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Quantum Entropy of Black Hole with Internal Global Monopole*

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Abstract *Using the generalized uncertainty relation, the new equation of state density is obtained, and then the entropy of black hole with an internal global monopole is discussed. The divergence that appears in black hole entropy calculation through original brick-wall model is overcome. The result of the direct proportion between black hole entropy and its event horizon area is drawn and given. The result shows that the black hole entropy must be the entropy of quantum state near the event horizon.*

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The problem about the origin of black hole entropy has always been focused since Bekenstein put forward the direct proportion between black hole entropy and its horizon in the 1970s.[1,2] From then on, many ways of studying black hole entropy emerge. The brick-wall pattern of calculating black hole entropy proposed by 't Hooft is one of the successful patterns.[3] 't Hooft supposed the heat radiation under temperature T_H around a black hole, then calculated and got its entropy. In order to avoid divergence, he also supposed that the matter field would disappear in a short distance from black hole horizon. When the distance satisfies a certain condition, the statistical entropy of the corresponding heat field and Bekenstein–Hawking entropy are unanimous. However, it was discovered that the brick-wall pattern had its own limit. The limit mainly lies in that it is only suitable for the calculation of black hole entropy in a thermoequivalence state. For this reason, Zhao Zheng *et al.* put forward thin film brick-wall pattern,[4,5] which showed that black hole entropy is from quantum field in a thin layer near horizon. Practice proves that the way of thin film birck-wall pattern is not only suitable for stationary and static black hole entropy's calculation, but also successful for the study of black hole entropy, which is in a non-thermoequivalence state.[6–9] However, either brick-wall model or thin film brick-wall way, in studying black hole entropy's practice does not appear very natural about acceptance or rejection of some items in calculation course and the discussion of result.

Recently, the researchers of quantum gravity have advanced a revised uncertainty relation.[10] We would like to try to introduce the relation on the base of film brick-wall way and study the entropy with an internal global monopole. The result is satisfactory as expected. We can not only get the same conclusion as film brick-wall way, but also avoid the divergence of state density near event horizon.

The space-time line element[11] with an internal global monopole is

$$\mathrm{d}s^2 = \Big(1 - 8\pi\eta^2 G - \frac{2GM}{r}\Big)\mathrm{d}t^2 - \Big(1 - 8\pi\eta^2 G - \frac{2GM}{r}\Big)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \qquad (1)$$

where M is the mass of the black hole, η is an internal global monopole, G takes nature coordinate. According to the definition of null hypersurface, we can get the event horizon position of spacetime as

$$r_H = \frac{2M}{1 - 8\pi\eta^2}\,. \qquad (2)$$

In general quantum mechanics, when dual sides of wave motion particle are considered, position x and momentum p is a pair of visible measure which cannot be defined at the same time. Its indefinite degree $\Delta x\Delta p$ needs to meet Heisenberg uncertainty relation,

$$\Delta x\Delta p \geq \frac{\hbar}{2}\,, \qquad (3)$$

where $\hbar$ is Planck constant.

However, thinking the quantum system under the Planck scale like the system described by quantum gravitation, equation (3) is not correct. In order to describe this kind of quantum system accurately, we need an improved uncertainty relation,[12]

$$\Delta x\Delta p \geq \frac{\hbar}{2}[1 + \lambda\langle\hat{p}^2\rangle + \beta\langle\hat{x}^2\rangle + \cdots]\,. \qquad (4)$$

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In order to simplify Eq. (4), taking into consideration of condition which momentum plays a leading role, we can write $\beta = 0$, $\lambda \neq 0$, equation (4) can be substituted into

$$\Delta x \Delta p = \frac{\hbar}{2}[1 + \lambda\langle \hat{p}^2\rangle] = \frac{\hbar}{2}[1 + \lambda(\Delta p^2 + \langle \hat{p}^2\rangle)]\,. \tag{5}$$

In order to use it to divide quantum state of phase space, we can further divide the phase space consisting of coordinate and momentum into several energy layers $E \sim E + \mathrm{d}E$ and make every energy layer be a phase cell. Thus, every phase cell representing phase space can stand for a quantum state of single system. According to Eq. (5), we can write line degree of every phase cell in phase as

$$2\pi\hbar(1 + \lambda p^2)\,, \tag{6}$$

where λ is a constant, which is of the order of the Planck scale.

Taking account of Eq. (5), Δx has an existence of a minimal length $2\sqrt{\lambda}$. Thinking of body element $\mathrm{d}^3x\mathrm{d}^3p$ in phase space, the number of quantum states is given by

$$\frac{\mathrm{d}^3x\mathrm{d}^3p}{(2\pi\hbar)^3(1 + \lambda p^2)^3}\,, \tag{7}$$

where $p^2 = p_ip^i$ $(i = 1, 2, 3)$. According to Eq. (7), we can write the density of internal energy of black body radiation as

$$u = \int_0^\infty \frac{\omega^3\,\mathrm{d}\omega}{(\,\mathrm{e}^{\beta\omega} - 1)(1 + \lambda\omega^2)^3} = \beta^{-4}\int_0^\infty \frac{x^3\,\mathrm{d}x}{(\,\mathrm{e}^{x} - 1)(1 + \lambda x^2)^3} = \beta^{-4}G(a)\,. \tag{8}$$

To obtain the result through integrating Eq. (8) directly is very difficult, but we can obtain the result needed through approximate calculation under an especial case. Thinking of the case that the temperature of black body radiation is far below that of Planck, when $a \ll 1$, we have

$$G(0) = \int_0^\infty \frac{x^3\,\mathrm{d}x}{\mathrm{e}^x - 1} = \frac{\pi^4}{15}\,, \qquad G'(0) = -3\int_0^\infty \frac{x^5\,\mathrm{d}x}{\mathrm{e}^x - 1} = -\frac{24\pi^6}{63}\,,$$

we obtain

$$u = \beta^{-4}[G(0) + aG'(0)] = \frac{\pi^4}{15}\beta^{-4}\Big(1 - \frac{40\pi^2\lambda}{7\beta^2}\Big)\,. \tag{9}$$

We have revised the density of radiation internal energy under an unexpected case by now. Generally speaking, the revision is infinitely small. Compared with the case below Planck temperature (10^{32} K), it is not enough to change the result of black body radiation. But when $a \gg 1$, equation (9) is not correct. Now, we can also calculate the maximum of energy density

$$u < \beta^{-4}\int_0^\infty \frac{x^2\,\mathrm{d}x}{(1 + \lambda x^2/\beta)^3} = \beta^{-4}\frac{\pi}{16}\Big(\frac{\lambda}{\beta^2}\Big)^{-3/2} = \frac{\pi}{16\lambda^{3/2}}\beta^{-1}\,. \tag{10}$$

Corresponding to quantum state density of Eq. (9), we call it new state density. It is shown as follows:

$$g(E) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}p_r\mathrm{d}p_\theta\mathrm{d}p_\varphi}{(1 + \lambda p^2)^3}\,. \tag{11}$$

Substituting the non-zero components of the metric tensor and components of determinant defined by spacetime (1) into Klein–Gordon equation of massless scalar field

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}\,g^{\mu\nu}\partial_\nu)\Phi = 0 \tag{12}$$

with $\Phi = \exp(-\mathrm{i}\omega t)\psi(r, \theta, \varphi)$, equation (12) becomes

$$\frac{\partial^2\psi}{\partial r^2} + \Big(\frac{2}{r} + \frac{R'}{R}\Big)\frac{\partial\psi}{\partial r} + \frac{1}{R}\Big[\frac{\omega^2}{R} + \frac{1}{r^2}\Big(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Big)\Big]\psi = 0\,, \tag{13}$$

where $R(r) = 1 - 8\pi\eta^2 - 2M/r$. By using the WKB approximation with $\psi = \exp[\mathrm{i}S(r, \theta, \varphi)]$ according to Eq. (13), we obtain

$$p_r^2 = \frac{1}{R}\Big[\frac{\omega^2}{R} - \frac{1}{r^2}p_\theta^2 - \frac{1}{r^2\sin^2\theta}p_\varphi^2\Big]\,. \tag{14}$$

Substituting Eq. (14) into Eq. (11), we get

$$\begin{aligned} g(\omega) &= \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}p_r\mathrm{d}p_\theta\mathrm{d}p_\varphi}{(1 + \lambda\omega^2/R)^3} = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi}{(1 + \lambda\omega^2/R)^3}\int \frac{2}{R^{1/2}}\Big[\frac{\omega^2}{R} - \frac{1}{r^2}p_\theta^2\frac{1}{r^2\sin^2\theta}p_\varphi^2\Big]^{1/2}\mathrm{d}p_\theta\mathrm{d}p_\varphi \\ &= \frac{2\omega^3}{3\pi}\int \frac{r^2\,\mathrm{d}r}{R^2(1 + \lambda\omega^2/R)^3}\,. \end{aligned} \tag{15}$$

The free energy is given by

$$F(\beta) = \frac{1}{\beta}\int \mathrm{d}g(\omega)\ln(1-\mathrm{e}^{-\beta\omega}) = -\frac{2}{3\pi}\int_{r_H}\frac{r^2\,\mathrm{d}r}{R^2}\int_0^\infty \frac{\omega^3\,\mathrm{d}\omega}{(\mathrm{e}^{\beta\omega}-1)(1+\alpha\omega^2 R)^3}\,. \tag{16}$$

and the entropy reads

$$S = \beta^2\frac{\partial F}{\partial\beta} = \frac{2\beta^{-3}}{3\pi}\int_{r_H}\frac{r^2\,\mathrm{d}r}{R^2}\int_0^\infty \frac{x^4\,\mathrm{d}x}{(1-\mathrm{e}^{-x})(\mathrm{e}^{x}-1)(1+\lambda x^2/\beta^2 R)^3}\,, \tag{17}$$

where $x=\beta\omega$. Taking into consideration of the inequality below $1-\mathrm{e}^{-x} > x/(1+x)$, and $\mathrm{e}^x - 1 > x$, according to Eq. (17), we have

$$S < \frac{2\beta^{-3}}{3\pi}\int_{r_H}\frac{r^2\,\mathrm{d}r}{R^2}\int_0^\infty \frac{(x^3+x^2)\,\mathrm{d}x}{(1+\lambda x^2/\beta^2 R)^3} = \frac{2\beta^{-3}}{3\pi}\int_{r_H}\frac{r^2\,\mathrm{d}r}{R^2}\Big(\frac{1}{4}\Big(\frac{\lambda}{\beta^2 R}\Big)^{-2} + \frac{\pi}{16}\Big(\frac{\lambda}{\beta^2 R}\Big)^{-3/2}\Big)$$

$$= \frac{\beta}{6\pi\lambda^2}\int_{r_H} r^2\,\mathrm{d}r + \frac{\lambda^{-3/2}}{24}\int_{r_H}\frac{r^2\,\mathrm{d}r}{R^{1/2}}\,. \tag{18}$$

For Eq. (18), we are only interested in the contribution from the vicinity near the horizon $[r_H, r_H+\varepsilon]$. Remembering Eq. (5), we can get a proper order of distance of the minimal length under Planck scale $2\sqrt{\lambda}$. Considering it to be a minimal length of pure space line element, we have

$$2\sqrt{\lambda} = \int_{r_H}^{r_H+\varepsilon}\sqrt{\gamma_{11}}\ \mathrm{d}r = \int_{r_H}^{r_H+\varepsilon}\frac{1}{R^{1/2}}\ \mathrm{d}r \approx \int_{r_H}^{r_H+\varepsilon}\frac{\mathrm{d}r}{\sqrt{2\kappa(r-r_H)}} = \sqrt{\frac{2\varepsilon}{\kappa}}\ , \tag{19}$$

where $\kappa = 2\pi\beta^{-1}$ is the surface gravity of the black hole horizon. According to Eq. (2), we have

$$S \approx \frac{\beta}{6\pi\lambda^2} r_H^2\varepsilon + \frac{\lambda^{-3/2}}{12} r_H^2\sqrt{\lambda} = \frac{3A}{16\pi\lambda}\,, \tag{20}$$

where $A = 4\pi r_H^2$ is the surface area of the horizon of black hole with an internal global monopole. From Eq. (20), we obtain the result of the direct proportion between black hole entropy and its event horizon area through calculating new state density with the generalized uncertainty relation. It is just what we expected.

Starting from Klein–Gordon equation under the spacetime background with an internal global monopole, we get a new state density equation with the generalized uncertainty relation. We have also studied the entropy of its black hole scalar field. The result is proportional to the black hole event horizon area, which is identical to Bekenstein conclusion. At the same time, during the calculation, we can remove the divergence occurred from brick-wall method without introducing cutoff. Although the new quantum state density from the generalized uncertainty relation can only provide the upper limit, it can also reflect the internal relation between black hole entropy and its horizon. Furthermore, it brings to light the black hole entropy is the entropy of quantum state near the event horizon, which is a quantum effect.

References

[1] S.W. Hawking, Commum. Math. Phys. **43** (1975) 199.

[2] J.D. Bekenstein, Phys. Rev. **D7** (1973) 2333.

[3] G. 't Hooft, Nucl. Phys. **B256** (1985) 727.

[4] X. Li and Z. Zhao, Chin. Phys. Lett. **18** (2001) 463.

[5] W.B. Liu and Z. Zhao, Chin. Phys. Lett. **18** (2001) 310.

[6] C.J. Gao and Y.G. Shen, Chin. Phys. Lett. **18** (2001) 1167.

[7] H. Feng, Z. Zheng, and W.K. Sung, Phys. Rev. **D64** (2001) 044025.

[8] H. He and Z. Zhao, J. Beijing Normal Univ. (Natural Science) **37** (2001) 785 (in Chinese).

[9] J.Y. Zhang, Acta. Phys. Sin. **52** (2003) 2354 (in Chinese).

[10] X. Li, Phys. Lett. **B540** (2002) 9.

[11] H.W. Yu and Y.J. Wang, Chin. Scin. Bull. **39** (1994) 1373 (in Chinese).

[12] W.B. Liu, Y.W. Han, and Z.A. Zhou, Int. J. Mod. Phys. **A18** (2003) 2681.