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Applications of *F*-expansion to Periodic Wave Solutions for Variant Boussinesq Equations*

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Abstract *We present an F-expansion method for finding periodic wave solutions of nonlinear evolution equations in mathematical physics, which can be thought of as a concentration of extended Jacobi elliptic function expansion method proposed recently. By using the F-expansion, without calculating Jacobi elliptic functions, we obtain simultaneously many periodic wave solutions expressed by various Jacobi elliptic functions for the variant Boussinesq equations. When the modulus m approaches 1 and 0, the hyperbolic function solutions (including the solitary wave solutions) and trigonometric solutions are also given respectively.*

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1 Introduction

In the present paper we will use the F-expansion to discuss the variant Boussinesq equations[1] in the form

$$H_t + (Hu)_x + u_{xxx} = 0\,, \tag{1a}$$

$$u_t + H_x + uu_x = 0\,. \tag{1b}$$

As a model for water waves, u is the velocity and H the total depth. Recently in Refs. [2] and [3] a number of periodic wave solutions to Eqs. (1) were obtained. It will be shown that more types of periodic wave solutions, than those obtained in Refs. [2] and [3], can be derived by using the F-expansion.

During the past three years many solutions expressed by various Jacobi elliptic functions for a wide class of nonlinear evolution equations in mathematical physics have been obtained by Jacobi elliptic function expansion method[4−8] and F-expansion method,[9−12] which can be thought of as a generalization of Jacobi elliptic function expansion since F here stands for any of Jacobi elliptic functions.

The rest of the paper is organized as follows. In Sec. 2, concentration formulas of the solutions to Eqs. (1) are derived. In Sec. 3, many periodic wave solutions expressed by various Jacobi elliptic functions to Eqs. (1) are obtained. In Sec. 4, under the limit case when the modulus $m \to 1$ and $m \to 0$, the solitary wave solutions and trigonometric solutions are given respectively. Finally in Sec. 5, some comments for the method are made.

2 Concentration Formulas of Solutions

(i) We seek traveling wave solutions for Eqs. (1) in the form

$$H(x,t) = H(\xi), \quad u(x,t) = u(\xi), \quad \xi = k(x - \lambda t)\,, \tag{2}$$

where k and λ are constants to be determined. Substituting Eqs. (2) into Eqs. (1) yields an ordinary differential equations (ODEs) for $H(\xi)$ and $u(\xi)$,

$$-\lambda H'(\xi) + (H(\xi)u(\xi))' + k^2 u'''(\xi) = 0\,, \tag{3a}$$

$$-\lambda u'(\xi) + H'(\xi) + u(\xi)u'(\xi) = 0\,. \tag{3b}$$

In order to simplify ODEs (3) further, integrating Eq. (3b) once yields

$$H(\xi) = \lambda u(\xi) - \frac{1}{2}u^2(\xi) + C\,, \tag{4}$$

where C is an integration constant.

Substituting expression (4) into Eq. (3a) we have an ODE for $u(\xi)$,

$$(C - \lambda^2)u' + 3\lambda uu' - \frac{3}{2}u^2u' + k^2u''' = 0\,. \tag{5}$$

Now the main goal is to solve the ODE (5).

(ii) Considering the homogeneous balance between u''' and u^2u' in ODE (5) we suppose that the solution of ODE (5) can be expressed by

$$u(\xi) = a_0 + a_{-1}F^{-1}(\xi) + a_1F(\xi) + b_1G(\xi), \quad a_1 \neq 0\,, \tag{6}$$

where a_0, a_{-1}, a_1, b_1 are constants to be determined, $F(\xi)$ and $G(\xi)$ satisfy the first order ODE,

$$F'^2 = P_1F^4 + Q_1F^2 + R_1\,,$$
$$F'' = 2P_1F^3 + Q_1F\,, \tag{7}$$

$$G'^2 = P_2G^4 + Q_2G^2 + R_2\,,$$
$$G'' = 2P_2G^3 + Q_2G\,, \tag{8}$$

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respectively, and

$$G^2 = \mu F^2 + \nu\,, \tag{9}$$

where P_i, Q_i, R_i $(i = 1, 2)$, and μ, ν are constants. In fact, ODEs (7) and (8) are all in Appendix B, if P_i, Q_i, R_i $(i = 1, 2)$ are so selected that the solutions of ODEs (7) and (8) are different Jacobi elliptic functions satisfying the relation (9) (see Appendix A).

Inserting Eq. (6) into ODE (5), cancelling $F'(\xi)$ and collecting all terms with F^iG^j, $(i = 0, 1, \ldots, 7;\ j = 0, 1)$ together, the left-hand side of Eq. (5) is converted into a finite series in F^iG^j, setting the coefficients of F^iG^j to zero yields a system of algebraic equations for a_0, a_{-1}, a_1, b_1, k, and λ as

$$F^7 : -3\mu b_1(3a_1^2 + \mu b_1^2 - 4k^2P_1) = 0\,, \tag{10a}$$

$$F^6G : -3a_1(a_1^2 + 3\mu b_1^2 - 4k^2P_1) = 0\,, \tag{10b}$$

$$F^6 : 12\mu(\lambda - a_0)a_1b_1 = 0\,, \tag{10c}$$

$$F^5G : 6(\lambda - a_0)(a_1^2 + \mu b_1^2) = 0\,, \tag{10d}$$

$$\begin{aligned} F^5 : b_1(&2C\mu - 2\lambda^2\mu + 6\lambda\mu a_0 - 3\mu a_0^2 \\ &- 6\mu a_{-1}a_1 - 6\nu a_1^2 - 3\mu\nu b_1^2 + 6k^2\nu P_1 \\ &+ 2k^2\mu Q_1) = 0\,, \end{aligned} \tag{10e}$$

$$\begin{aligned} F^4G : &-3a_{-1}a_1^2 - 3\mu a_{-1}b_1^2 + a_1(2C - 2\lambda^2 + 6\lambda a_0 \\ &- 3a_0^2 - 3\nu b_1^2 + 2k^2Q_1) = 0\,, \end{aligned} \tag{10f}$$

$$F^4 : 6\nu(\lambda - a_0)a_1b_1 = 0\,, \tag{10g}$$

$$F^3 : 3\mu a_{-1}^2 b_1 = 0\,, \tag{10h}$$

$$\begin{aligned} F^2G : a_{-1}(&-2C + 2\lambda^2 - 6\lambda a_0 + 3a_0^2 \\ &+ 3a_{-1}a_1 + 3\nu b_1^2 - 2k^2Q_1) = 0\,, \end{aligned} \tag{10i}$$

$$F^2 : 6\nu(-\lambda + a_0)a_{-1}b_1 = 0\,, \tag{10j}$$

$$FG : 6a_{-1}^2(-\lambda + a_0) = 0\,, \tag{10k}$$

$$F : 6\nu a_{-1}^2 b_1 = 0\,, \tag{10l}$$

$$G : 3a_{-1}(a_{-1}^2 - 4k^2R_1) = 0\,. \tag{10m}$$

It should be noted that we have used the identities $P_1 = P_2\mu$ and $3P_2\nu = Q_1 - Q_2$ (see Appendix C) for obtaining Eqs. (10).

(iii) Solving the algebraic equations (10) we have three solutions, which are

$$\text{a)}\quad a_0 = \lambda, \quad a_{-1} = b_1 = 0, \quad a_1 = \pm 2k\sqrt{P_1}\,, \tag{11a}$$

λ and k satisfy the relation

$$\lambda^2 + 2C + 2k^2Q_1 = 0\,, \tag{11b}$$

$$\text{b)}\quad a_0 = \lambda, \quad a_1 = \pm 2k\sqrt{P_1}\,,$$

$$a_{-1} = \pm 2k\sqrt{R_1}, \quad b_1 = 0\,, \tag{12a}$$

where λ and k satisfy the relation

$$\lambda^2 + 2C + 2k^2(Q_1 \pm 6\sqrt{P_1R_1}) = 0\,; \tag{12b}$$

(if $a_{-1}a_1 > 0$, taking "$-$"; otherwise taking "$+$");

$$\text{c)}\quad a_0 = \lambda, \quad a_1 = \pm k\sqrt{P_1}\,,$$

$$a_{-1} = 0, \quad b_1 = \pm\frac{k\sqrt{\mu P_1}}{\mu}\,, \tag{13a}$$

where λ and k satisfy the relation

$$\lambda^2 + 2C + 2k^2\Big(Q_1 - \frac{3\nu}{2\mu}P_1\Big) = 0\,. \tag{13b}$$

Substituting Eq. (11a) into Eqs. (6) and (4) we obtain the first kind of concentration formulas of solutions to Eqs. (1), which are expressed by single $F(\xi)$,

$$u(x,t) = \lambda \pm 2k\sqrt{P_1}F(\xi)\,, \tag{14a}$$

$$H(x,t) = -k^2Q_1 - 2k^2P_1F^2(\xi)\,, \tag{14b}$$

where $P_1 > 0, \lambda$ and k satisfy Eq. (11b), $\xi = k(x - \lambda t)$.

Substituting Eq. (12a) into Eqs. (6) and (4), we obtain the second kind of concentration formulas of solutions to Eqs. (1), which are expressed by single $F(\xi)$,

$$u(x,t) = \lambda \pm 2k\sqrt{P_1}F(\xi) \pm 2k\sqrt{R_1}F^{-1}(\xi)\,, \tag{15a}$$

$$\begin{aligned} H(x,t) = &\pm 2k^2\sqrt{P_1R_1} - k^2Q_1 - 2k^2P_1F^2(\xi) \\ &- 2k^2R_1F^{-2}(\xi)\,, \end{aligned} \tag{15b}$$

where $P_1 > 0$, $R_1 > 0$, λ and k satisfy Eq. (12b), $\xi = k(x - \lambda t)$. The sign "$\pm$" in formula Eq. (15a) means that all possible combinations of "$+$" and "$-$" can be taken; but the sign "$\pm$" in the formula (15b) depends upon the sign in the first formula, to be more precise, if the two terms in first formula are of the same sign, then it is taken for "$+$" in the second formula, and if the two terms in the first formula are of different signs, then it is taken for "$-$" in the second formula. Hereafter, the sign "$\pm$" always stands for this meaning in the similar circumstances.

Substituting Eq. (13a) into Eqs. (6) and (4), we obtain the third kind of concentration formulas of solutions to Eqs. (1), which are expressed by $F(\xi)$ and $G(\xi)$,

$$u(x,t) = \lambda \pm k\sqrt{P_1}F(\xi) \pm \frac{k\sqrt{\mu P_1}}{\mu}G(\xi)\,, \tag{16a}$$

$$\begin{aligned} H(x,t) = \frac{k^2}{\mu}[\,&\nu P_1 - \mu Q_1 - \mu P_1F^2(\xi) \\ &\mp \sqrt{\mu}P_1F(\xi)G(\xi)]\,, \end{aligned} \tag{16b}$$

where $P_1 > 0$, $\mu > 0$, λ and k satisfy Eq. (13b), $\xi = k(x - \lambda t)$. The sign "$\pm$" in formula (16a) means that all possible combinations of "$+$" and "$-$" can be taken; but the sign "$\pm$" in the formula (16b) depends upon the sign in the first formula. To be more precise, if the two terms in the first formula are of the same sign, then it is taken for "$-$" in the second formula, and if the two terms in the first formula are of different signs, then it is taken for "$+$" in the second formula. Hereafter, the signs "$\pm$" always stand for this meaning in the similar circumstances.

3 Periodic Wave Solutions

In this section, with the aid of Appendices A and B, from the concentration formulas of the solutions we will dissociate many periodic wave solutions expressed by various Jacobi elliptic functions to Eqs. (1).

(i) From Appendix B, taking $F(\xi) = \operatorname{sn}\xi$, $P_1 = m^2$, $Q_1 = -(1+m^2)$, and inserting them into Eqs. (14) yields

$$u_1(x,t) = \lambda \pm 2km\,\operatorname{sn}(\xi)\,,$$
$$H_1(x,t) = k^2[1+m^2-2m^2\operatorname{sn}^2(\xi)]\,.$$

Similarly, we have other periodic wave solutions to Eqs. (1).

$$u_2(x,t) = \lambda \pm 2km\,\operatorname{cd}(\xi)\,,$$
$$H_2(x,t) = k^2[1+m^2-2m^2\operatorname{cd}^2(\xi)]\,,$$
$$u_3(x,t) = \lambda \pm 2k\,\operatorname{ns}(\xi)\,,$$
$$H_3(x,t) = k^2[1+m^2-2\,\operatorname{ns}^2(\xi)]\,,$$
$$u_4(x,t) = \lambda \pm 2k\,\operatorname{dc}(\xi)\,,$$
$$H_4(x,t) = k^2[1+m^2-2\,\operatorname{dc}^2(\xi)]\,.$$

In u_i and H_i $(i=1,2,3,4)$,

$$\lambda = \pm\sqrt{2k^2(1+m^2)-2C}\,,$$

λ and k are arbitrary constants provided that $C \le k^2(1+m^2)$, $\xi = k(x-\lambda t)$.

$$u_5(x,t) = \lambda \pm 2k\sqrt{1-m^2}\,\operatorname{nc}(\xi)\,,$$
$$H_5(x,t) = -k^2[2m^2-1+2(1-m^2)\,\operatorname{nc}^2(\xi)]\,,$$
$$u_6(x,t) = \lambda \pm 2k\,\operatorname{ds}(\xi)\,,$$
$$H_6(x,t) = -k^2[2m^2-1+2\,\operatorname{ds}^2(\xi)]\,.$$

In (u_5, H_5) and (u_6, H_6), $\lambda = \pm\sqrt{2k^2(1-2m^2)-2C}$, λ and k are arbitrary constants provided that $C \le k^2(1-2m^2)$, $\xi = k(x-\lambda t)$.

$$u_7(x,t) = \lambda \pm 2k\,\operatorname{cs}(\xi)\,,$$
$$H_7(x,t) = -k^2[2-m^2+2\,\operatorname{cs}^2(\xi)]\,,$$
$$u_8(x,t) = \lambda \pm 2k\sqrt{1-m^2}\,\operatorname{sc}(\xi)\,,$$
$$H_8(x,t) = -k^2[2-m^2+2(1-m^2)\,\operatorname{sc}^2(\xi)]\,.$$

In (u_7, H_7) and (u_8, H_8), $\lambda = \pm\sqrt{2k^2(m^2-2)-2C}$, λ and k are arbitrary constants provided that $C \le k^2(m^2-2), \xi = k(x-\lambda t)$.

(ii) With the aid of Appendix B, inserting

$$F = \operatorname{sn}\xi\,, \qquad F^{-1} = \operatorname{ns}\xi\,,$$
$$P_1 = m^2, \quad Q_1 = -(1+m^2), \quad R_1 = 1$$

into Eqs. (15) yields

$$u_9(x,t) = \lambda \pm 2k\,m\,\operatorname{sn}(\xi) \pm 2k\,\operatorname{ns}(\xi)\,,$$
$$H_9(x,t) = \pm 2k^2m + k^2(1+m^2) - 2k^2m^2\operatorname{sn}^2(\xi) - 2k^2\operatorname{ns}^2(\xi)\,.$$

Similarly, we have other periodic wave solutions to Eqs. (1),

$$u_{10}(x,t) = \lambda \pm 2k\,m\,\operatorname{cd}(\xi) \pm 2k\,\operatorname{dc}(\xi)\,,$$
$$H_{10}(x,t) = \pm 2k^2m + k^2(1+m^2) - 2k^2m^2\operatorname{cd}^2(\xi) - 2k^2\operatorname{dc}^2(\xi)\,.$$

In (u_9, H_9) and (u_{10}, H_{10}),

$$\lambda = \pm\sqrt{2k^2(1+m^2\mp 6m)-2C}\,,$$

λ and k are arbitrary constants provided that $C \le k^2(1+m^2\mp 6m)$, $\xi = k(x-\lambda t)$.

$$u_{11}(x,t) = \lambda \pm 2k\sqrt{1-m^2}\,\operatorname{sc}(\xi) \pm 2k\,\operatorname{cs}(\xi)\,,$$
$$H_{11}(x,t) = \pm 2k^2\sqrt{1-m^2} - k^2(2-m^2) - 2k^2(1-m^2)\,\operatorname{sc}^2(\xi) - 2k^2\operatorname{cs}^2(\xi)\,.$$

In (u_{11}, H_{11}),

$$\lambda = \pm\sqrt{2k^2(m^2-2\mp 6\sqrt{1-m^2})-2C}\,,$$

λ and k are arbitrary constants provided that $C \le k^2(m^2-2\mp 6\sqrt{1-m^2})$, $\xi = k(x-\lambda t)$.

(iii) From Appendix A, choose $\operatorname{ns}^2\xi = \operatorname{cs}^2\xi + 1$ and set

$$F(\xi) = \operatorname{cs}(\xi), \quad G(\xi) = \operatorname{ns}(\xi), \quad \mu = 1, \quad \nu = 1\,, \tag{17}$$

and from Appendix B we find

$$P_1 = 1, \qquad Q_1 = 2-m^2\,. \tag{18}$$

Inserting Eqs. (17) and (18) into Eqs. (16) yields

$$u_{12}(x,t) = \lambda \pm k\,\operatorname{cs}(\xi) \pm k\,\operatorname{ns}(\xi)\,,$$
$$H_{12}(x,t) = k^2[m^2-1-\operatorname{cs}^2(\xi) \pm \operatorname{cs}(\xi)\operatorname{ns}(\xi)]\,.$$

After the aforementioned manner we can obtain the other results:

$$u_{13}(x,t) = \lambda \pm k\sqrt{1-m^2}\,\operatorname{sc}(\xi) \pm k\,\operatorname{dc}(\xi)\,,$$
$$H_{13}(x,t) = k^2[m^2-1-(1-m^2)\,\operatorname{sc}^2(\xi) \pm \sqrt{1-m^2}\,\operatorname{sc}(\xi)\operatorname{dc}(\xi)]\,.$$

In (u_{12}, H_{12}) and (u_{13}, H_{13}),

$$\lambda = \pm\sqrt{2k^2\Bigl(m^2-\frac{1}{2}\Bigr)-2C}\,,$$

λ, and k are arbitrary constants provided that $C \le k^2(m^2-1/2)$, $\xi = k(x-\lambda t)$.

$$u_{14}(x,t) = \lambda \pm k\,\operatorname{ds}(\xi) \pm k\,\operatorname{ns}(\xi)\,,$$
$$H_{14}(x,t) = k^2[1-m^2-\operatorname{ds}^2(\xi) \pm \operatorname{ds}(\xi)\operatorname{ns}(\xi)]\,.$$
$$u_{15}(x,t) = \lambda \pm k\sqrt{1-m^2}\,\operatorname{nc}(\xi) \pm k\,\operatorname{dc}(\xi)\,,$$
$$H_{15}(x,t) = k^2[1-m^2-(1-m^2)\operatorname{nc}^2(\xi) \pm \sqrt{1-m^2}\,\operatorname{nc}(\xi)\operatorname{dc}(\xi)]\,.$$

In (u_{14}, H_{14}) and (u_{15}, H_{15}),

$$\lambda = \pm\sqrt{2k^2(1-m^2/2)-2C}\,,$$

λ and k are arbitrary constants provided that $C \le k^2(1-m^2/2)$, $\xi = k(x-\lambda t)$.

$$u_{16}(x,t) = \lambda \pm k\,\operatorname{cs}(\xi) \pm k\,\operatorname{ds}(\xi)\,,$$
$$H_{16}(x,t) = k^2[-1-\operatorname{cs}^2(\xi) \pm \operatorname{cs}(\xi)\operatorname{ds}(\xi)]\,,$$
$$u_{17}(x,t) = \lambda \pm k\sqrt{1-m^2}\,\operatorname{sc}(\xi) \pm k\sqrt{1-m^2}\,\operatorname{nc}(\xi)\,,$$
$$H_{17}(x,t) = k^2[-1-(1-m^2)\,\operatorname{sc}^2(\xi) \pm (1-m^2)\operatorname{sc}(\xi)\,\operatorname{nc}(\xi)]\,.$$

In (u_{16}, H_{16}) and (u_{17}, H_{17}),

$$\lambda = \pm\sqrt{-k^2(1+m^2)-2C}\,,$$

λ and k are arbitrary constants provided that $C \le -(k^2/2)(1+m^2)$, $\xi = k(x-\lambda t)$.

4 Solitary Wave Solutions and Trigonometric Solutions

Some solitary wave solutions can be obtained if the modulus $m \to 1$:

$$(u_1, H_1) \to (u_{18}(x,t) = \lambda \pm 2k\tanh(\xi)\,,$$
$$H_{18}(x,t) = 2k^2\mathrm{sech}^2(\xi))\,.$$

The solitary wave solution $(u_{18}(x,t), H_{18}(x,t))$ is similar to the result in Ref. [2].

$$(u_3, H_3) \to (u_{19}(x,t) = \lambda \pm 2k\coth(\xi)\,,$$
$$H_{19}(x,t) = -2k^2\mathrm{csch}^2(\xi))\,.$$

In (u_{18}, H_{18}) and (u_{19}, H_{19}), $\lambda = \pm\sqrt{4k^2-2C}$, λ and k are arbitrary constants provided that $C \le 2k^2$, $\xi = k(x-\lambda t)$.

$$(u_6,H_6),(u_7,H_7),(u_{11},H_{11}),\quad \text{and}\quad (u_{16},H_{16}) \to$$
$$(u_{20}(x,t) = \lambda \pm 2k\,\mathrm{csch}(\xi)\,,$$
$$H_{20}(x,t) = -k^2[1+2\mathrm{csch}^2(\xi)])\,.$$

In (u_{20}, H_{20}), $\lambda = \pm\sqrt{-2k^2-2C}$, λ and k are arbitrary constants provided that $C \le -k^2$, $\xi = k(x-\lambda t)$.

$$(u_9,H_9) \to$$
$$(u_{21}(x,t) = \lambda \pm 2k\tanh(\xi) \pm 2k\coth(\xi)\,,$$
$$H_{21}(x,t) = 2k^2[\pm 1 + 1 - \tanh^2(\xi) - \coth^2(\xi)])\,.$$

In (u_{21}, H_{21}), $\lambda = \pm\sqrt{2k^2(2\mp 6)-2C}$, λ and k are arbitrary constants provided that $C \le k^2(2\mp 6)$, $\xi = k(x-\lambda t)$.

$$(u_{12}H_{12})\quad \text{and}\quad (u_{14},H_{14}) \to$$
$$(u_{22}(x,t) = \lambda \pm k\,\mathrm{csch}(\xi) \pm k\coth(\xi)\,,$$
$$H_{22}(x,t) = k^2[-\mathrm{csch}^2(\xi) \pm \mathrm{csch}(\xi)\coth(\xi)])\,.$$

In (u_{22}, H_{22}), $\lambda = \pm\sqrt{k^2-2C}$, λ and k are arbitrary constants provided that $C \le k^2/2$, $\xi = k(x-\lambda t)$.

Some trigonometric function solutions can also be obtained if the modulus $m \to 0$,

$$(u_3,H_3),(u_6,H_6),(u_9,H_9)\quad \text{and}\quad (u_{14},H_{14}) \to$$
$$(u_{23}(x,t) = \lambda \pm 2k\csc(\xi)\,,$$
$$H_{23}(x,t) = k^2[1-2\csc^2(\xi)])\,,$$
$$(u_4,H_4),(u_5,H_5),(u_{10},H_{10})\quad \text{and}\quad (u_{15},H_{15}) \to$$
$$(u_{24}(x,t) = \lambda \pm 2k\sec(\xi)\,,$$
$$H_{24}(x,t) = k^2[1-2\sec^2(\xi)])\,.$$

In (u_{23}, H_{23}) and (u_{24}, H_{24}), $\lambda = \pm\sqrt{2k^2-2C}$, λ and k are arbitrary constants provided that $C \le k^2, \xi = k(x-\lambda t)$.

$$(u_7,H_7) \to (u_{25}(x,t) = \lambda \pm 2k\,\cot(\xi)\,,$$
$$H_{25}(x,t) = -2k^2\csc^2(\xi))\,.$$
$$(u_8,H_8) \to (u_{26}(x,t) = \lambda \pm 2k\tan(\xi)\,,$$
$$H_{26}(x,t) = -2k^2\sec^2(\xi))\,.$$

In (u_{25}, H_{25}) and (u_{26}, H_{26}), $\lambda = \pm\sqrt{-4k^2-2C}$, λ and k are arbitrary constants provided that $C \le -2k^2$, $\xi = k(x-\lambda t)$.

$$(u_{11},H_{11}) \to$$
$$(u_{27}(x,t) = \lambda \pm 2k\tan(\xi) \pm 2k\cot(\xi)\,,$$
$$H_{27}(x,t) = 2k^2[\pm 1 - 1 - \tan^2(\xi) - \cot^2(\xi)])\,.$$

In (u_{27}, H_{27}), $\lambda = \pm\sqrt{2k^2(-2\mp 6)-2C}$, λ and k are arbitrary constants provided that $C \le k^2(-2\mp 6)$, $\xi = k(x-\lambda t)$.

$$(u_{12},H_{12})\quad \text{and}\quad (u_{16},H_{16}) \to$$
$$(u_{28}(x,t) = \lambda \pm k\cot(\xi) \pm k\csc(\xi)\,,$$
$$H_{28}(x,t) = k^2[-\csc^2(\xi) \pm \cot(\xi)\csc(\xi)])\,.$$
$$(u_{13},H_{13})\quad \text{and}\quad (u_{17},H_{17}) \to$$
$$(u_{29}(x,t) = \lambda \pm k\tan(\xi) \pm k\sec(\xi)\,,$$
$$H_{29}(x,t) = k^2[-\sec^2(\xi) \pm \tan(\xi)\sec(\xi)])\,.$$

In (u_{28}, H_{28}) and (u_{29}, H_{29}), $\lambda = \pm\sqrt{-k^2-2C}$, λ and k are arbitrary constants provided that $C \le -k^2/2$, $\xi = k(x-\lambda t)$.

5 Conclusion

The F-expansion includes every one of extended Jacobi elliptic function expansions, so the former can be regarded as a concentration of the latter. The main advantages of the F-expansion are that firstly, by calculating $F(\xi)$ and $G(\xi)$ which are different solutions satisfying relation (9) of ODEs (7) and (8), one can derive the general form of traveling wave solutions to the considered equations, which is called concentration formulas of the solutions, instead of performing tedious and repeated calculations of Jacobi elliptic functions; Secondly, with the aid of Appendices A and B, from the concentration formulas of the solutions, one can simultaneously dissociate many periodic wave solutions expressed by various Jacobi elliptic functions. By using the F-expansion, many kinds of exact solutions are obtained. It seems that the F-expansion is more effective and simple than extended Jacobi elliptic function expansion proposed more recently. The method can be applied to other nonlinear evolution equations.

Appendix A

Jacobi elliptic functions with modulus m $(0 < m < 1)$ have the identity relations in the form $G^2 = \mu F^2 + \nu$:

$$\mathrm{cn}^2\xi = -\mathrm{sn}^2\xi + 1, \qquad \mathrm{dn}^2\xi = -m^2\mathrm{sn}^2\xi + 1\,,$$
$$\mathrm{cd}^2\xi = \frac{m^2-1}{m^2}\mathrm{nd}^2\xi + \frac{1}{m^2}, \qquad \mathrm{cd}^2\xi = (m^2-1)\,\mathrm{sd}^2\xi + 1\,,$$
$$\mathrm{dn}^2\xi = m^2\mathrm{cn}^2\xi + (1-m^2), \qquad \mathrm{nd}^2\xi = m^2\,\mathrm{sd}^2\xi + 1\,,$$
$$\mathrm{ns}^2\xi = \mathrm{cs}^2\xi + 1, \qquad \mathrm{ns}^2\xi = \mathrm{ds}^2\xi + m^2\,,$$

$\mathrm{ds}^2\xi = \mathrm{cs}^2\xi + (1-m^2)$, $\mathrm{nc}^2\xi = \mathrm{sc}^2\xi + 1$, $\mathrm{dc}^2\xi = (1-m^2)\,\mathrm{nc}^2\xi + m^2$, $\mathrm{dc}^2\xi = (1-m^2)\,\mathrm{sc}^2\xi + 1$.

Appendix B

Talbe 1 Relations between values of (P, Q, R) and corresponding $F(\xi)$ in ODE $F'^2 = PF^4 + QF^2 + R$.

P	Q	R	$F'^2 = PF^4 + QF^2 + R$	$F(\xi)$
m^2	$-(1+m^2)$	1	$F'^2 = (1-F^2)(1-m^2F^2)$	$\mathrm{sn}\,\xi$, $\mathrm{cd}\xi = \dfrac{\mathrm{cn}\,\xi}{\mathrm{dn}\,\xi}$
$-m^2$	$2m^2-1$	$1-m^2$	$F'^2 = (1-F^2)(m^2F^2+1-m^2)$	$\mathrm{cn}\,\xi$
-1	$2-m^2$	m^2-1	$F'^2 = (1-F^2)(F^2+m^2-1)$	$\mathrm{dn}\,\xi$
1	$-(1+m^2)$	m^2	$F'^2 = (1-F^2)(m^2-F^2)$	$\mathrm{ns}\,\xi$, $\mathrm{dc}\,\xi = \dfrac{\mathrm{dn}\,\xi}{\mathrm{cn}\,\xi}$
$1-m^2$	$2m^2-1$	$-m^2$	$F'^2 = (1-F^2)[(m^2-1)F^2-m^2]$	$\mathrm{nc}\,\xi = (\mathrm{cn}\,\xi)^{-1}$
m^2-1	$2-m^2$	-1	$F'^2 = (1-F^2)[(1-m^2)F^2-1]$	$\mathrm{nd}\,\xi = (\mathrm{dn}\,\xi)^{-1}$
$1-m^2$	$2-m^2$	1	$F'^2 = (1+F^2)[(1-m^2)F^2+1]$	$\mathrm{sc}\,\xi = \dfrac{\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi}$
$-m^2(1-m^2)$	$2m^2-1$	1	$F'^2 = (1+m^2F^2)[1+(m^2-1)F^2]$	$\mathrm{sd}\,\xi = \dfrac{\mathrm{sn}\,\xi}{\mathrm{dn}\,\xi}$
1	$2-m^2$	$1-m^2$	$F'^2 = (1+F^2)[F^2+1-m^2]$	$\mathrm{cs}\,\xi = \dfrac{\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi}$
1	$2m^2-1$	$-m^2(1-m^2)$	$F'^2 = (F^2+m^2)[F^2+m^2-1]$	$\mathrm{ds}\,\xi = \dfrac{\mathrm{dn}\,\xi}{\mathrm{sn}\,\xi}$

Appendix C Identities Among μ, ν in Appendix A and P_i, Q_i, R_i $(i = 1, 2)$ in Appendix B

If $F(\xi)$ and $G(\xi)$ satisfy the first order ODE

$$F'^2 = P_1F^4 + Q_1F^2 + R_1 \quad \text{and} \quad G'^2 = P_2G^4 + Q_2G^2 + R_2\,,$$

respectively, and

$$G^2 = \mu F^2 + \nu\,,$$

where P_i, Q_i, R_i $(i = 1, 2)$, μ, and ν are constants, then the following identities hold,

$$P_1 = P_2\mu, \qquad Q_1 = 3P_2\nu + Q_2, \qquad \mu R_1 = \nu(2P_2\nu + Q_2), \qquad R_2 = -P_2\nu^2 - Q_2\nu\,,$$

which can be examined by using Appendices A and B. The identities are very useful when solving the algebraic equations obtained by using extended F-expansion method.

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