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Torsion Axial-Vector in an Alternative Kerr Tetrad Field

ZHANG Cheng-Min*

National Astronomical Observatories, the Chinese Academy of Sciences, Beijing 100012, China

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Abstract *In the framework of parallelism general relativity, the torsion axial-vector in the rotating gravitational field is studied in terms of the alternative Kerr tetrad. In the case of the weak field and slow rotation approximation, we obtain that the torsion axial-vector possesses the dipole-like structure. Furthermore, the effect of massive neutrino spin precession in this field is mentioned.*

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1 Introduction

As an extension of Einstein's general relativity (GR) and the fundamental exploration of spacetime, the tetrad theory of gravitation has been paid more attention by many people,[1−7] where the spacetime is characterized by the torsion tensor and the vanishing curvature, and the relevant spacetime is the Weitzenböck spacetime,[1] which is a special case of the Riemann–Cartan spacetime.[8] The tetrad theory of gravitation will be equivalent to general relativity when the convenient choice of the parameters of the Lagrangian is made.[1]

We will use the Greek alphabet ($\mu, \nu, \rho, \ldots = 1, 2, 3, 4$) to denote tensor indices, that is, indices related to spacetime. The Latin alphabet ($a, b, c, \ldots = 1, 2, 3, 4$) will be used to denote local Lorentz (or tangent space) indices. Of course, being of the same kind, tensor and local Lorentz indices can be changed into each other with the use of the tetrad h^a_μ, which satisfies

$$h^a{}_\mu h_a{}^\nu = \delta_\mu{}^\nu\,, \qquad h^a{}_\mu h_b{}^\mu = \delta^a{}_b\,. \tag{1}$$

A nontrivial tetrad field can be used to define the linear Cartan connection,[1,5]

$$\Gamma^\sigma{}_{\mu\nu} = h_a{}^\sigma \partial_\nu h^a{}_\mu\,, \tag{2}$$

with respect to which the tetrad is parallel:

$$\nabla_\nu h^a{}_\mu \equiv \partial_\nu h^a{}_\mu - \Gamma^\rho{}_{\mu\nu} h^a{}_\rho = 0\,. \tag{3}$$

The Cartan connection can be decomposed according to

$$\Gamma^\sigma{}_{\mu\nu} = \mathring{\Gamma}^\sigma{}_{\mu\nu} + K^\sigma{}_{\mu\nu}\,, \tag{4}$$

where

$$\mathring{\Gamma}^\sigma{}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}] \tag{5}$$

is the Levi–Civita connection of the metric

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu\,, \tag{6}$$

where η^{ab} is the metric in flat space with the line element

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu\,, \tag{7}$$

and

$$K^\sigma{}_{\mu\nu} = \frac{1}{2} [T_\mu{}^\sigma{}_\nu + T_\nu{}^\sigma{}_\mu - T^\sigma{}_{\mu\nu}] \tag{8}$$

is the contorsion tensor with

$$T^\sigma{}_{\mu\nu} = \Gamma^\sigma{}_{\mu\nu} - \Gamma^\sigma{}_{\nu\mu}\,, \tag{9}$$

the torsion of the Cartan connection.[1,5] The irreducible torsion vectors, i.e., the torsion vector and the torsion axial-vector, can then be constructed as[1,5]

$$V_\mu = T^\nu{}_{\nu\mu}\,, \tag{10}$$

$$A_\mu = \frac{1}{6} \epsilon_{\mu\nu\rho\sigma} T^{\nu\rho\sigma} \tag{11}$$

with $\epsilon_{\mu\nu\rho\sigma}$ being the completely antisymmetric tensor normalized as $\epsilon_{0123} = \sqrt{-g}$ and $\epsilon^{0123} = 1/\sqrt{-g}$, where g is the determinant of metric.

The spacetime dynamic effects on the spin is incorporated into Dirac equation through the "spin connection" appearing in the Dirac equation in gravitation.[1] In Weitzenböck spacetime, as well as the general version of torsion gravity, it has been shown by many authors[1,2,9−12] that the spin precession of a Dirac particle is intimately related to the torsion axial-vector, and it is interesting to note that the torsion axial-vector represents the deviation of the axial symmetry from the spherical symmetry.[2] *In PGR, the particle spin vector S^μ is defined by the spinor function ψ'_o, the Dirac matrices γ^k ($k = 0, 1, 2, 3, 4, 5$), and the tetrad $h_k{}^\mu$ as follows:*[1]

$$S^\mu = -(1/2) h_k{}^\mu \bar{\psi}'_o \gamma^5 \gamma^k \psi'_o\,, \tag{12}$$

so the spin vector depends on the choice of tetrad. Furthermore, the spin precession equation is given as[1]

$$\frac{d\boldsymbol{S}}{dt} = -\frac{3}{2} \boldsymbol{A} \times \boldsymbol{S}\,, \tag{13}$$

*E-mail: zhangcm@bao.ac.cn

where $\boldsymbol{S}$ is the space component of spin vector S^μ of a Dirac particle, and $\boldsymbol{A}$ is the spacelike part of the torsion axial-vector A^μ. Therefore, the corresponding extra Hamiltonian energy is of the form

$$\delta H = -\frac{3}{2}\boldsymbol{A}\cdot\boldsymbol{S}\,. \tag{14}$$

The purpose of the paper is to derive the torsion axial-vector with the alternative Kerr tetrad, which is performed in Sec. 2. In the weak field and slow rotation approximation, the analytical expression of the torsion axial-vector is obtained in Sec. 3. Discussions and conclusions are given in the last section. Throughout this paper we use the unit with $c = 1$.

2 Torsion Axial-Vector in Kerr Spacetime

In PGR, the description of Kerr spacetime is the same as that in general relativity,[6] so the gravitational field of a rotating mass is described by the axially symmetric stationary Kerr metric,[13]

$$\mathrm{d}\tau^2 = g_{00}\mathrm{d}t^2 + g_{11}\mathrm{d}r^2 + g_{22}\mathrm{d}\theta^2 + g_{33}\mathrm{d}\phi^2 + 2g_{03}\mathrm{d}\phi\mathrm{d}t\,, \tag{15}$$

where

$$g_{00} = 1 - \frac{r_s r}{\Sigma}\,,\quad g_{11} = -\frac{\Sigma}{\Delta}\,,\quad g_{22} = -\Sigma\,, \tag{16}$$

$$g_{33} = -\Big(r^2 + a^2 + \frac{r_s r a^2}{\Sigma}\sin^2\theta\Big)\sin^2\theta\,, \tag{17}$$

$$g_{03} = g_{30} = \frac{r_s r a}{\Sigma}\sin^2\theta \tag{18}$$

with

$$\Delta = r^2 - r_s r + a^2 \quad\text{and}\quad \Sigma = r^2 + a^2\cos^2\theta\,. \tag{19}$$

In these expressions, r_s is Schwarzschild radius and a is the angular momentum of a gravitational unit mass source. If $a = 0$, the Kerr metric becomes the Schwarzschild metric in the standard form. In Kerr spacetime, the tetrad can be expressed by the dual basis of the differential one-form[14] through choosing a coframe of the coordinate system,

$$\vartheta^{\hat 0} = A[\mathrm{d}t - a\sin^2\theta\mathrm{d}\phi]\,, \tag{20}$$

$$\vartheta^{\hat 1} = A^{-1}\mathrm{d}r\,, \tag{21}$$

$$\vartheta^{\hat 2} = \sqrt{\Sigma}\,\mathrm{d}\theta\,, \tag{22}$$

$$\vartheta^{\hat 3} = B[(-a\mathrm{d}t + (r^2 + a^2)\mathrm{d}\phi]\,, \tag{23}$$

where $A = \sqrt{\Delta/\Sigma}$, $B = \sin\theta/\sqrt{\Sigma}$. Therefore, the tetrad can be obtained with the subscript μ denoting the column index,

$$h^a{}_\mu = \begin{pmatrix} A & 0 & 0 & -aA\sin^2\theta \\ 0 & 1/A & 0 & 0 \\ 0 & 0 & \sqrt{\Sigma} & 0 \\ -aB & 0 & 0 & (r^2+a^2)B \end{pmatrix} \tag{24}$$

with the inverse

$$hh_a{}^\mu = \begin{pmatrix} (r^2+a^2)\sin\theta/A & 0 & 0 & a\sin\theta/A \\ 0 & A\Sigma\sin\theta & 0 & 0 \\ 0 & 0 & B\Sigma & 0 \\ a\sqrt{\Sigma}\sin^2\theta & 0 & 0 & \sqrt{\Sigma} \end{pmatrix}, \tag{25}$$

where $h = \det(h^a{}_\mu) = \Sigma\sin\theta = \sqrt{-g}$ with $g = \det(g_{\mu\nu})$. We can inspect that equations (24) and (25) satisfy the conditions in Eq. (1). From Eqs. (24) and (25), we can now construct the Cartan connection, whose nonvanishing components are

$$h\Gamma^0{}_{01} = \Big(\frac{A'}{A}\Big)(r^2 + a^2)\sin\theta - B'a^2\sqrt{\Sigma}\sin^2\theta\,, \tag{26}$$

$$h\Gamma^0{}_{31} = -\Big(\frac{A'}{A}\Big)a(r^2 + a^2)\sin^3\theta + [B(r^2 + a^2)]'a\sqrt{\Sigma}\sin^2\theta\,, \tag{27}$$

$$h\Gamma^0{}_{02} = \Big(\frac{A'_\theta}{A}\Big)(r^2 + a^2)\sin\theta - a^2\sin^2\theta\cos\theta = a^4\sin^4\theta\cos\theta/\Sigma\,, \tag{28}$$

$$h\Gamma^0{}_{32} = -[A\sin^2\theta]'_\theta\frac{a}{A}\sin\theta(r^2 + a^2) + B'_\theta(r^2 + a^2)a\sqrt{\Sigma}\sin^2\theta\,, \tag{29}$$

$$h\Gamma^1{}_{12} = -a^2\sin^2\theta\cos\theta\,,\quad h\Gamma^2{}_{21} = r\sin\theta\,, \tag{30}$$

$$h\Gamma^3{}_{01} = \Big(\frac{A'}{A}\Big)a\sin\theta - B'a\sqrt{\Sigma}\,, \tag{31}$$

$$h\Gamma^3{}_{31} = -\Big(\frac{A'}{A}\Big)a^2\sin^3\theta + [B(r^2 + a^2)]'\sqrt{\Sigma}\,, \tag{32}$$

$$h\Gamma^3{}_{02} = \Big(\frac{A'_\theta}{A}\Big)a\sin\theta - B'_\theta\sqrt{\Sigma}a = (a^2/\Sigma - 1)a\cos\theta\,, \tag{33}$$

$$h\Gamma^3{}_{32} = -(A\sin^2\theta)'_\theta\frac{a^2}{A}\sin\theta + B'_\theta(a^2 + r^2)\sqrt{\Sigma}\,, \tag{34}$$

where

$$A' = \frac{\partial A}{\partial r} = \frac{r - r_s/2}{A\Sigma} - \frac{Ar}{\Sigma}\,, \quad A'_\theta = \frac{\partial A}{\partial \theta} = \frac{a^2 A}{\Sigma}\sin\theta\cos\theta\,, \tag{35}$$

$$B' = \frac{\partial B}{\partial r} = -\frac{Br}{\Sigma}\,, \quad B'_\theta = \frac{\partial B}{\partial \theta} = \frac{\cos\theta}{\sqrt{\Sigma}} + \frac{a^2 B}{\Sigma}\sin\theta\cos\theta\,. \tag{36}$$

The nonzero torsion axial-vector components are

$$A^{(1)} \times (6h) = -2(g_{00}T^0{}_{23} + g_{03}T^3{}_{23} + g_{30}T^0{}_{02} + g_{33}T^3{}_{02}) = -2(g_{00}\Gamma^0{}_{32} + g_{03}\Gamma^3{}_{32} - g_{30}\Gamma^0{}_{02} - g_{33}\Gamma^3{}_{02})\,, \tag{37}$$

$$A^{(2)} \times (6h) = 2[g_{00}T^0{}_{13} + g_{03}T^3{}_{13} + g_{30}T^0{}_{01} + g_{33}T^3{}_{01}] = 2[g_{00}\Gamma^0{}_{31} + g_{03}\Gamma^3{}_{31} - g_{30}\Gamma^0{}_{01} - g_{33}\Gamma^3{}_{01}]\,. \tag{38}$$

3 Slow Rotation and Weak Field Approximations

In the case of slow rotation and weak field, we keep the terms up to the first order in the angular momentum a and in r_s/r. The related quantities are simplified as follows:

$$\Delta = r^2 - r_s r\,, \quad \Sigma = r^2\,, \tag{39}$$

$$g_{00} = (-g_{11})^{-1} = 1 - \frac{r_s}{r}\,, \quad g_{22} = -r^2\,, \tag{40}$$

$$g_{33} = -r^2\sin^2\theta\,, \quad g_{03} = \frac{r_s a}{r}\sin^2\theta\,, \tag{41}$$

$$h = r^2\sin\theta\,, \quad A = \sqrt{1 - r_s/r}\,, \quad B = \frac{\sin\theta}{r}\,. \tag{42}$$

In this approximation, all terms reduce to the values of the Schwarzschild solution except g_{03}. On the other hand, in the weak field limit, characterized by keeping terms up to the first order in r_s/r, the nonzero components of the axial-vector torsion become

$$h\Gamma^0{}_{32} = -ar^2\sin^2\theta\cos\theta\,, \quad h\Gamma^3{}_{32} = r^2\cos\theta\,,$$
$$h\Gamma^0{}_{02} = 0\,, \quad h\Gamma^3{}_{02} = -a\cos\theta\,, \tag{43}$$

and

$$h\Gamma^0{}_{31} = ar\sin^3\theta\Big(1 - \frac{r_s}{2r}\Big)\,, \quad h\Gamma^3{}_{31} = r\sin\theta\,,$$
$$h\Gamma^0{}_{01} = \frac{r_s}{2}\sin\theta\,, \quad h\Gamma^3{}_{01} = a\frac{\sin\theta}{r}\Big(1 + \frac{r_s}{2r}\Big). \tag{44}$$

Substituting Eqs. (39) ∼ (42) into Eqs. (37) and (38), we obtain

$$A^{(1)} = \frac{2}{3}\Big(1 - \frac{r_s}{r}\Big)\frac{a}{r^2}\cos\theta\,, \tag{45}$$

$$A^{(2)} = \frac{2}{3}\frac{a}{r^3}\sin\theta\,. \tag{46}$$

In spacelike vector form, the axial-vector becomes

$$-\boldsymbol{A} = A^{(1)}\sqrt{-g_{11}}\,\boldsymbol{e}_r + A^{(2)}\sqrt{-g_{22}}\,\boldsymbol{e}_\theta\,, \tag{47}$$

where $\boldsymbol{e}_r = \sqrt{-g_{11}}\,\mathrm{d}r$ and $\boldsymbol{e}_\theta = \sqrt{-g_{22}}\,\mathrm{d}\theta$ are unit vectors in (r, θ) directions, and then we have

$$-\boldsymbol{A} = \frac{2a}{3r^2}\Big[\sqrt{1 - \frac{r_s}{r}}\cos\theta\boldsymbol{e}_r + \sin\theta\boldsymbol{e}_\theta\Big]. \tag{48}$$

It has been shown by many authors[1,10] that the spin precession of a Dirac particle in torsion gravity is intimately related to the axial-vector,

$$\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = \boldsymbol{b} \times \boldsymbol{s}\,, \tag{49}$$

where $\boldsymbol{s}$ is the spin vector, and $\boldsymbol{b} = -3\boldsymbol{A}/2$. Therefore

$$\boldsymbol{b} = \frac{J}{Mr^2}\Big[\sqrt{1 - \frac{r_s}{r}}\cos\theta\boldsymbol{e}_r + \sin\theta\,\boldsymbol{e}_\theta\Big] \tag{50}$$

with $J = Ma$ the angular momentum.

4 Discussions and Conclusions

The torsion axial-vector Dirac spin coupling by the special choice of the Kerr tetrad in the framework of the torsion spacetime without curvature has been derived. We employ the given Kerr tetrad to derive the torsion axial-vector, as one of the three irreducible quantities in Weitzenböck spacetime, which will couple with the Dirac spin. We find that $\boldsymbol{b}$ field is still a dipole-like field, which is on account of the axisymmetric property of Kerr spacetime, although it is not a standard dipole gravitomagnetic field as obtained from the standard Kerr tetrad.[6] Furthermore, if there is no rotation, say $a = 0$, then the $\boldsymbol{b}$ field disappears, which is reasonable because the axial-vector represents the measurement of the axial symmetry deviated from the spherical symmetry.[2]

The application of the coupling between the massive neutrino spin and rotating gravitational field is worthy of mention because it may be important for understanding some phenomena around black hole, and it has been considered before[15] in terms of the approximated Kerr tetrad. It is expected that the coupling energy from neutrino spin precession induced by the rotating gravitational field will give rise to the neutrino spin flip transition, and this effect may be detected in strong gravitational fields, such as in the vicinity of a rotating neutron star, black hole, or around the gravitational lens.[16] Therefore the study on the neutrino spin precession effect by the rotating gravitational field is important because the massive neutrino may pass by many galaxy black holes before its propagating to the earth from the deep universe.

Moreover, on how to measure the axial-torsion spin effect, we have the following further arguments. Generally, unlike the macroscopic spin object, we will measure the spin effect by the axial-torsion spin coupling energy of

Dirac particle. If the spin object is an electron, we will measure the photon splitting energy. In fact, this kind of measurement will be very difficult in the astrophysical environments because the existence of the strong astronomical magnetic field will diminish the chance of observing this coupling energy. As for the case of massive neutrino, in principle, the spin-flavor oscillation will be influenced by the axial-torsion spin coupling although the probability also depends on the masses of neutrinos. However, we have to assume that the massive neutrino has no or so little magnetic moment that the magnetic induced spin-flavor oscillation is as low as possible. Moreover, the black hole itself has no magnetic field, but the accreted matters around black hole may have weak magnetic field. Therefore, in order to avoid the magnetic influence, we need to observe the massive neutrinos to come from the region close to the horizon and not to pass through the accretion disk. On the status of observer, it is in a remote distance from the gravitational source.

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