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Conservation Laws of a Class of Combined Equations

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Abstract *In this paper, we investigate conservation laws of a class of partial differential equations, which combines the nonlinear telegraph equations and the nonlinear diffusion-convection equations. Moreover, some special conservation laws of the combined equations are obtained by means of symmetry classifications of wave equations* $u_{xx} = H(x)u_{tt}$.

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1 Introduction

A class of partial differential equations

$$\alpha u_{tt} + \beta u_t = [F(u)u_x]_x + [G(u)]_x \,, \tag{1}$$

where $F(u)$ and $G(u)$ are arbitrary differentiable functions, contains a number of physically important subclasses, which have been widely studied and may be applied in many situations. For example, when $\beta = 0$, it becomes the nonlinear telegraph equations

$$u_{tt} = [F(u)u_x]_x + [G(u)]_x \,, \tag{2}$$

which has been investigated in detail by Bluman *et al.*[1,2] In particular, the symmetry and conservation law of this equation are exhaustively described in Ref. [1]. When $\alpha = 0$, it reduces to a class of nonlinear diffusion-convection equation

$$u_t = [F(u)u_x]_x + [G(u)]_x \,, \tag{3}$$

which has many applications in mathematical physics.[3–8] Here we will consider the case $\alpha\beta \neq 0$, under which equation (1) can be rewritten as

$$u_{tt} + \lambda u_t = [F(u)u_x]_x + [G(u)]_x \,. \tag{4}$$

Introducing a potential v, equation (4) is equivalent to

$$\begin{aligned} H_1[u,v] &= v_t - F(u)u_x - G(u) = 0 \,, \\ H_2[u,v] &= u_t + \lambda u - v_x = 0 \,, \end{aligned} \tag{5}$$

whose conservation laws will be investigated in this paper.

In Ref. [9], it is shown how to find the multipliers for conservation laws of a given system of partial differential equations (PDEs). In Refs. [10] and [11], the direct construction method is presented to obtain the multipliers and the corresponding conservation laws for a given system of PDEs through an integral formula.

Two functions $\xi = \xi(x,t,U,V)$ and $\phi = \phi(x,t,U,V)$ are multipliers of a conservation law of system (5) if they satisfy

$$\xi H_1[U,V] + \phi H_2[U,V] = D_x X + D_t T \tag{6}$$

for all differentiable functions $U(x,t)$ and $V(x,t)$ and some differentiable functions $X = X(x,t,U,V)$ and $T = T(x,t,U,V)$. Consequently, the conservation law

$$D_x X + D_t T = 0 \tag{7}$$

holds for all solutions $U = u(x,t)$, $V = v(x,t)$ of the combined equations (5) with flux $X(x,t,U,V)$ and density $T(x,t,U,V)$. Throughout this paper, (U,V) denotes arbitrary functions of x and t; (u,v) denotes solutions of Eq. (5). The necessary and sufficient conditions for $\xi(x,t,U,V)$, $\phi(x,t,U,V)$ to yield multipliers for a conservation law of (5) are that the Euler operators E_U and E_V with respect to U and V, respectively, annihilate the left hand side of (6), i.e.,

$$\begin{aligned} &E_U[\phi(x,t,U,V)(U_t + \lambda U - V_x) \\ &\quad + \xi(x,t,U,V)(V_t - F(U)U_x - G(U))] = 0 \,, \\ &E_V[\phi(x,t,U,V)(U_t + \lambda U - V_x) \\ &\quad + \xi(x,t,U,V)(V_t - F(U)U_x - G(U))] = 0 \,, \end{aligned} \tag{8}$$

for all differentiable functions $U(x,t)$ and $V(x,t)$, where

$$\begin{aligned} E_U &= \frac{\partial}{\partial U} - D_x \frac{\partial}{\partial U_x} - D_t \frac{\partial}{\partial U_t} \,, \\ E_V &= \frac{\partial}{\partial V} - D_x \frac{\partial}{\partial V_x} - D_t \frac{\partial}{\partial V_t} \,, \end{aligned} \tag{9}$$

D_x and D_t are total derivative operators with respect to the independent variables x and t. In particular,

$$\begin{aligned} D_x = \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} \\ + U_{xt} \frac{\partial}{\partial U_t} + V_x x \frac{\partial}{\partial V_x} + V_{xt} \frac{\partial}{\partial V_t} + \cdots \,. \end{aligned} \tag{10}$$

Consequently, we derive the determining equations about ξ and ϕ,

$$\begin{aligned} &\phi_V - \xi_U = 0, \quad \phi_U - F(U)\xi_V = 0 \,, \\ &\lambda U \phi_V - G(U)\xi_V + \phi_x - \xi_t = 0 \,, \\ &\lambda U \phi_U + \lambda \phi - \phi_t + F(U)\xi_x - G(U)\xi_U - G'(U)\xi = 0 \end{aligned} \tag{11}$$

with x, t, U, V as independent variables and ξ, ϕ as dependent variables.

In order to solve the determining system (11) with two arbitrary functions $F(U)$ and $G(U)$, one works completely

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on the solution space of the given PDEs. Hence one can use the same algorithmic procedures and symbolic manipulation programs[12−15] as symmetry analysis for PDEs in order to solve the conservation law determining system. Moreover, for each solution one can directly obtain the resulting conservation law by evaluating the construction formula working entirely on the solution space of the given PDEs. In this study, we adopt the Wu's differential characteristic set[12,13] arising from the algebraic polynomial system[16] to derive the solutions.

The rest of the paper is organized as follows. In Sec. 2, we present the new conservation laws based on the complete classical and potential symmetry classifications of wave equations $u_{xx} = H(x)u_{tt}$.[17] And we give a conclusion of this paper in Sec. 3.

2 Conservation Law

Firstly, note that the point transformation

$$x' = x + a, \quad t' = t + b, \quad u' = cu, \quad v' = cv + d\,, \tag{12}$$

where a, b, c, and d are constants and $c \neq 0$, is an equivalent transformation of (5). This means that scalings and translations of x, t, u, and v may be used to simplify the analysis with the understanding that these equivalent transformations are included in the conclusions. In particular, $F(u)(\neq 0)$ may be scaled in order to simplify the form of $F(u)$ without any loss of generality. For example, if $F(u)$ is a non-zero constant it may be assumed that $F(u) = 1$.

Now, we consider the problem of solving the system of determining equations (11) and finding the corresponding fluxes and densities. In particular, we find compatibility conditions such that $F(u)$ and $G(u)$ yield solutions of Eq. (11). Then we determine some conservation laws of each combined system (5) in terms of multipliers. We first derive a formula to determine the flux and density (X, T) for a known set of multipliers (ξ, ϕ). Expanding the right side of Eq. (6), one obtains

$$\begin{aligned}&\phi(x,t,U,V)(U_t + \lambda U - V_x)\\&\quad+\xi(x,t,U,V)(V_t - F(U)U_x - G(U))\\&\equiv X_x + X_U U_x + X_V V_x + T_t + T_U U_t + T_V V_t\,,\end{aligned} \tag{13}$$

which must hold for all differentiable functions $U(x,t)$, $V(x,t)$. Comparing the coefficients of U_t, V_t, U_x, V_x and the remaining terms of both sides of Eq. (13), one obtains

$$\begin{aligned}&T_U = \phi, \quad T_V = \xi, \quad X_U = -F(U)\xi\,,\\&X_V = \phi, \quad X_x + T_t = -G(U)\xi + \lambda U\phi\,.\end{aligned} \tag{14}$$

By direct calculations, one can obtain solutions of system (14),

$$\begin{aligned}X &= -\int_a^U \xi(x,t,s,b)F(s)\,\mathrm{d}s - \int_b^V \phi(x,t,U,s)\,\mathrm{d}s\\&\quad - G(a)\int^x \xi(s,t,a,b)\,\mathrm{d}s + \lambda a\int^x \phi(s,t,a,b)\,\mathrm{d}s\,,\\T &= \int_a^U \phi(x,t,s,b)\,\mathrm{d}s + \int_b^V \xi(x,t,U,s)\,\mathrm{d}s\,,\end{aligned} \tag{15}$$

where $\xi(x,t,U,V)$, $\phi(x,t,U,V)$ solving the determining system (11). Moreover, $X = X(x,t,u,v)$ and $T = T(x,t,u,v)$ defined by (15) yield the flux and density of the corresponding conservation law of the combined system (5) for any solution $U = u(x,t)$, $V = v(x,t)$ of (5). In Eq. (15), constants a and b are chosen such that the integrals are not singular.

Now we consider the problem of finding all pairs $(F(u), G(u))$ such that the determining system (11) for multipliers $\xi(x,t,U,V)$, $\phi(x,t,U,V)$ has a solution.

In order to proceed, the following cases about the solution of (11) should be considered.

Case 1 *$F(u)$ and $G(u)$ are arbitrary functions.*

In this case, we have $\xi = 0$, $\phi = \mathrm{e}^{\lambda t}$, corresponding conserved flux and conserved density are $X = -v\,\mathrm{e}^{\lambda t}$, $T = u\,\mathrm{e}^{\lambda t}$.

Case 2 *$G(u)$ is arbitrary function.*

We get the same results as ***Case 1***.

Case 3 *$F(u)$ is arbitrary.*

In this case, when $F(u)$ is an arbitrary function, one has four cases about $G(u)$, which are $G(u) = 0$, $G(u) = 1$, $G(u) = u$, $G(u) = 1/u$. Using the Characteristic Set algorithm for differential polynomial systems[12,13] and its computer algebra program, the flux and density are obtained in Table 1.

Table 1 $F(u)$ are arbitrary functions.

$F(u)$	$G(u)$	Multipliers (ξ, ϕ); flux and density (X, T)
arbitrary	0	$\xi = \mathrm{e}^{\lambda t}u$, $\phi = \mathrm{e}^{\lambda t}v$, $X = -\mathrm{e}^{\lambda t}(v^2 + \frac{1}{\lambda}\int^u s\,F(s)\,\mathrm{d}s)$, $T = \mathrm{e}^{\lambda t}\,u\,v$
		$\xi = \frac{1}{\lambda}\mathrm{e}^{\lambda t}$, $\phi = \mathrm{e}^{\lambda t}x$, $X = \frac{\mathrm{e}^{\lambda t}}{\lambda}(v + \lambda\,u\,x)$, $T = -\mathrm{e}^{\lambda}t\,v\,x - \frac{\mathrm{e}^{\lambda t}}{\lambda^2}\int^u F(s)\,\mathrm{d}s$
		$\xi = 1$, $\phi = 0$, $X = -\int^u F(s)\,\mathrm{d}s$, $T = v$
arbitrary	1	$\xi = \mathrm{e}^{\lambda t}u$, $\phi = \mathrm{e}^{\lambda t}(v - t)$, $X = \frac{-\mathrm{e}^{\lambda t}}{2}(-2\,t\,v + v^2) - \mathrm{e}^{\lambda t}\int^u s\,F(s)\,\mathrm{d}s$, $T = \mathrm{e}^{\lambda t}\,u\,(-t + v)$
		$\xi = \frac{1}{\lambda}\mathrm{e}^{\lambda t}$, $\phi = \mathrm{e}^{\lambda t}x$, $X = \frac{\mathrm{e}^{\lambda t}}{\lambda}(-x - \lambda\,v\,x + \int^u F(s)\,\mathrm{d}s)$, $T = \frac{\mathrm{e}^{\lambda t}}{\lambda}(v + \lambda\,u\,x)$
	u	$\xi = \frac{\mathrm{e}^{\lambda t}}{\lambda}$, $\phi = \mathrm{e}^{\lambda t}(x - \frac{t}{\lambda})$, $X = \frac{\mathrm{e}^{\lambda t}}{\lambda}(t\,v - \lambda\,v\,x - \lambda\int^u F(s)\,\mathrm{d}s)$, $T = \frac{\mathrm{e}^{\lambda t}}{\lambda}(-t\,u + v + \lambda\,u\,x)$
		$\xi = 1$, $\phi = \frac{1}{\lambda}$, $X = -\frac{v}{\lambda} - \int^u F(s)\,\mathrm{d}s$, $T = \frac{u}{\lambda} + v$
	$\frac{1}{u}$	$\xi = \mathrm{e}^{\lambda t}u$, $\phi = \mathrm{e}^{\lambda t}v$, $X = \frac{\mathrm{e}^{\lambda t}}{2}(-v^2 - 2\,x) - \mathrm{e}^{\lambda t}\int^u s\,F(s)\,\mathrm{d}s$, $T = \mathrm{e}^{\lambda t}\,u\,v$

Case 4 *$F(u)$ and $G(u)$ are special functions.*

In this case, in order to solve the conservation law determining equations, we introduce the compatibility conditions of the system (11). It is easy to find that if the system has solutions, the first and the second equation must satisfy compatible conditions, i.e. $\phi_{vu} = \phi_{uv}$, which suggests $\xi_{uu} - F(u)\xi_{vv} = 0$, it has the form of

$$u_{xx} = H(x)u_{tt}\,, \tag{16}$$

whose complete symmetry classifications have been obtained in Ref. [17]. Its results are listed in Table 2.

Table 2 Classical and Potential Symmetry Classifications of Eq. (16).

$H(x)$	$\alpha,\beta,\gamma,\delta$	Conditions
	$\alpha = p(x,t) + (c_4 t + c_5)u + (c_4 x + c_6)v$	$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}$
	$\beta = q(x,t) + (c_4 t + c_5)v + H(x)\,(c_4 x + c_6)u$	$H(x)\frac{\partial f}{\partial t} = \frac{\partial g}{\partial x}$
$\tilde{I}_6 = 0$	$\gamma = f(x,t) + (c_1 t + c_2)u + (c_1 x + c_3)v$	$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial t}$
	$\theta = g(x,t) + (c_1 t + c_2)v + H(x)\,(c_1 x + c_3)u$	$H(x)\frac{\partial p}{\partial t} = \frac{\partial q}{\partial x}$
	$\alpha = H(x)\,(c_1 t + c_2)$	$\frac{\partial g}{\partial t} = u\frac{\partial f}{\partial x}$
	$\beta = \int c_1 H(x)^2\,\mathrm{d}x + \frac{3\,(c_1 t^2/2 + c_2 t + c_3)\,H'(x)}{2}$	$\frac{\partial g}{\partial x} = H(x)\frac{\partial f}{\partial t}$
$\tilde{I}_5 = 0$	$\gamma = f(x,t) - \frac{c_1 H(x) v}{2} + (c_4 - c_1 t\,H'(x))u$	
$\tilde{I}_6 \neq 0$	$\theta = g(x,t) - \frac{c_1 H(x)^2 u}{2} + \left(c_4 + \frac{(-c_1 t + c_2)\,H'(x)}{2}\right)v$	
	$\alpha = \frac{c_1 t + c_2}{H(x)}$	$\frac{\partial g}{\partial x} = H(x)\frac{\partial f}{\partial t}$
	$\beta = c_1 x + c_3 - \frac{(c_1 t^2/2 + c_2 t + c_3)H'(x)}{2\,H(x)^2}$	$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}$
$\tilde{I}_2 = 0$	$\gamma = f(x,t) - \frac{c_1 v}{2\,H(x)} + c_4 u$	
$\tilde{I}_6 \neq 0$	$\theta = g(x,t) - \frac{c_1 u}{2} + \left(c_4 + \frac{(c_1 t + c_2)\,H'(x)}{2\,H(x)^2}\right)v$	
	$\alpha = \frac{c_1 t + c_2}{\sqrt{H(x)}}$	$\frac{\partial g}{\partial x} = H(x)\frac{\partial f}{\partial t}$
	$\beta = \int c_1\sqrt{H(x)}\,\mathrm{d}x + c_3$	$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}$
$\tilde{I}_3 = 0$	$\gamma = f(x,t) - \frac{c_1 v}{2\sqrt{H(x)}} + \left(c_4 - \frac{c_1\,H'(x)t}{4\,H(x)^{3/2}}\right)u$	
$\tilde{I}_6 \neq 0$	$\theta = g(x,t) - \frac{c_1\sqrt{H(x)}u}{2} + \left(c_4 + \frac{(c_1 t + 2\,c_2)\,H'(x)}{4\,H(x)^{3/2}}\right)v$	
	$\alpha = \frac{c_1 t + c_2}{H(x)^2}$	$\frac{\partial g}{\partial x} = H(x)\frac{\partial f}{\partial t}$
	$\beta = \int c_1\frac{1}{H(x)}\,\mathrm{d}x + \frac{3(c_1 t^2/2 + c_2 t + c_3)\,H'(x)}{2H(x)^3}$	$\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x}$
$\tilde{I}_4 = 0$	$\gamma = f(x,t) - \frac{c_1 v}{2\,H(x)^2} + \left(c_4 + \frac{c_1 t\,H'(x)}{2\,H(x)^3}\right)u$	
$\tilde{I}_6 \neq 0$	$\theta = g(x,t) + \frac{c_1 u}{H(x)} + \left(c_4 + \frac{(c_1 t + c_2)\,H'(x)}{2\,H(x)^3}\right)v$	
	$\alpha = \frac{k(t)\,H(x)}{H'(x)}$	$a(x,t) = -k(t) + \frac{k(t)\,H(x)\,H''(x)}{2\,H'(x)^2} + c_1$
	$\beta = \frac{1}{2}\left(3 - \frac{2\,H(x)\,H''(x)}{H'(x)^2}\right)\int k(t)\,\mathrm{d}t + c_2\int\frac{H(x)^2}{H'(x)}\,\mathrm{d}x$	$\frac{H'(x)^2\,H''(x) - 2\,H(x)\,H''(x)^2 + H(x)\,H'(x)\,H^{(3)}(x)}{-H(x)^2\,H'(x)^2}$
$\tilde{I}_1 = 0$	$\gamma = a(x,t)u - \frac{H(x)\,k'(t)}{2\,H'(x)v} + b(x,t)$	$= \lambda = \frac{k''(t)}{k(t)}$
$\tilde{I}_6 \neq 0$	$\theta = \left(a(x,t) + \frac{k(t)}{2} + c_1\right)v - \frac{H(x)^2\,k'(t)}{2\,H'(x)}u + c(x,t)$	$b(x,t), c(x,t)$ are solutions of (16)
	$\alpha = \chi_1 x^2$	
	$\beta = \int\frac{2\int\frac{\partial\chi_1}{\partial x}\mathrm{d}t - x\int\frac{\partial^2\chi_1}{\partial x^2}\mathrm{d}t + x\frac{\partial\chi_1}{\partial t}}{x^3}\,\mathrm{d}x$	
	$+\frac{1}{x^2}\int\frac{\partial\chi_1}{\partial x}\mathrm{d}t - c_1 t + c_2,$	
$\tilde{I}_7 = 0$	$\gamma = \chi_1 x u + \chi_2 x$	$\frac{\partial^2\chi_1}{\partial t^2} - x^3\left(2\frac{\partial\chi_1}{\partial x} + x\frac{\partial^2\chi_1}{\partial x^2}\right) = 0$

In Table 2,

$$\tilde{I}_1 = 2\,H'(x)^4\,H''(x) - 2\,H(x)\,H'(x)^2\,H''(x)^2 - 4\,H(x)^2\,H''(x)^3 - H(x)^2\,H'(x)^2\,H^{(4)}(x) + 5\,H(x)^2\,H'(x)\,H''(x)\,H^{(3)}(x)\,,$$

$$\tilde{I}_2 = -2\,H'(x)^2 + H(x)\,H''(x), \quad \tilde{I}_3 = -3\,H'(x)^2 + 2\,H(x)\,H''(x), \quad \tilde{I}_4 = -3\,H'(x)^2 + H(x)\,H''(x)\,,$$

$$\tilde{I}_5 = H''(x), \qquad \tilde{I}_6 = H'(x), \quad \tilde{I}_7 = -5H'(x)^2 + 4H(x)H''(x)\,.$$

Thus, let F correspond to H, then the determining system can be simplified according to different expressions of F. Next, based on the differential characteristic set method, for some special $F(u)$ and $G(u)$, the determining system (11) can be solved exactly and finally the multipliers are derived. Thus, some special cases are listed in Table 3.

Table 3 $F(u)$ and $G(u)$ are special functions.

$F(u)$	$G(u)$	Multipliers (ξ,ϕ); flux and density (X,T)
e^u	e^u	$\xi=\frac{1}{\lambda}e^{\lambda t+x}$, $\phi=e^{\lambda t+x}$, $X=-\frac{e^{t\lambda}}{\lambda}(-1+e^{u+x}+e^x v\lambda)$, $T=\frac{e^{x+t\lambda}}{\lambda}(v+u\lambda)$
		$\xi=e^x$, $\phi=0$, $X=1-e^{u+x}$, $T=e^x v$, $\xi=\frac{1}{\lambda}e^{\lambda t+x/2}$, $\phi=e^{\lambda t+x/2}$
$\frac{1}{\sqrt{u}}$	$\sqrt{u}$	$X=-\frac{2}{\lambda}e^{x/2+t\lambda}(\sqrt{u}+v\lambda)$, $T=\frac{1}{\lambda}e^{x/2+t\lambda}(v+2u\lambda)$
		$\xi=e^{x/2}$, $\phi=0$, $X=-\frac{2}{\lambda}e^{x/2}\sqrt{u}$, $T=e^{x/2}v$
$\frac{1}{u}$	$\log u$	$\xi=\frac{1}{\lambda}e^{\lambda t+x}$, $\phi=e^{\lambda t+x}$, $X=\frac{e^{t\lambda}}{\lambda}(-\lambda e^x(v-\lambda)-\lambda^2-e^x\log u)$, $T=\frac{e^{x+t\lambda}}{\lambda}(v+(-1+u)\lambda)$
		$\xi=e^x$, $\phi=0$, $X=-e^x\log u$, $T=e^x v$
$\frac{1}{u^4}$	$\frac{1}{u^3}$	$\xi=\frac{1}{\lambda}e^{\lambda t-3x}$, $\phi=-\frac{1}{3}e^{\lambda t-3x}$, $X=\frac{e^{-3x+t\lambda}}{9u^3\lambda}(3+u^3\lambda(3v+\lambda)-e^{3x}u^3(3+\lambda^2))$, $T=\frac{e^{-3x+t\lambda}}{3\lambda}(3v+(1-u)\lambda)$
		$\xi=e^{-3x}$, $\phi=0$, $X=\frac{1}{3}(-1+\frac{e^{-3x}}{u^3})$, $T=v\,e^{-3x}$
1	0 or 1	$\xi=e^{\lambda t}u$, $\phi=e^{\lambda t}v$, $X=\frac{-e^{t\lambda}}{2}(u^2+v^2)$, $T=e^{t\lambda}uv$
	0 or u	$\xi=f(x,t)$, $\phi=g(x,t)$, $X=-f(x,t)u-g(x,t)v$, $T=f(x,t)v+g(x,t)u$, where $\frac{\partial f}{\partial t}=\frac{\partial g}{\partial x}$, $f-\frac{\partial f}{\partial x}=\lambda g-\frac{\partial g}{\partial t}$
1	1	$\xi=f(x,t)$, $\phi=g(x,t)$, $X=-f(x,t)u-g(x,t)v-\int^x f(s,t)\,ds$, $T=f(x,t)v+g(x,t)u$, where $\frac{\partial f}{\partial t}=\frac{\partial g}{\partial x}$, $e^{\lambda t}-\frac{\partial f}{\partial x}=\lambda g-\frac{\partial g}{\partial t}$

3 Conclusion

In this paper, we present conservation laws of the equations which combine the nonlinear telegraph equations and the diffusion-convection equations. Some ordinary conservation laws are derived. The cornerstone of presented investigation is the application of symmetry classifications of the variable-coefficient wave equations $u_{xx}=H(x)u_{tt}$ and the differential characteristic set software package. The differential characteristic set method seems to be an efficient tool in solving the differential polynomial system. Furthermore, Bluman *et al*[18] have shown how to obtain new conservation laws from known ones when a system of PDEs admits the symmetry transformation. In the future work, we will adopt these ideas to get more conservation laws.

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