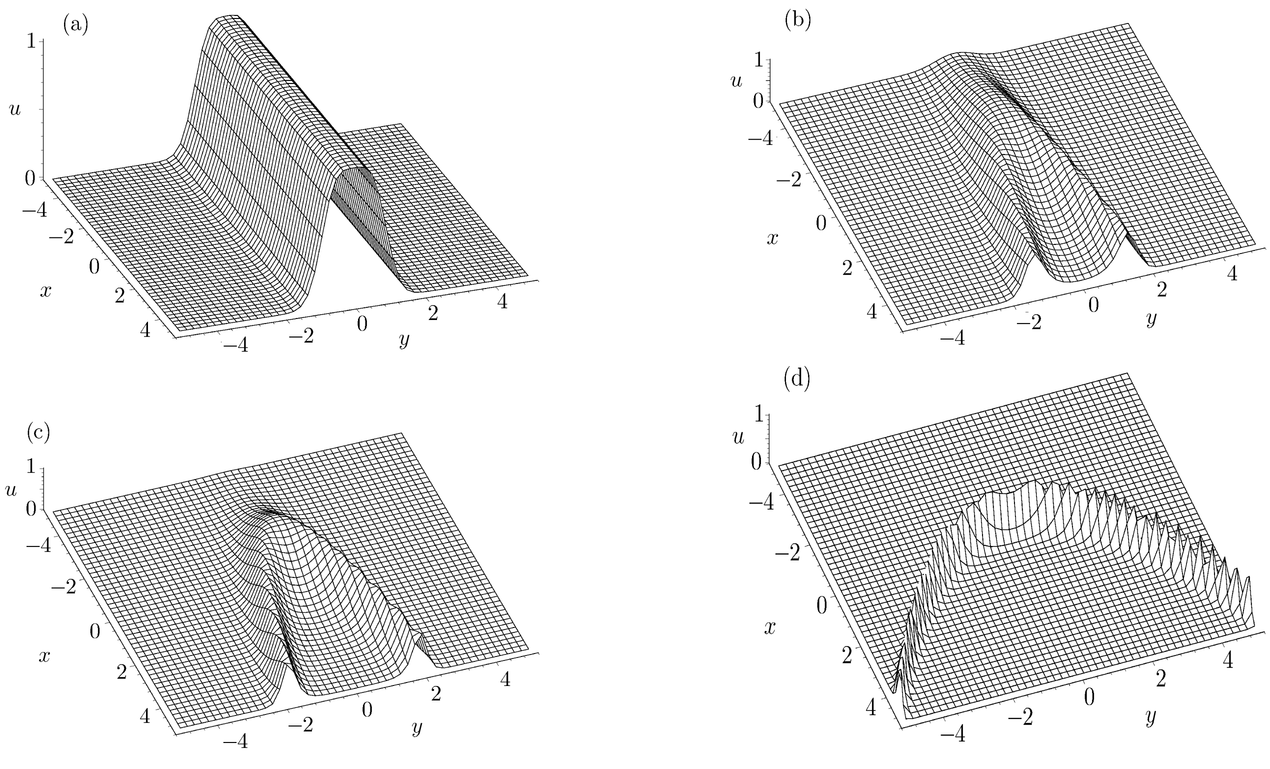

Fig. 2 Evolution of a straight line solution to a curve solution at $t = 0,\ 0.4,\ 0.5,\ 1$.

Figure 1(a) shows a typical structure of the single straight line soliton structure given by Eq. (17) with the parameter selections

$$c_1 = c_2 = 0, \qquad c_3 = 1$$

at time $t = 0$.

The application of the symmetry group theorem on the single straight line soliton (17) yields the group-invariant solution

$$u = \eta_y\tau_t^{1/2}\frac{\exp(2\xi)\exp(2\eta)\exp(2\tau)(c_3 - c_1c_2)}{(1 + c_1\exp(2\xi)\exp(2\tau) + c_2\exp(2\eta) + c_3\exp(2\xi)\exp(2\eta)\exp(2\tau))^2}$$

with the arbitrary function of $\eta(y)$ and $\tau(t)$ and $\xi = \tau_t^{1/2} x + \xi_1(t)$ for arbitrary $\xi_1(t)$.

Now by selecting the arbitrary functions, $\eta(y)$ and $\tau(t)$, introduced by the transformation group, one can obtain various types of solutions. In the following we just plot some special group deformations of the straight line soliton.

Figure 1(b) exhibits a plot of the curve soliton with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = (2\cos(t))^{1/2} x + \sin(t), \quad \eta = -y^2, \quad \tau = 2\sin(t)$$

at time $t = -\pi/6$.

Figure 2 displays evolution of a straight line solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = \sqrt{6} t x + \sin(t), \quad \eta = -y^2, \quad \tau = 2t^3 .$$

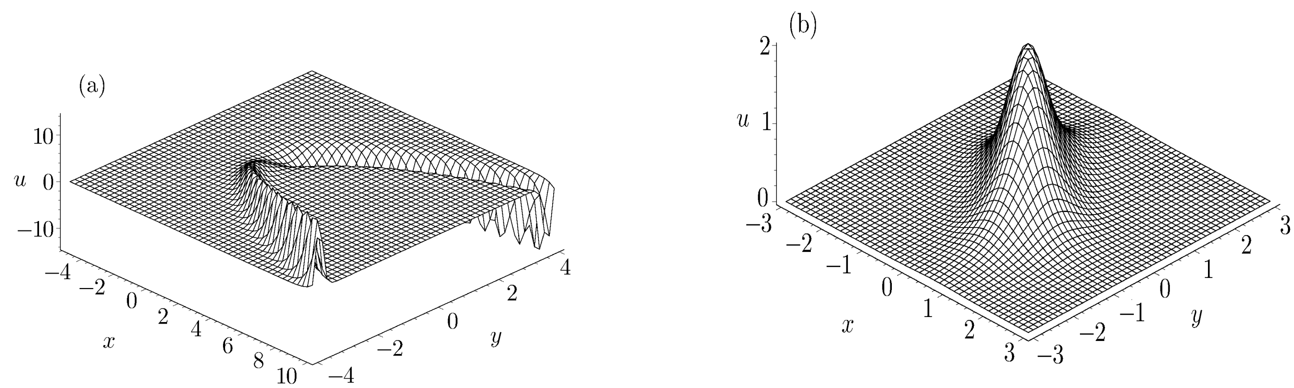

Fig. 3 The special curve solution (a) and the dromion solution (b) at $t = 0$.

Figure 3(a) displays the special curve solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = (2\cos(t))^{1/2} x + \sin(t), \quad \eta = -y^2, \quad \tau = 2\sin(t)$$

at time $t = 0$.

Figure 3(b) displays the dromion solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = \sqrt{2 - 2\tanh(t)^2} x, \quad \eta = \tanh(y), \quad \tau = 2\tanh(t)$$

at time $t = 0$.

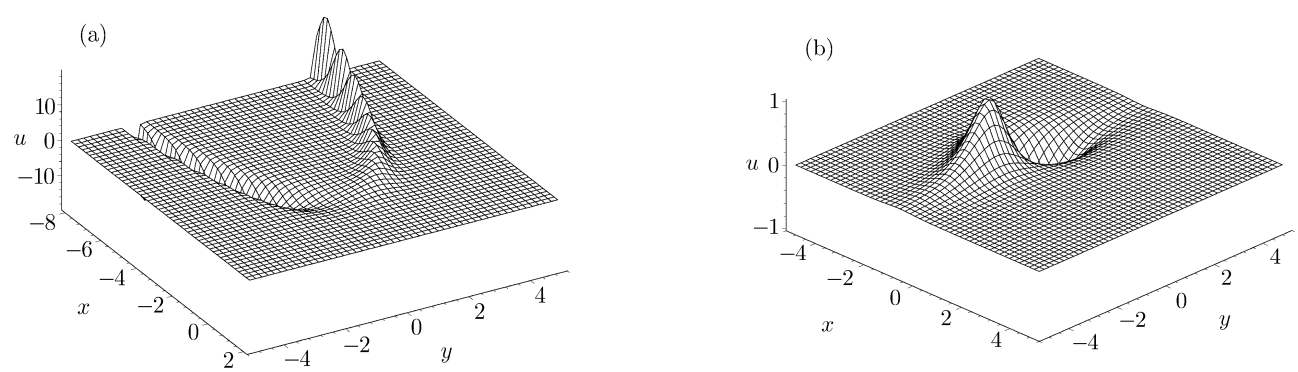

Fig. 4 The special curve solution at $t = -\pi/30$ (a) and the two dromion solution at $t = 0$ (b).

Figure 4(a) displays the special curve solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = (2\cos(t))^{1/2} x + \cos(t), \quad \eta = \cosh(y), \quad \tau = 2\sin(t)$$

at time $t = -\pi/30$.

Figure 4(b) displays the two-dromion solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = (2\cos(t))^{1/2} x + \exp(t), \quad \eta = \mathrm{sech}(y), \quad \tau = 2\sin(t)$$

at time $t = 0$.

Figure 5 displays the evolution of a kink solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = \sqrt{3} t x + \sin(t), \quad \eta = \mathrm{sech}(y), \quad \tau = 2\sin(t) .$$

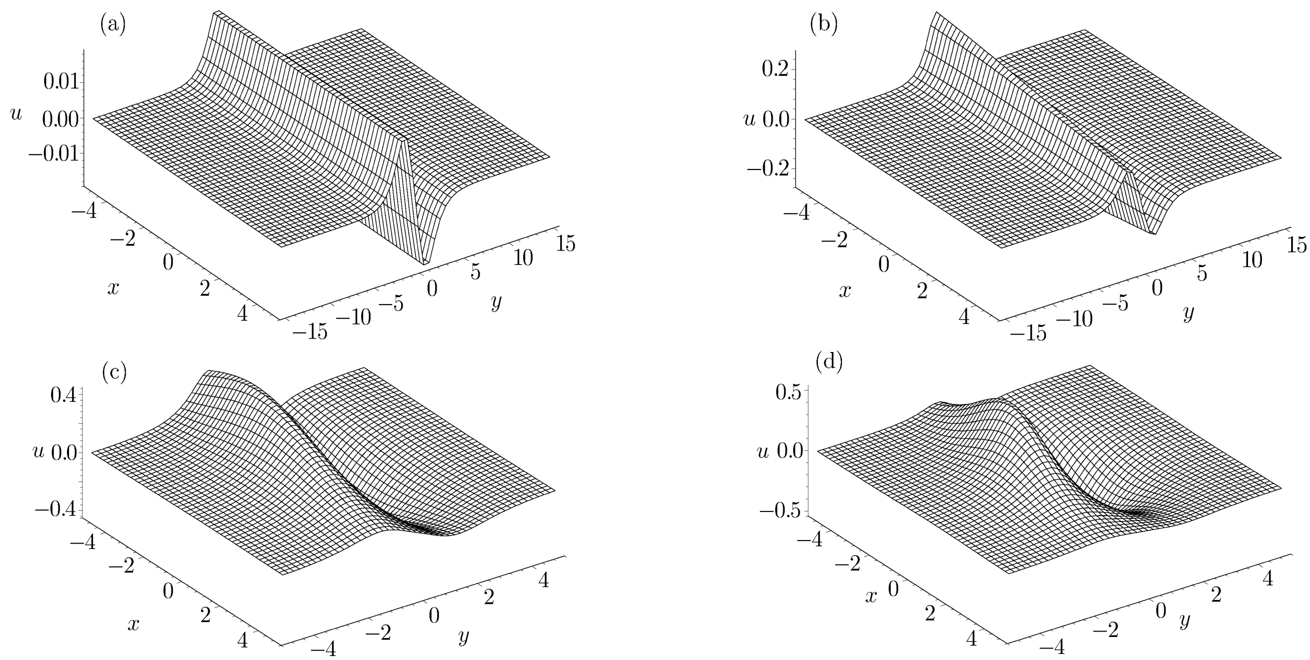

Fig. 5 Evolution of a kink solution to two-dromion solution at $t = -1/30,\ -1/3,\ -1/2,\ -0.6$.

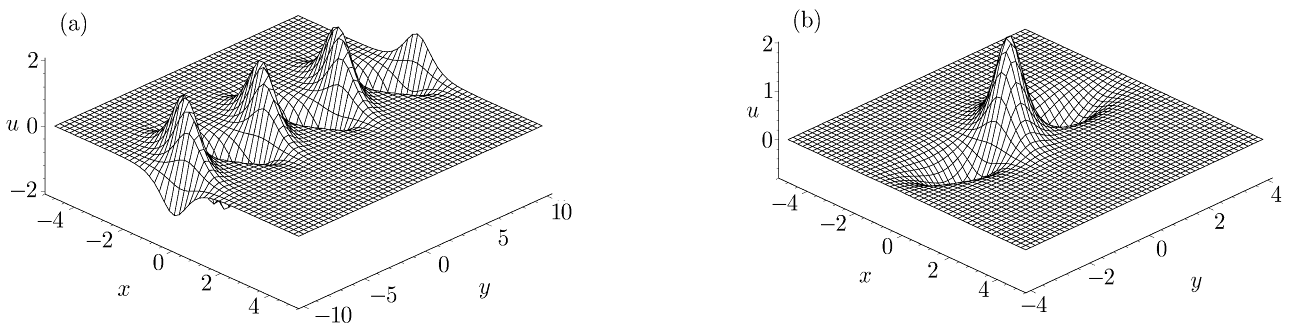

Fig. 6 The periodic solution (a) and the multi-dromion solution (b) at $t = 0$.

Figure 6(a) displays the periodic solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = (2\cos(t))^{1/2}x + \exp(t^3), \quad \eta = \cos(y), \quad \tau = 2\sin(t)$$

at time $t = 0$.

Figure 6(b) displays the multi-dromion solution with

$$c_1 = c_2 = 0, \quad c_3 = 1, \quad \xi = (2\cos(t))^{1/2}x + \exp(t^3), \quad \eta = \operatorname{sech}(y)\sin(y), \quad \tau = 2\sin(t)$$

at time $t = 0$.

In summary, taking the Broer–Kaup equation as a simple example, the direct method is developed to directly get the Lie point symmetry groups (finite invariant transformations) of nonlinear systems. To find the related symmetry algebras becomes only a simple limiting procedure. For the Lax integrable models, the invariant finite transformation groups can also be directly derived by a combined gauge and space-time transformation.[25] The method reported here can be applied in principle to all nonlinear systems including both integrable and non-integrable ones and more details will be reported in a future full paper.[26]

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