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Symmetry Reductions and Explicit Solutions for a Generalized Zakharov–Kuznetsov Equation*

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Abstract *Two types of symmetry of a generalized Zakharov–Kuznetsov equation are obtained via a direct symmetry method. By selecting suitable parameters occurring in the symmetries, we also find some symmetry reductions and new explicit solutions of the generalized Zakharov–Kuznetsov equation.*

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1 Introduction

In order to describe complex phenomena in various fields of science, some important nonlinear evolution equations have been established, such as Kadomtsev–Petviashvilli (KP) equation, Korteweg-de Vries (KdV) equation, and Zakharov–Kuznetson (ZK) equation and so on.[1−4] The KdV equation is a model to describe and identify mechanisms for atmospheric blocking, cyclogenesis, meandering of the oceanic screams and so on. The ZK equation investigated by Zakharov–Kuzentsov[5] governs the behavior of weakly nonlinear ion-acoustic waves in plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field. Moreover ZK equation supports stable solitary waves, which makes the ZK equation a very attractive model equation for the study of vortices in geophysical flows. In this letter, we shall consider the following generalized Zakharov–Kuznetson (GZK) equation,[6]

$$u_t + auu_x + bu^2u_x + cu_{xxx} + du_{xyy} = 0\,, \tag{1}$$

where a, b, c, and d are arbitrary constants. Equation (1) includes considerable interesting equations, such as KdV equation, mKdV equation, ZK equation, and mZK equation. For the special case $a = 1$, $b = 0$, and $c = 1$, Moussa[7] derived the similarity reductions and some explicit solutions of Eq. (1). Exact solutions with solitons and periodic structures of Eq. (1) with $b = 0$ were obtained by sine-cosine algorithm in Ref. [8]. Our main aim is to present more abundant solutions of the GZK equation (1), which will be helpful to a better understanding of objective laws described by the nonlinear evolution equation.

Using a direct method,[9,10] we arrive at two types of symmetry of Eq. (1), which include results in Ref. [7]. At the same time some symmetry reductions and new similarity solutions are obtained, including polynomial solutions, triangular function solutions, hyperbolic function solutions and so on.

2 Symmetry of GZK Equation

To find symmetry $\sigma(x,y,t,u)$ of Eq. (1), we set

$$\sigma = \alpha(x,y,t)u_t + \beta(x,y,t)u_x + \gamma(x,y,t)u_y + \delta(x,y,t)u + e(x,y,t)\,, \tag{2}$$

where α, β, γ, δ, and e are functions to be determined later. Substituting Eq. (2) into the following equation

$$\sigma_t + au\sigma_x + a\sigma u_x + 2buu_x\sigma + bu^2\sigma_x + c\sigma_{xxx} + d\sigma_{xyy} = 0\,. \tag{3}$$

Requiring $u = u(x,y,t)$ in the above equation satisfies Eq. (1), we expand Eq. (3) with the help of *Maple*. To ensure the expansion of Eq. (3) is true for an arbitrary solution u, it is necessary to take the coefficients of u_{xyyyy} and u_{xxyyy} to be zero, so we have

$$\alpha_x = 0\,, \quad \alpha_y = 0\,, \quad \text{i.e.} \quad \alpha = \alpha(t)\,. \tag{4}$$

Substituting $\alpha = \alpha(t)$ into the expansion of Eq. (3), we set the coefficient of u_{yyy} to be zero, then

$$\gamma_x = 0\,, \quad \text{i.e.} \quad \gamma = \gamma(y,t)\,. \tag{5}$$

According to the same method step by step, it leads to

$$\beta_y = 0\,, \quad \delta_x = 0\,, \quad \beta_{xx} = 0\,, \quad \gamma_t = 0\,, \quad e_x = 0\,. \tag{6}$$

Equation (3) can be reduced to

$$(3\beta_x - \alpha_t)u_{xxx} + (d\delta_{yy} + \beta_t + ae)u_x + (2d\delta_y + d\gamma_{yy})u_{xy} + (\beta_x - \alpha_t + 2\gamma_y)u_{xyy} + (\delta_t + ae_x)u + b(\beta_x - \alpha_t + 2\delta)u^2u_x + (2be + a\beta_x + a\delta - a\alpha_t)uu_x = 0\,. \tag{7}$$

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From Eqs. (4) $\sim$ (7), we arrive at

(i) $\alpha = c_1 t + c_2\,,\quad \beta = \frac{1}{3}c_1 x - \frac{a^2}{6b}c_1 t + c_4\,,\quad \gamma = \frac{1}{3}c_1 y + c_3\,,\quad \delta = \frac{1}{3}c_1\,,\quad e = \frac{a}{6b}c_1\,,\quad (b \neq 0)\,,$

(ii) $\alpha = c_1 t + c_2\,,\quad \beta = \frac{1}{3}c_1 x - a c_5 t + c_4\,,\quad \gamma = \frac{1}{3}c_1 y + c_3\,,\quad \delta = \frac{2}{3}c_1\,,\quad e = c_5\,,\quad (b = 0)\,,$

where c_1, c_2, c_3, c_4, and c_5 are arbitrary constants. Therefore we obtain two types of symmetry of GZK equation (1),

$$\sigma_1 = (c_1 t + c_2)u_t + \Big(\frac{1}{3}c_1 x - a c_5 t + c_4\Big)u_x + \Big(\frac{1}{3}c_1 y + c_3\Big)u_y + \frac{2}{3}c_1 u + c_5\,,\quad (b = 0)\,, \tag{8}$$

$$\sigma_2 = (c_1 t + c_2)u_t + \Big(\frac{1}{3}c_1 x - \frac{a^2}{6b}c_1 t + c_4\Big)u_x + \Big(\frac{1}{3}c_1 y + c_3\Big)u_y + \frac{1}{3}c_1 u + \frac{a}{6b}c_1\,,\quad (b \neq 0)\,. \tag{9}$$

Hence σ_1 and σ_2 can generate groups of fibre-preserving transformations, since α and β are independent of u.

Remark 1 It is easy to see that the symmetry given in Ref. [7] is the special case of σ_1 with $a = 1$, $c_5 = -c_0$, and the symmetry σ_2 is a new one.

3 Symmetry Reductions of GZK Equation

In this section, we will discuss how to construct similarity reductions and solutions of Eq. (1) by using the compatibility of $\sigma = 0$ and Eq. (1). One first solves the equation $\sigma = 0$ to obtain invariant transformations and then substitutes these results into Eq. (1) to determine the corresponding reduced equations. Finally similarity solutions can be obtained. The special case $a = 1$ and $c_5 = -c_0$ of σ_1 has been considered in detail in Ref. [7], so we only discuss the symmetry σ_2. In order to obtain invariant variables ξ, η, and $f(\xi,\eta)$, we should solve the following characteristic equations of $\sigma_2 = 0$,

$$\frac{\mathrm{d}t}{c_1 t + c_2} = \frac{\mathrm{d}x}{c_1 x/3 - a^2 c_1 t/6b + c_4} = \frac{\mathrm{d}y}{c_1 y/3 + c_3} = -\frac{\mathrm{d}u}{c_1 u/3 + a c_1/6b}\,. \tag{10}$$

In terms of different parameters, we get the following Table 1.

Table 1 Reduced equations of GZK equation.

Case	Parameters	The invariant variables	Reduced equations (Re)
1	$c_2 = 0,\ c_1 \neq 0,\ c_3 \neq 0,\ c_4 \neq 0$	$\xi = xt^{-1/3} + (a^2/4b)t^{2/3} + (3c_4/c_1)t^{-1/3}$, $\eta = yt^{-1/3} + (3c_3/c_1)t^{-1/3}$, $u = f(\xi,\eta)t^{-1/3} - a/2b$	$3cf_{\xi\xi\xi} + 3df_{\xi\eta\eta} + 3bf^2 f_\xi$ $-\eta f_\eta - \xi f_\xi - f = 0$
2	$c_2 = c_3 = c_4 = 0,\ c_1 \neq 0$	$\xi = xt^{-1/3} + (a^2/4b)t^{2/3},\ \eta = yt^{-1/3}$, $u = f(\xi,\eta)t^{-1/3} - a/2b$	$3cf_{\xi\xi\xi} + 3df_{\xi\eta\eta} + 3bf^2 f_\xi$ $-\eta f_\eta - \xi f_\xi - f = 0$
3	$c_1 = 0,\ c_2 \neq 0,\ c_3 \neq 0,\ c_4 \neq 0$	$\xi = x - (c_4/c_2)t,\ \eta = y - (c_3/c_2)t$, $u = f(\xi,\eta)$	$cf_{\xi\xi\xi} + df_{\xi\eta\eta} + bf^2 f_\xi$ $-(c_3/c_2)f_\eta - (c_4/c_2)f_\xi + aff_\xi = 0$
4	$c_1 = c_2 = 0,\ c_3 \neq 0,\ c_4 \neq 0$	$\xi = x - (c_4/c_3)y,\ \eta = t$, $u = f(\xi,\eta)$	$[c + d(c_4^2/c_3^2)]f_{\xi\xi\xi} + bf^2 f_\xi$ $+f_\eta + aff_\xi = 0$
5	$c_1 = c_4 = 0,\ c_2 \neq 0,\ c_3 \neq 0$	$\xi = x,\ \eta = y - (c_3/c_2)t$, $u = f(\xi,\eta)$	$cf_{\xi\xi\xi} + df_{\xi\eta\eta} + bf^2 f_\xi$ $-(c_3/c_2)f_\eta + aff_\xi = 0$

Solving reduced equations (Re) in Table 1, we can get the corresponding explicit solutions.

4 Explicit Solutions of GZK Equation

Set $w = l\xi + m\eta$, where l and m are nonzero constants. Then Re1 can be written as

$$f''' + k_1 f^2 f' + k_2 w f' + k_2 f = 0\,, \tag{11}$$

where $f = f(w)$, $k_1 = b/(cl^2 + dm^2)$ and $k_2 = -1/(3cl^3 + 3dlm^2)$. It is not difficult to see that equation (11) has a solution $f(w) = 1/w$, so we can obtain the following solution of GZK equation (1),

$$u(x,y,t) = \frac{1}{(l\xi + m\eta)t^{1/3}} - \frac{a}{2b}\,, \tag{12}$$

where

$$\xi = xt^{-1/3} + \frac{a^2}{4b}t^{2/3} + \frac{3c_4}{c_1}t^{-1/3},$$
$$\eta = yt^{-1/3} + \frac{3c_3}{c_1}t^{-1/3},$$
$$b + 6(cl^2 + dm^2) = 0.$$

Using the same method, we have rational function solution of GZK in case 2

$$u(x,y,t) = \frac{1}{(l\xi + m\eta)t^{1/3}} - \frac{a}{2b}, \tag{13}$$

where

$$\xi = xt^{-1/3} + \frac{a^2}{4b}t^{2/3}, \quad \eta = yt^{-1/3},$$
$$b + 6(cl^2 + dm^2) = 0.$$

Set $w = l\xi + m\eta$, where l and m are nonzero constants. Thus Re3 can be written as the following equation with integrating twice,

$$f' = \varepsilon\sqrt{a_0 + a_1 f + a_2 f^2 + a_3 f^3 + a_4 f^4}, \tag{14}$$

where $f = f(w)$, $\varepsilon = \pm 1$, a_0 and a_1 are integral constants, and

$$a_2 = \frac{c_4 l + c_3 m}{c_2(cl^3 + dlm^2)},$$
$$a_3 = -\frac{a}{3(cl^2 + dm^2)},$$
$$a_4 = -\frac{b}{6(cl^2 + dm^2)}. \tag{15}$$

Considerable explicit solutions of Eq. (14) have been given in Ref. [11]. Therefore we can derive some new explicit solutions of GZK equation (1), including polynomial solutions, triangular function solutions, elliptic periodic solutions and so on.

Case A Polynomial Solutions

If $a_0 > 0$, $a_1 = a_2 = a_3 = a_4 = 0$, then $f = \varepsilon\sqrt{a_0}w$, and the corresponding solution of GZK equation is that

$$u(x,y,t) = \varepsilon\sqrt{a_0}(l\xi + m\eta). \tag{16}$$

If $a_1 \neq 0$, $a_2 = a_3 = a_4 = 0$, then $f = -(a_0/a_1) + (a_1/4)w^2$, and the solution of GZK equation is expressed by

$$u(x,y,t) = -\frac{a_0}{a_1} + \frac{a_1}{4}(l\xi + m\eta)^2. \tag{17}$$

Case B Rational Solutions

If $a_0 = a_1 = a_2 = a_3 = 0$, $a_4 > 0$, then $f = -\varepsilon/(\sqrt{a_4}w)$, so we have

$$u(x,y,t) = -\frac{\varepsilon}{\sqrt{a_4}(l\xi + m\eta)}. \tag{18}$$

If $a_0 = a_1 = a_2 = 0$, $a_3 \neq 0$, then $f = 1/(a_3 w^2)$, and one can get

$$u(x,y,t) = \frac{1}{a_3(l\xi + m\eta)^2}. \tag{19}$$

Case C Triangular Function Solutions

If $a_0 = a_3 = a_4 = 0$, $a_2 < 0$, then $f = -(a_1/2a_2) + (\varepsilon a_1/2a_2)\sin(\sqrt{-a_2}w)$, and equation (1) has the following solution,

$$u(x,y,t) = -\frac{a_1}{2a_2} + \frac{\varepsilon a_1}{2a_2}\sin(\sqrt{-a_2}(l\xi + m\eta)). \tag{20}$$

If $a_0 = a_1 = a_3 = 0$, $a_2 < 0$, $a_4 > 0$, then $f = \sqrt{-a_2/a_4}\sec(\sqrt{-a_2}w)$, so we get

$$u(x,y,t) = \sqrt{-\frac{a_2}{a_4}}\sec(\sqrt{-a_2}(l\xi + m\eta)). \tag{21}$$

If $a_1 = a_3 = 0$, $a_0 = a_2^2/4a_4$, $a_2 > 0$, $a_4 > 0$, then $f = \varepsilon\sqrt{a_2/a_4}\tan(\sqrt{a_2/2}\,w)$, so one can derive

$$u(x,y,t) = \varepsilon\sqrt{\frac{a_2}{a_4}}\tan\Big(\sqrt{\frac{a_2}{2}}(l\xi + m\eta)\Big). \tag{22}$$

If $a_0 = a_1 = a_4 = 0$, $a_2 < 0$, $a_3 \neq 0$, then $f = -(a_2/a_3)\sec^2(\sqrt{-a_2}w/2)$, and equation (1) has the following solution:

$$u(x,y,t) = -\frac{a_2}{a_3}\sec^2\Big(\frac{\sqrt{-a_2}}{2}(l\xi + m\eta)\Big). \tag{23}$$

If $a_0 = a_1 = 0$, $a_2 < 0$, then

$$f = -\frac{a_2\sec^2(\sqrt{-a_2}w/2)}{2\varepsilon\sqrt{-a_2 a_4}\tan(\sqrt{-a_2}w/2) + a_3}$$

and we have

$$u(x,y,t) = -\frac{a_2\sec^2(\sqrt{-a_2}(l\xi + m\eta)/2)}{2\varepsilon\sqrt{-a_2 a_4}\tan(\sqrt{-a_2}(l\xi + m\eta)/2) + a_3}. \tag{24}$$

Case D Hyperbolic Function Solutions

If $a_0 = a_3 = a_4 = 0$, $a_2 > 0$, then $f = -(a_1/2a_2) + (\varepsilon a_1/2a_2)\sinh(2\sqrt{a_2}w)$, and one has the following solution

$$u(x,y,t) = -\frac{a_1}{2a_2} + \frac{\varepsilon a_1}{2a_2}\sinh(2\sqrt{a_2}(l\xi + m\eta)). \tag{25}$$

If $a_0 = a_1 = a_3 = 0$, $a_2 > 0$, and $a_4 < 0$, then $f = \sqrt{-a_2/a_4}\sinh(w)$, so we derive

$$u(x,y,t) = \sqrt{-\frac{a_2}{a_4}}\sinh(l\xi + m\eta). \tag{26}$$

If $a_1 = a_3 = 0$, $a_0 = a_2^2/4a_4$, $a_2 < 0$, $a_4 > 0$, then $f = \varepsilon\sqrt{-a_2/a_4}\tanh(\sqrt{-a_2/2}\,w)$, so one can get

$$u(x,y,t) = \varepsilon\sqrt{-\frac{a_2}{a_4}}\tanh\Big(\sqrt{-\frac{a_2}{2}}(l\xi + m\eta)\Big). \tag{27}$$

If $a_1 = a_3 = 0$, $a_0 = a_2^2/4a_4$, $a_2 < 0$, $a_4 > 0$, then $f = -(a_2/a_3)\,\mathrm{sech}^2(\sqrt{a_2}w/2)$, and equation (1) has the following solution

$$u(x,y,t) = -\frac{a_2}{a_3}\mathrm{sech}^2\Big(\frac{\sqrt{a_2}}{2}(l\xi + m\eta)\Big). \tag{28}$$

Case E Elliptic Periodic Solutions

If $a_1 = a_3 = 0$, $a_2 > 0$, $a_4 < 0$, and $a_0 = a_2^2 k^2(1 - k^2)/(a_4(2k^2 - 1)^2)$, then

$$f = \sqrt{-\frac{a_2 k^2}{a_4(2k^2 - 1)}}\,\mathrm{cn}\Big(\sqrt{\frac{a_2}{(2k^2 - 1)}}w\Big),$$

and we can derive solution of Eq. (1),

$$u(x,y,t)=\sqrt{-\frac{a_2k^2}{a_4(2k^2-1)}}\,\mathrm{cn}\Big(\sqrt{\frac{a_2}{2k^2-1}}(l\xi+m\eta)\Big)\,. \quad (29)$$

If $a_1 = a_3 = 0$, $a_2 > 0$, $a_4 < 0$, $a_0 = a_2^2(1-k^2)/(a_4(2-k^2)^2)$, then

$$f=\sqrt{-\frac{k^2}{a_4(2-k^2)}}\,\mathrm{dn}\Big(\sqrt{\frac{a_2}{2-k^2}}w\Big)\,,$$

and one can get

$$u(x,y,t)=\sqrt{-\frac{k^2}{a_4(2-k^2)}}\,\mathrm{dn}\Big(\sqrt{\frac{a_2}{2-k^2}}(l\xi+m\eta)\Big)\,. \quad (30)$$

If $a_1 = a_3 = 0$, $a_2 < 0$, $a_4 > 0$, $a_0 = a_2^2k^2/(a_4(k^2+1)^2)$, then

$$f=\varepsilon\sqrt{-\frac{a_2k^2}{a_4(k^2+1)}}\,\mathrm{sn}\Big(\sqrt{-\frac{a_2}{k^2+1}}w\Big),$$

so we have

$$u(x,y,t)=\varepsilon\sqrt{-\frac{a_2k^2}{a_4(k^2+1)}}\,\mathrm{sn}\Big(\sqrt{-\frac{a_2}{k^2+1}}(l\xi+m\eta)\Big), \quad (31)$$

where $\xi = x-(c_4/c_2)t$, $\eta = y-(c_3/c_2)t$ and k is modulus of elliptic function.

With the same method we can get explicit solutions in case 4 and case 5. Here we omit them.

Remark 2 It can be seen that when setting $l = m = 1$, $(c_3+c_4)/c_2 = c$, exact solutions (23) and (28) can be reduced to solutions (30) and (32) in Ref. [8]. It shows that we generalize the results of the paper [8].

5 Conclusions

The basic goal of this work has been the study of a generalized Zakharov–Kuznetson equation, which is important in mathematics and physics. According to the compatibility of symmetry $\sigma = 0$ and Eq. (1), we obtain several families of explicit solutions of GZK equation, which include triangular type solutions, kink shaped solitary wave solutions, bell shaped solitary wave solutions, elliptic periodic solutions and so on. These exact solutions might provide a useful help for physicists to study more complicated physical phenomena and for mathematicians to check on the accuracy and reliability of numerical algorithm.

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