Commun. Theor. Phys. (Beijing, China) **45** (2006) pp. 405–410
© International Academic Publishers
Vol. 45, No. 3, March 15, 2006

A Hierarchy of Integrable Lattice Soliton Equations and New Integrable Symplectic Map*

SUN Ye-Peng,[1] CHEN Deng-Yuan,[1] and XU Xi-Xiang[2]
[1]Department of Mathematics, Shanghai University, Shanghai 200444, China
[2]Collega of Science, Shandong University of Science and Technology, Qingdao 266510, China

(Received June 17, 2005)

Abstract *Starting from a discrete spectral problem, a hierarchy of integrable lattice soliton equations is derived. It is shown that the hierarchy is completely integrable in the Liouville sense and possesses discrete bi-Hamiltonian structure. A new integrable symplectic map and finite-dimensional integrable systems are given by nonlinearization method. The binary Bargmann constraint gives rise to a Bäcklund transformation for the resulting integrable lattice equations. At last, conservation laws of the hierarchy are presented.*

PACS numbers: 02.30.Ik
Key words: lattice soliton equation, discrete Hamiltonian structure, integrable symplectic map

1 Introduction

It almost becomes a curiosity to find new nonlinear integrable models due to difficulty in the construction of integrable models. Many nonlinear integrable lattice equations have been proposed and discussed, for instance Ablowitz–Ladik lattice,[1] Toda lattice,[2] the differential-difference KdV equation,[3] and so on.[4−15]

Finite-dimensional integrable systems are closely connected with infinite-dimensional integrable systems described by soliton equations. A systemic approach, so-called nonlinearization method of Lax pairs, has been developed to generate finite-dimensional integrable systems from soliton equations.[16−19] Then, the binary nonlinearization method has been proposed,[20−23] which involves both the Lax pairs and the adjoint Lax pairs of soliton equations. Up to now, much work has been carried out in the study of the nonlinearization of the Lax pairs for continuous soliton equations.[16−27] However, there has been little discussion about the nonlinearization of the Lax pairs for lattice soliton equations. In addition, the conservation law plays an important role in studying integrability of soliton equations. Some methods constructing conservation laws of soliton equations have been presented.[28,29]

This paper is organized as follows. In Sec. 2, we derive a hierarchy of lattice soliton equations from a discrete spectral problem. It is shown that the hierarchy is completely integrable in the Liouville sense and possesses discrete bi-Hamiltonian structure. In Sec. 3, the Lax pairs and the adjoint Lax pairs of the resulting hierarchy are binary nonlinearized by means of the binary Bargmann constraints. The spatial part is nonlinearized into a new integrable symplectic map.[30−32] The temporal parts are nonlinearized into a family of finite-dimensional Liouville integrable systems. Moreover, the binary Bargmann constraint provides a Bäcklund transformation for each lattice soliton equation in the resulting hierarchy.[33] In Sec. 4, we construct many conservation laws for the hierarchy.

2 A Hierarchy of Liouville Integrable Lattice Soliton Equations

First, we specify some fundamental conceptions. The shift operator E, the inverse of E, and difference operator D are defined as follows:

$$(Ef)(n) = f(n+1)\,,$$

$$(E^{-1}f)(n) = f(n-1), \quad n \in Z\,, \tag{1}$$

$$(Df)(n) = f(n+1) - f(n) = (E-1)f(n), \quad n \in Z\,, \tag{2}$$

and write $f^{(j)} = E^j f$, $j \in Z$. Assume that $u = (q, r)^{\mathrm{T}}$, $q = q(n,t)$, $r = r(n,t)$ are real functions defined over $Z \times R$, and u is required to vanish rapidly at the infinity. λ is the spectral parameter and $\lambda_t = 0$.

A lattice equation

$$u_t = K(u, Eu, E^{-1}u, \ldots) \tag{3}$$

is said to be Lax integrable, if it can be rewritten as a compatibility condition

$$U_t = (EV)U - UV \tag{4}$$

of a discrete spatial spectral problem

$$E\varphi = U(u,\lambda)\varphi\,, \tag{5}$$

and a corresponding continuous time evolution equation

$$\varphi_t = V(u,\lambda)\varphi\,, \tag{6}$$

where $U(u,\lambda)$ and $V(u,\lambda)$ are of same order square matrices. Equations (5) and (6) are said to be a Lax pairs of Eq. (3). Equation (4) is called a discrete zero curvature representation of Eq. (3).

The Gateaux derivative, the variational derivative and the inner product are defined by

$$J'(u)[v] = \frac{\partial}{\partial \varepsilon} J(u + \varepsilon v)\,|_{\varepsilon=0}\,, \tag{7}$$

$$\frac{\delta \tilde{H}}{\delta u} = \sum_{m \in Z} E^{-m}\Big(\frac{\partial H}{\partial u^{(m)}}\Big)\,, \tag{8}$$

$$\langle f, g\rangle = \sum_{n \in Z} (f(n), g(n))\,, \tag{9}$$

where $f = f(n)$ and $g = g(n)$ are required to be rapidly vanishing at the infinity, $(f(n), g(n))$ denotes the standard inner product of $f = f(n)$ and $g = g(n)$ in the Euclidean space R^2. $\tilde{H} = \sum_{n\in Z} H(u(n))$. Operator J^* is defined by $\langle f, J^* g\rangle = \langle Jf, g\rangle$, which is called the adjoint operator of J with respect to Eq. (9).

If an operator J has the property $J = -J^*$, then J is called to be skew-symmetric. A linear operator J is called

---

*The project supported by National Natural Science Foundation of China under Grant No. 10371070

a Hamiltonian operator, if J is a skew-symmetric operator satisfying the Jacobi identity, i.e.,

$$\langle J'(u)[Jf]g, h\rangle + \mathrm{Cycle}(f,g,h) = 0\,.$$

The associated Poisson bracket with a given Hamiltonian operator J is given by

$$\{f,g\}_J = \Big\langle \frac{\delta f}{\delta u}, J\frac{\delta g}{\delta u}\Big\rangle\,. \tag{10}$$

In virtue of Ref. [2], the discrete Hamiltonian system

$$u_{t_m} = J\frac{\delta \tilde{H}}{\delta u}, \quad m \geq 0\,,$$

is called to be Liouville integrable if an infinite number of conserved functionals $\{\tilde{H}\}_{m=0}^{\infty}$ are in involution in pairs with respect to the Poisson bracket (10), i.e., $\{\tilde{H}_m, \tilde{H}_l\}_J = 0$, $m, l \geq 0$.

We introduce a discrete spectral problem

$$E\varphi = U(u,\lambda)\varphi, \quad U(u,\lambda) = \begin{pmatrix} 0 & -1/q \\ q & -\lambda - r \end{pmatrix},$$

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}\,. \tag{11}$$

In order to derive a hierarchy of integrable lattice equations, we first solve

$$(E\Gamma)U - U\Gamma = 0\,. \tag{12}$$

Set $\Gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, then equation (12) is equivalent to

$$q^2 b^{(1)} + c = 0\,, \tag{13a}$$

$$\lambda q b^{(1)} + qrb^{(1)} + a^{(1)} + a = 0\,, \tag{13b}$$

$$\lambda c + rc - q(a^{(1)} + a) = 0\,, \tag{13c}$$

$$\lambda q(a^{(1)} - a) + qr(a^{(1)} - a) - q^2 b - c^{(1)} = 0\,. \tag{13d}$$

On setting

$$a = \sum_{m\geq 0} a_m\lambda^{-m}, \quad b = \sum_{m\geq 0} b_m\lambda^{-m}, \quad c = \sum_{m\geq 0} c_m\lambda^{-m},$$

equation (13) yields the initial data

$$b_0^{(1)} = 0, \quad c_0 = 0, \quad a_0^{(1)} - a_0 = 0$$

and the recursion relation

$$q^2 b_m^{(1)} + c_m = 0\,, \tag{14a}$$

$$q b_{m+1}^{(1)} + qrb_m^{(1)} + a_m^{(1)} + a_m = 0\,, \tag{14b}$$

$$c_{m+1} + rc_m - q(a_m^{(1)} + a_m) = 0\,, \tag{14c}$$

$$q(a_{m+1}^{(1)} - a_{m+1}) + qr(a_m^{(1)} - a_m) - q^2 b_m - c_m^{(1)} = 0. \tag{14d}$$

We choose $a_0 = -1/2$, $b_0 = 0$ and require selecting zero constants for the inverse operation of the difference operator D in computing a_m, $m \geq 1$. Thus, the recursion relation (14) uniquely determines a_m, b_m, c_m, $m \geq 1$ and the first a few quantities are given as

$$a_1 = 0, \quad b_1 = \frac{1}{q^{(-1)}}\,, \quad c_1 = -q\,,$$

$$a_2 = -\frac{q}{q^{(-1)}}\,, \quad b_2 = -\frac{r^{(-1)}}{q^{(-1)}}, \quad c_2 = qr, \ldots$$

For any integer $m \geq 0$, we choose

$$(\Gamma\lambda^m)_+ = \begin{pmatrix} \sum_{i=0}^m a_i\lambda^{m-i} & \sum_{i=0}^m b_i\lambda^{m-i} \\ \sum_{i=0}^m c_i\lambda^{m-i} & -\sum_{i=0}^m a_i\lambda^{m-i} \end{pmatrix}\,,$$

then the following equation holds

$$(E(\Gamma\lambda^m)_+)U - U(\Gamma\lambda^m)_+$$

$$= \begin{pmatrix} 0 & b_{m+1}^{(1)} \\ -c_{m+1} & a_{m+1} - a_{m+1}^{(1)} \end{pmatrix}\,. \tag{15}$$

We let $V^{(m)} = (\Gamma\lambda^m)_+$ and introduce the following continuous time evolution equation

$$\varphi_{t_m} = V^{(m)}\varphi\,. \tag{16}$$

Then the compatibility conditions of Eqs. (11) and (16) are

$$U_{t_m} = (EV^{(m)})U - UV^{(m)}\,, \tag{17}$$

which give rise to a hierarchy of lattice soliton equations

$$u_{t_m} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_m} = X_m = \begin{pmatrix} -c_{m+1} \\ a_{m+1}^{(1)} - a_{m+1} \end{pmatrix}\,. \tag{18}$$

The first nonlinear lattice equations in Eq. (18) is

$$q_{t_1} = -qr, \quad r_{t_1} = \frac{q}{q^{(-1)}} - \frac{q^{(1)}}{q}\,. \tag{19}$$

The following theorem serves to develop the Hamiltonian structure of the hierarchy (18).

Theorem 2.1 The hierarchy (18) possesses the bi-Hamiltonian structure

$$u_{t_m} = J\frac{\delta \tilde{H}_m}{\delta u} = K\frac{\delta \tilde{H}_{m-1}}{\delta u} = \cdots = J\Theta^m \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \tag{20}$$

where the Hamiltonian operators J, K, the recursive operator Θ and the Hamiltonian function are given by

$$J = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}\,,$$

$$K = J\Theta = \begin{pmatrix} -q(1+E)(1-E)^{-1}q & -qr \\ qr & -\frac{1}{q}Eq + qE^{-1}\frac{1}{q} \end{pmatrix}\,,$$

$$\Theta = \begin{pmatrix} -r & \frac{1}{q^2}Eq - E^{-1}\frac{1}{q} \\ -(1+E)(1-E)^{-1}q & -r \end{pmatrix}\,,$$

$$\tilde{H}_m = \sum_{n\in Z} \frac{c_{m+2}}{(m+1)q}(n)\,. \tag{21}$$

Proof We introduce

$$G_m = \Big(\frac{a_{m+1} - a_{m+1}^{(1)}}{q}, -\frac{c_{m+1}}{q}\Big)^{\mathrm{T}}\,.$$

By means of the following recursion relations

$$G_{-1} = (0,0) \in \mathrm{Ker}\,J\,,$$

$$KG_{j-1} = JG_j, \quad j = 0,1,2,\ldots\,,$$

then equation (18) can be rewritten as

$$u_{t_m} = X_m = JG_m = KG_{m-1} = J\Theta^m G_0$$

$$= J\Theta^m \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,. \tag{22}$$

We define

$$V = \Gamma U^{-1} = \begin{pmatrix} -(\lambda+r)a - qb & a/q \\ -(\lambda+r)c + qa & c/q \end{pmatrix}\,,$$

and $\langle M, N\rangle \equiv \mathrm{Tr}(MN)$, where M and N are of the same order square matrices. A direct calculation yields

$$\Big\langle V, \frac{\partial U}{\partial \lambda}\Big\rangle = -\frac{c}{q}\,,$$

$$\Big\langle V, \frac{\partial U}{\partial q}\Big\rangle = -\frac{(\lambda+r)c}{q^2} + \frac{2a}{q} = \frac{a - a^{(1)}}{q}\,,$$

$$\Big\langle V, \frac{\partial U}{\partial r}\Big\rangle = -\frac{c}{q}\,.$$

By the trace identity,[2]

$$\frac{\delta}{\delta u}\sum_{n\in Z}\Big\langle V,\frac{\partial U}{\partial\lambda}\Big\rangle(n)=\lambda^{-\varepsilon}\Big(\frac{\partial}{\partial\lambda}\Big)\lambda^{\varepsilon}\Big\langle V,\frac{\partial U}{\partial u}\Big\rangle\,,$$

we have

$$\frac{\delta}{\delta u}\sum_{n\in Z}\Big(-\frac{c}{q}\Big)(n)=\lambda^{-\varepsilon}\Big(\frac{\partial}{\partial\lambda}\Big)\lambda^{\varepsilon}\Big(\frac{a-a^{(1)}}{q},-\frac{c}{q}\Big)^{\mathrm{T}}\,.$$

Substituting $a=\sum_{m\geq 0}a_m\lambda^{-m}$, $b=\sum_{m\geq 0}b_m\lambda^{-m}$, $c=\sum_{m\geq 0}c_m\lambda^{-m}$ into the above equation, and comparing the coefficients of λ^{-m-1} yields

$$\frac{\delta}{\delta u}\sum_{n\in Z}\Big(-\frac{c_{m+2}}{q}\Big)(n)=(\varepsilon-m-1)G_m\,.$$

Set $m=0$, then $\varepsilon=0$. So, we obtain

$$G_m=\delta\tilde{H}_m/\delta u\,,\tag{23}$$

where $\tilde{H}_m$ is given by Eq. (21). Combining Eq. (22) with Eq. (23), we obtain the desired Hamiltonian formulation (20) of the hierarchy (18).

In the following, we will study the integrability of the hierarchy (18).

Theorem 2.2 $\{\tilde{H}_m\}_{m\geq 0}$ defined by Eq. (21) forms an infinite set of conserved functionals of the hierarchy (18), and $\tilde{H}_m$, $m\geq 0$ are in involution. In other words, the hierarchy (18) is a completely integrable discrete Hamiltonian system in the Liouville sense.

Proof It is easy to see that $\Theta^*J=J\Theta$. We have

$$\begin{aligned}\{\tilde{H}_m,\tilde{H}_l\}_J&=\Big\langle\frac{\delta\tilde{H}_m}{\delta u},J\frac{\delta\tilde{H}_l}{\delta u}\Big\rangle=\Big\langle\Theta^m\frac{\delta\tilde{H}_0}{\delta u},J\Theta^l\frac{\delta\tilde{H}_0}{\delta u}\Big\rangle\\&=\Big\langle\Theta^m\frac{\delta\tilde{H}_0}{\delta u},\Theta^*J\Theta^{l-1}\frac{\delta\tilde{H}_0}{\delta u}\Big\rangle\\&=\Big\langle\Theta^{m+1}\frac{\delta\tilde{H}_0}{\delta u},J\Theta^{l-1}\frac{\delta\tilde{H}_0}{\delta u}\Big\rangle\\&=\{\tilde{H}_{m+1},\tilde{H}_{l-1}\}_J=\cdots=\{\tilde{H}_{m+l},\tilde{H}_0\}_J\,.\end{aligned}$$

Similarly, we also get $\{\tilde{H}_l,\tilde{H}_m\}_J=\{\tilde{H}_{m+l},\tilde{H}_0\}_J$. Thus, $\{\tilde{H}_m,\tilde{H}_l\}_J=\{\tilde{H}_l,\tilde{H}_m\}_J$. On the other hand, $\{\tilde{H}_m,\tilde{H}_l\}_J=-\{\tilde{H}_l,\tilde{H}_m\}_J$. This implies $\{\tilde{H}_m,\tilde{H}_l\}_J=0$ and

$$\begin{aligned}(\tilde{H}_m)_{t_l}&=\Big(\sum_{n\in Z}H_m\Big)_{t_l}=\Big\langle\frac{\delta\tilde{H}_m}{\delta u},u_{t_l}\Big\rangle\\&=\Big\langle\frac{\delta\tilde{H}_m}{\delta u},J\frac{\delta\tilde{H}_l}{\delta u}\Big\rangle=\{\tilde{H}_m,\tilde{H}_l\}_J=0\,.\end{aligned}$$

This implies that $\tilde{H}_m$, $m\geq 0$, are also conserved functionals. The proof is completed.

3 Binary Nonlinearization of Lax Pairs and Adjoint Lax Pairs

Let us consider the adjoint spectral problem of spectral problem (11)

$$E^{-1}\psi=(E^{-1}U^{\mathrm{T}}(u,\lambda))\psi\,,\quad\psi=(\psi_1,\psi_2)^{\mathrm{T}}\,,\tag{24}$$

and the continuous time evolution equations

$$\psi_{t_m}=-(V^{(m)}(u,\lambda))^{\mathrm{T}}\psi\,.\tag{25}$$

From $(E^{-1}\psi)_{t_m}=E^{-1}\psi_{t_m}$, we find that

$$\begin{aligned}E^{-1}U^{\mathrm{T}}_{t_m}&=(E^{-1}U^{\mathrm{T}})(V^{(m)})^{\mathrm{T}}\\&\quad-(E^{-1}(V^{(m)})^{\mathrm{T}})(E^{-1}U^{\mathrm{T}})\,.\end{aligned}\tag{26}$$

It is easy to verify that equation (26) can be rewritten as

$$U_{t_m}=(EV^{(m)})U-UV^{(m)}\,.$$

Therefore, equation (18) has another kind of discrete zero curvature representation (26). Equations (24) and (25) are called the adjoint Lax pairs of Eq. (18). The adjoint Lax pairs (24) and (25) can help determine the variational derivative of the spectral parameter λ with respect to the potential u.[31]

Let $\lambda_1,\lambda_2,\ldots,\lambda_N$ be N distinct eigenvalues of spectral problem (11), we have

$$\begin{pmatrix}E\varphi_{1j}\\E\varphi_{2j}\end{pmatrix}=U(u,\lambda_j)\begin{pmatrix}\varphi_{1j}\\\varphi_{2j}\end{pmatrix}\,,\tag{27a}$$

$$\begin{pmatrix}E^{-1}\psi_{1j}\\E^{-1}\psi_{2j}\end{pmatrix}=E^{-1}U^{\mathrm{T}}(u,\lambda_j)\begin{pmatrix}\psi_{1j}\\\psi_{2j}\end{pmatrix}\,,\tag{27b}$$

$$\begin{pmatrix}\varphi_{1j}\\\varphi_{2j}\end{pmatrix}_{t_m}=V^{(m)}(u,\lambda_j)\begin{pmatrix}\varphi_{1j}\\\varphi_{2j}\end{pmatrix}\,,\tag{28a}$$

$$\begin{pmatrix}\psi_{1j}\\\psi_{2j}\end{pmatrix}_{t_m}=-(V^{(m)}(u,\lambda_j))^{\mathrm{T}}\begin{pmatrix}\psi_{1j}\\\psi_{2j}\end{pmatrix}\,,\tag{28b}$$

$$(E\varphi_{1j},E\varphi_{2j})=(\varphi_{1j},\varphi_{2j})U^{\mathrm{T}}(u,\lambda_j)\,,\tag{29}$$

$$(E\psi_{1j},E\psi_{2j})=(\psi_{1j},\psi_{2j})U^{-1}(u,\lambda_j)\,,\tag{30}$$

for $1\leq j\leq N$. From Ref. [31], it is known that

$$\begin{aligned}\frac{\delta\lambda_j}{\delta q}&=\alpha_j\Big(\varphi_{1j}\psi^{(1)}_{2j}+\frac{\varphi_{2j}\psi^{(1)}_{1j}}{q^2}\Big)\,,\\\frac{\delta\lambda_j}{\delta r}&=-\alpha_j\varphi_{2j}\psi^{(1)}_{2j}\,,\end{aligned}\tag{31}$$

where $\delta\lambda_j/\delta u$ is a variational derivative for eigenvalue λ_j, while α_j, $1\leq j\leq N$, are constants and φ_i, ψ_i, $i=1,2$ are required to be rapidly vanishing at the infinity. Equation (31) will play a central role in the binary nonlinearization method. We can deduce that

$$\Theta\frac{\delta\lambda_j}{\delta u}=\lambda_j\frac{\delta\lambda_j}{\delta u}\,.$$

We will use the following notations,

$$\begin{aligned}&\Phi_i=(\varphi_{i1},\varphi_{i2},\ldots,\varphi_{iN})\,,\\&\Psi_i=(\psi_{i1},\psi_{i2},\ldots,\psi_{iN}),\quad i=1,2\,,\\&\Lambda=\mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_N)\,.\end{aligned}$$

And the inner product in R^N is denoted by $\langle\cdot,\cdot\rangle$.

Consider the discrete constraint

$$J\frac{\delta\tilde{H}_0}{\delta u}=J\sum_{j=1}^{N}\frac{\delta\lambda_j}{\delta u}\,,\tag{32}$$

where $\alpha_j=1$. From Eq. (32), we have

$$\frac{\delta\tilde{H}_0}{\delta u}=\begin{pmatrix}\dfrac{a_1-a_1^{(1)}}{q}\\-\dfrac{c_1}{q}\end{pmatrix}=\begin{pmatrix}\langle\Phi_1,\Psi_2^{(1)}\rangle+\dfrac{\langle\Phi_2,\Psi_1^{(1)}\rangle}{q^2}\\-\langle\Phi_2,\Psi_2^{(1)}\rangle\end{pmatrix}\,.$$

Then the symmetry constraint (32) yields

$$\begin{aligned}&q=-\langle\Phi_2,\Psi_1\rangle\,,\\&r=\langle\Phi_2,\Psi_2\rangle-\langle\Phi_1,\Psi_1\rangle-\frac{\langle\Lambda\Phi_2,\Psi_1\rangle}{\langle\Phi_2,\Psi_1\rangle}\,.\end{aligned}\tag{33}$$

Because it is possible to solve Eq. (32) for u, and u depends on Φ_i, Ψ_i, $i=1,2$, equation (32) or (33) is called a binary Bargmann symmetry constraint. Under the constraint (33), putting the N copies of Eqs. (29) and (30) in vector form, we obtain a discrete Bargmann system,

$$E\Phi_1 = \frac{\Phi_2}{\langle\Phi_2,\Psi_1\rangle}\,, \tag{34a}$$

$$\begin{aligned}E\Phi_2 = {}& -\langle\Phi_2,\Psi_1\rangle\Phi_1 - \Lambda\Phi_2 - (\langle\Phi_2,\Psi_2\rangle \\ & - \langle\Phi_1,\Psi_1\rangle - \frac{\langle\Lambda\Phi_2,\Psi_1\rangle}{\langle\Phi_2,\Psi_1\rangle})\Phi_2\,,\end{aligned} \tag{34b}$$

$$\begin{aligned}E\Psi_1 = {}& \langle\Phi_2,\Psi_1\rangle\Psi_2 - \Lambda\Psi_1 - (\langle\Phi_2,\Psi_2\rangle \\ & - \langle\Phi_1,\Psi_1\rangle - \frac{\langle\Lambda\Phi_2,\Psi_1\rangle}{\langle\Phi_2,\Psi_1\rangle})\Psi_1\,,\end{aligned} \tag{34c}$$

$$E\Psi_2 = -\frac{\Psi_1}{\langle\Phi_2,\Psi_1\rangle}\,. \tag{34d}$$

Setting

$$\begin{aligned}& f_i = f_i(\Phi_1,\Phi_2,\Psi_1,\Psi_2)\,,\\ & g_i = g_i(\Phi_1,\Phi_2,\Psi_1,\Psi_2), \quad 1\le i\le 2N\,,\\ & f_j = \langle\Phi_2,\Psi_1\rangle\psi_{2j} - \lambda_j\psi_{1j} - (\langle\Phi_2,\Psi_2\rangle - \langle\Phi_1,\Psi_1\rangle \\ & \qquad - \frac{\langle\Lambda\Phi_2,\Psi_1\rangle}{\langle\Phi_2,\Psi_1\rangle})\psi_{1j}, \quad 1\le j\le N\,,\end{aligned} \tag{35a}$$

$$f_{N+j} = -\frac{\psi_{1j}}{\langle\Phi_2,\Psi_1\rangle}, \quad 1\le j\le N\,, \tag{35b}$$

$$g_j = \frac{\varphi_{2j}}{\langle\Phi_2,\Psi_1\rangle}, \quad 1\le j\le N\,, \tag{35c}$$

$$\begin{aligned}g_{N+j} = {}& -\langle\Phi_2,\Psi_1\rangle\varphi_{1j} - \lambda_j\varphi_{2j} - (\langle\Phi_2,\Psi_2\rangle - \langle\Phi_1,\Psi_1\rangle \\ & - \frac{\langle\Lambda\Phi_2,\Psi_1\rangle}{\langle\Phi_2,\Psi_1\rangle})\varphi_{2j}, \quad 1\le j\le N\,.\end{aligned} \tag{35d}$$

We define the Poisson bracket for two functions f and g in symplectic space $(R^{4N}, \sum_{i=1}^{2}\mathrm{d}\Psi_i\Lambda\mathrm{d}\Phi_i)$ as follows:

$$\begin{aligned}\{f,g\} &= \sum_{i=1}^{2}\sum_{j=1}^{N}\Big(\frac{\partial f}{\partial\psi_{ij}}\frac{\partial g}{\partial\varphi_{ij}} - \frac{\partial f}{\partial\varphi_{ij}}\frac{\partial g}{\partial\psi_{ij}}\Big)\\ &= \sum_{i=1}^{2}\Big(\Big\langle\frac{\partial f}{\partial\Psi_i},\frac{\partial g}{\partial\Phi_i}\Big\rangle - \Big\langle\frac{\partial f}{\partial\Phi_i},\frac{\partial g}{\partial\Psi_i}\Big\rangle\Big)\,,\end{aligned} \tag{36}$$

which is skew-symmetric, bilinear, and satisfies the Jacobi identity. In particular, f and g are called to be involutive if $\{f,g\}=0$.

Proposition 3.1 Equation (34) determines a symplectic map H:[29]

$$H(\Psi_1,\Psi_2,\Phi_1,\Phi_2) = (E\Psi_1,E\Psi_2,E\Phi_1,E\Phi_2)\,. \tag{37}$$

Proof Through laborious, but direct computations, we have

$$\{f_i,f_j\} = \{g_i,g_j\} = 0\,, \quad \{f_i,g_j\} = \delta_{ij},$$

for $1\le i,\, j\le 2N$. This completes the proof.

We can solve Eq. (14) as follows:

$$\begin{aligned}& \tilde{a}_0 = -\tfrac{1}{2}, \quad \tilde{b}_0 = 0, \quad \tilde{c}_0 = 0, \quad \tilde{a}_1 = 0\,,\\ & \tilde{a}_m = \tfrac{1}{2}(\langle\Lambda^{m-1}\Phi_1,\Psi_1\rangle - \langle\Lambda^{m-1}\Phi_2,\Psi_2\rangle), \quad m\ge 2\,,\\ & \tilde{b}_m = \langle\Lambda^{m-1}\Phi_1,\Psi_2\rangle\,,\\ & \tilde{c}_m = \langle\Lambda^{m-1}\Phi_2,\Psi_1\rangle, \quad m\ge 1\,.\end{aligned} \tag{38}$$

Set

$$\begin{aligned}& \tilde{F}_0 = \tfrac{1}{4}, \quad \tilde{F}_1 = 0\,,\\ & \tilde{F}_m = -\tfrac{1}{2}(\langle\Lambda^{m-1}\Phi_1,\Psi_1\rangle - \langle\Lambda^{m-1}\Phi_2,\Psi_2\rangle)\\ & \qquad + \sum_{i=1}^{m}\langle\Lambda^{i-1}\Phi_1,\Psi_2\rangle\langle\Lambda^{m-i-1}\Phi_2,\Psi_1\rangle\\ & \qquad + \frac{1}{4}\sum_{i=1}^{m-1}(\langle\Lambda^{i-1}\Phi_1,\Psi_1\rangle - \langle\Lambda^{i-1}\Phi_2,\Psi_2\rangle)\\ & \qquad \times(\langle\Lambda^{m-i-1}\Phi_1,\Psi_1\rangle - \langle\Lambda^{m-i-1}\Phi_2,\Psi_2\rangle)\,,\end{aligned} \tag{39}$$

then, we have the following proposition.

Proposition 3.2

$$D\tilde{F}_m = 0, \quad m\ge 0\,, \tag{40}$$

i.e., $\tilde{F}_m$, $m\ge 0$, constitute a hierarchy of integrals of motion for symplectic map H (37) (or the discrete flow (34)).

Proof Because expressions (38) are the solutions of Eq. (14), we have

$$\tilde{a} = \sum_{m=0}^{\infty}\tilde{a}_m\lambda^{-m}\,,$$

$$\tilde{b} = \sum_{m=0}^{\infty}\tilde{b}_m\lambda^{-m}, \quad \tilde{c} = \sum_{m=0}^{\infty}\tilde{c}_m\lambda^{-m}$$

are a set of solutions of Eq. (13), thus $D(\tilde{a}^2+\tilde{b}\tilde{c}) = 0$. This implies $D\tilde{F}_m = 0$, $m\ge 0$. This completes the proof.

In the following, we would like to discuss the Liouville integrable on the nonlinearized temporal parts of the Lax pairs and adjoint Lax pairs.

Under the control of Eq. (33), the temporal parts of the Lax pairs and the adjoint Lax pairs by substituting Eq. (38) into Eq. (28) become

$$\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix}_{t_m} = \tilde{V}^{(m)}(u,\lambda_j)|_B\begin{pmatrix}\varphi_{1j}\\ \varphi_{2j}\end{pmatrix}, \tag{41a}$$

$$\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}_{t_m} = -(\tilde{V}^{(m)}(u,\lambda_j))^{\mathrm{T}}|_B\begin{pmatrix}\psi_{1j}\\ \psi_{2j}\end{pmatrix}, \tag{41b}$$

for $j=1,2,\ldots,N$, where the subscript B means substitution of Eq. (33) into the expression.

Through a direct calculation, we find that equation (41) can be rewritten as

$$\Phi_{it_m} = \frac{\partial\tilde{F}_{m+1}}{\partial\Psi_i}, \quad \Psi_{it_m} = -\frac{\partial\tilde{F}_{m+1}}{\partial\Phi_i}, \quad i=1,2\,. \tag{42}$$

Proposition 3.3 $\tilde{F}_m$, $m\ge 0$, are in involution in pairs with respect to the Poisson bracket (36), and constitute a hierarchy of involutive integrals of motion for Eq. (42).

Proof By a direct calculation, we get $\tilde{\Gamma}_{t_m} = [\tilde{V}^{(m)},\tilde{\Gamma}]$, where

$$\tilde{\Gamma} = \begin{pmatrix}\tilde{a} & \tilde{b}\\ \tilde{c} & -\tilde{a}\end{pmatrix} = \sum_{m=0}^{\infty}\begin{pmatrix}\tilde{a}_m & \tilde{b}_m\\ \tilde{c}_m & -\tilde{a}_m\end{pmatrix}\lambda^{-m}\,.$$

This yields[32]

$$2\frac{\mathrm{d}}{\mathrm{d}t_m}(\tilde{a}^2+\tilde{b}\tilde{c}) = \frac{\mathrm{d}}{\mathrm{d}t_m}\mathrm{Tr}\tilde{\Gamma}^2 = \frac{\mathrm{d}}{\mathrm{d}t_m}\mathrm{Tr}[\tilde{V}^{(m)},\tilde{\Gamma}^2] = 0\,,$$

i.e., $(\mathrm{d}/\mathrm{d}t_m)\tilde{F}_l = 0$, $l,m\ge 1$.

Therefore, we have

$$\{\tilde{F}_l,\tilde{F}_m\} = \frac{\mathrm{d}}{\mathrm{d}t_m}\tilde{F}_l = 0, \quad l,m\ge 1\,.$$

This elucidates that $\tilde{F}_m$, $m\ge 0$, constitute a hierarchy of involutive integrals of motion for Eq. (42). $\tilde{F}_m$,

$m \geq 0$, are in involution in pairs with respect to the Poisson bracket (36). This completes the proof.

Set $\bar{F}_j = \varphi_{1j}\psi_{1j} + \varphi_{2j}\psi_{2j}$, $1 \leq j \leq N$. It is easy to calculate

$$\begin{aligned} & D\bar{F}_m = 0\,, \\ & \frac{\mathrm{d}}{\mathrm{d}t_m}\bar{F}_j = 0, \quad 1 \leq j \leq N, \quad m \geq 0\,, \\ & \{\tilde{F}_m, \bar{F}_j\} = 0, \quad m \geq 0, \quad 1 \leq j \leq N\,, \\ & \{\bar{F}_i, \bar{F}_j\} = 0, \quad 1 \leq i, j \leq N\,. \end{aligned} \tag{43}$$

Proposition 3.4 $\tilde{F}_{m+1}$, $1 \leq m \leq N$, $\bar{F}_j$, $1 \leq j \leq N$, are functionally independent over some region of R^{4N}.

Proof A straightforward computation gives

$$\frac{\partial \tilde{F}_{m+1}}{\partial \Phi_1} = -\frac{1}{2}\Lambda^m \Psi_1 + \sum_{i=1}^{m} \tilde{a}_i \Lambda^{m-i}\Psi_1 + \sum_{i=1}^{m+1} \tilde{c}_i \Lambda^{m-i}\Psi_2\,, \tag{44a}$$

$$\frac{\partial \tilde{F}_{m+1}}{\partial \Phi_2} = \frac{1}{2}\Lambda^m \Psi_2 - \sum_{i=1}^{m} \tilde{a}_i \Lambda^{m-i}\Psi_2 + \sum_{i=1}^{m+1} \tilde{b}_i \Lambda^{m-i}\Psi_1\,. \tag{44b}$$

Thus, we have

$$\frac{\partial \tilde{F}_{m+1}}{\partial \Phi_1}\,|_{\Phi_1=\Phi_2=0} = -\frac{1}{2}\Lambda^m \Psi_1, \quad m \geq 1\,, \tag{45a}$$

$$\frac{\partial \tilde{F}_{m+1}}{\partial \Phi_2}\,|_{\Phi_1=\Phi_2=0} = \frac{1}{2}\Lambda^m \Psi_2, \quad m \geq 1\,. \tag{45b}$$

Assume that the result on the functional independence is not true. Then there exist $2N$ real numbers $\xi_1, \xi_2, \ldots, \xi_N, \eta_1, \eta_2, \ldots, \eta_N$, satisfying $\sum_{i=1}^N \xi_i^2 + \sum_{j=1}^N \eta_j^2 \neq 0$, such that

$$\sum_{i=1}^N \xi_i \bar{F}_i + \sum_{j=1}^N \eta_j \tilde{F}_j = 0\,.$$

A direct calculation gives rise to

$$\frac{\partial \bar{F}_j}{\partial \varphi_{il}} = \psi_{ij}\delta_{jl}, \quad j, l = 1, 2, \ldots, N\,.$$

We can compute

$$\det\begin{pmatrix} \dfrac{\partial \bar{F}_1}{\partial \Phi_1} & \cdots & \dfrac{\partial \bar{F}_N}{\partial \Phi_1} & \dfrac{\partial \tilde{F}_2}{\partial \Phi_1} & \cdots & \dfrac{\partial \tilde{F}_{N+1}}{\partial \Phi_1} \\ \dfrac{\partial \bar{F}_1}{\partial \Phi_2} & \cdots & \dfrac{\partial \bar{F}_N}{\partial \Phi_2} & \dfrac{\partial \tilde{F}_2}{\partial \Phi_2} & \cdots & \dfrac{\partial \tilde{F}_{N+1}}{\partial \Phi_2} \end{pmatrix} = \det\begin{pmatrix} \psi_{11} & \cdots & 0 & -\dfrac{1}{2}\lambda_1\psi_{11} & \cdots & -\dfrac{1}{2}\lambda_1^N\psi_{11} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_{1N} & -\dfrac{1}{2}\lambda_N\psi_{1N} & \cdots & -\frac{1}{2}\lambda_N^N\psi_{1N} \\ \psi_{21} & \cdots & 0 & \dfrac{1}{2}\lambda_1\psi_{21} & \cdots & \dfrac{1}{2}\lambda_1^N\psi_{21} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_{2N} & \frac{1}{2}\lambda_N\psi_{2N} & \cdots & \dfrac{1}{2}\lambda_N^N\psi_{2N} \end{pmatrix}$$

$$= \Big(\prod_{j=1}^N \psi_{1j}\Big)\Big(\prod_{j=1}^N \psi_{2j}\Big)\Big(\prod_{j=1}^N \lambda_j\Big)\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^{N-1} \\ \cdots & \cdots & \cdots & \cdots & \\ 1 & \lambda_N & \lambda_N^2 & \cdots & \lambda_N^{N-1} \end{vmatrix}.$$

By observing that the Vandermode determinant $V(\lambda_1, \lambda_2, \cdots, \lambda_N) \neq 0$, this means that all ξ_i, η_j must be zero. Therefore the functions $\bar{F}_j$, $\tilde{F}_{j+1}$, $j = 1, 2, \ldots, N$, are functionally independent at least on certain region of R^{4N}. This completes the proof.

Summing up, according to the Propositions 3.2 ∼ 3.4, we have the following theorem.

Theorem 3.1 Symplectic map (37) is integrable in the Liouville sense. Nonlinearized temporal parts of the Lax pairs and the adjoint Lax pairs (42) are all finite-dimensional integrable systems in the Liouville sense.

In virtue of theorem 3.1, construction of solutions (33) to the hierarchy (18) is split into finding $\Phi_i(n, t_m)$, $\Psi_i(n, t_m)$, $i = 1, 2$, to the integrable symplectic map (37) and the finite-dimensional integrable systems (42). In fact, in view of Bäcklund transformation,[33]

$$q(n, t_m) = -\langle \Phi_2(n, t_m), \Psi_1(n, t_m)\rangle\,, \tag{46a}$$

$$r(n, t_m) = \langle \Phi_2(n, t_m), \Psi_2(n, t_m)\rangle - \langle \Phi_1(n, t_m), \Psi_1(n, t_m)\rangle - \frac{\langle \Lambda\Phi_2(n, t_m), \Psi_1(n, t_m)\rangle}{\langle \Phi_2(n, t_m), \Psi_1(n, t_m)\rangle}\,, \tag{46b}$$

is a Bäcklund transformation between the lattice soliton equation (18) and the integrable symplectic map (37) (or discrete flow (34)) and the finite-dimensional completely integrable systems (42). As a result, we have the following conclusion.

Theorem 3.2 Let $(\Phi_1(n, t_1)$, $\Phi_2(n, t_1)$, $\Psi_1(n, t_1)$, $\Psi_2(n, t_1))$ be a solution of the discrete flow (34) and the following continuous time evolution equation

$$\Phi_{it_1} = \frac{\partial \tilde{F}_2}{\partial \Psi_i}, \quad \Psi_{it_1} = \frac{\partial \tilde{F}_2}{\partial \Phi_i}, \quad i = 1, 2\,.$$

Then, equation (46) is a solution of the discrete soliton equation (19).

4 Conservation Laws

In this section, we will derive conservation laws of the hierarchy (18), by a simple and direct way.[32,33] We set $Ef(n, t) = f(n+1, t)$, $E^{-1}f(n, t) = f(n-1, t)$. From the Lax pairs (11) and (16) of the hierarchy (18), we have

$$\varphi_2(n+1) = -\frac{q(n)}{q(n-1)}\varphi_2(n-1)$$

$$-\,(\lambda + r(n))\varphi_2(n)\,, \qquad (47)$$

$$\varphi_2(n)_{t_m} = \Big(-\frac{1}{q(n-1)}\sum_{i=0}^{m} c_i(n)\lambda^{m-i}\Big)\varphi_2(n-1) - \Big(\sum_{i=0}^{m} a_i(n)\lambda^{m-i}\Big)\varphi_2(n)\,.$$

Assume that $\theta(n) = \varphi_2(n)/\varphi_2(n+1)$, a direct calculation gives

$$-(\ln\theta(n))_{t_m} = (E-1)\Big\{\Big[-\frac{1}{q(n-1)}\sum_{i=0}^{m} c_i(n)\lambda^{m-i}\Big] \times \theta(n-1) - \Big[\sum_{i=0}^{m} a_i(n)\lambda^{m-i}\Big]\Big\}\,. \qquad (48)$$

Therefore, we obtain the conservation laws (48) of the hierarchy (18). Especially, for the lattice equation (19), the conservation laws (48) become

$$-(\ln\theta(n))_t = (E-1)\Big(\frac{q(n)}{q(n-1)}\theta(n-1) + \frac{1}{2}\lambda\Big)\,. \qquad (49)$$

In virtue of Eq. (47), we have

$$\lambda\theta(n) = -1 - \frac{q(n)}{q(n-1)}\theta(n)\theta(n-1) - r(n)\theta(n)\,. \qquad (50)$$

Then, expanding

$$\theta(n) = \sum_{j=1}^{\infty}\frac{\theta_j(n)}{\lambda_j}\,, \qquad (51)$$

and substituting Eq. (51) into Eq. (50), we get the following recursion relation:

$$\theta_1(n) = -1\,,$$

$$\theta_{j+1}(n) = -r(n)\theta_j(n) - \frac{q(n)}{q(n-1)}\sum_{k=1}^{j-1}\theta_k(n-1)\theta_{j-k}(n), \quad j \geq 1\,. \qquad (52)$$

At this moment, we have

$$\theta_2(n) = r(n), \quad \theta_3(n) = -\frac{q(n)}{q(n-1)} - r^2(n), \quad \cdots$$

Substituting Eq. (51) into Eq. (49), we obtain

$$\Big[\sum_{i=1}^{\infty}(-1)^i\frac{1}{i}\Big(\sum_{j=1}^{\infty}\frac{\theta_{j+1}(n)}{\lambda^j}\Big)^i\Big]_t = (E-1)\Big[\frac{q(n)}{q(n-1)}\sum_{j=1}^{\infty}\frac{\theta_j(n-1)}{\lambda^j}\Big]\,. \qquad (53)$$

Equating the power of $1/\lambda$ in Eq. (53), we can get an infinite number of conservation laws of the lattice equation (19). The first two are pointed out as

$$[r(n)]_t = (E-1)\Big[\frac{q(n)}{q(n-1)} - 1\Big]\,,$$

$$\Big[\frac{3}{2}r^2(n) + \frac{q(n)}{q(n-1)} - 1\Big]_t = (E-1)\Big[\frac{q(n)}{q(n-1)}r(n-1)\Big]\,.$$

Similarly, we can get conservation laws of other lattice equations in the hierarchy (18).

References

[1] M. Ablowitz and J. Ladik, J. Math. Phys. **16** (1975) 598.

[2] G.Z. Tu, J. Phys. A: Math. Gen. **23** (1990) 3903.

[3] Y. Ohta and R. Hirota, J. Phys. Soc. Jpn. **60** (1991) 2059.

[4] O.I. Bogoyavlensky, Phys. Lett. A **134** (1988) 34.

[5] W.X. Ma, *et al.*, J. Math. Phys. **40** (1999) 2400.

[6] K. Narita, J. Phys. Soc. Jpn. **71** (2002) 2401.

[7] W. Oevel, *et al.*, Prog. Theor. Phys. **81** (1989) 294.

[8] X.X. Xu, Nonlinear Analysis **61** (2005) 1225.

[9] X.X. Xu and H.H. Dong, Commun. Theor. Phys. (Beijing, China) **38** (2002) 523.

[10] X.X. Xu and Y.F. Zhang, Commun. Theor. Phys. (Beijing, China) **41** (2004) 321.

[11] X.X. Xu, *et al.*, Acta. Math. Sci. A **23** (2003) 298.

[12] Y.T. Wu and X.G. Geng, J. Math. Phys. **37** (1996) 2338.

[13] Y.T. Wu and X.G. Geng, J. Phys. A: Math. Gen. **31** (1998) L677.

[14] D.J. Zhang and D.Y. Chen, J. Phys. A: Math. Gen. **35** (2002) 7225.

[15] H.W. Zhang, G.Z. Tu, W. Oevel, and B. Fuchssteiner, J. Math. Phys. **32** (1991) 1908.

[16] C.W. Cao, Sci. China Ser. A **33** (1990) 528.

[17] C.W. Cao and X.G. Geng, *Nonlinear Physics*, eds. C.H. Gu, *et al.*, Springer, Berlin (1990) p. 68.

[18] C.W. Cao, X.G. Geng, and Y.T. Wu, J. Phys. A: Math. Gen. **32** (1999) 8059.

[19] R.G. Zhou and W.X. Ma, Phys. Lett. A **269** (2000) 103.

[20] Y.S. Li and W.X. Ma, Chaos, Solitons and Fractrals **11** (2000) 697.

[21] W.X. Ma, Chinese Ann. Math. Ser. B **18** (1997) 79.

[22] W.X. Ma, Q. Ding, W.G. Zhang, and B.Q. Liu, IL Nuovo Cimento B **111** (1996) 1135.

[23] W.X. Ma and W. Strmpp, Phys. Lett. A **185** (1994) 277.

[24] X.X. Xu, Phys. Lett. A **301** (2002) 250.

[25] D.Y. Chen, D. Levi, and Y.S. Li, J. Phys. A: Math. Gen. **20** (1987) 313.

[26] D.Y. Chen, H.W. Xing, and D.J Zhang, Chaos, Solitons and Fractrals **15** (2003) 761.

[27] X.X. Xu, Chin. Phys. Lett. **12** (1995) 513.

[28] M. Wadati, K. Konno, and H. Ichikawa, J. Phys. Soc. Jpn. **47** (1979) 1698.

[29] D.J. Zhang and D.Y. Chen, Chaos, Soliton and Fractrals **14** (2002) 573.

[30] Y.B. Zeng and Rauch-Wojciechowski, J. Phys. A: Math. Gen. **28** (1995) 113.

[31] M. Bruschi, O. Ragnisco, and G.Z. Tu, Phys. D **49** (1991) 273.

[32] Y.B. Zeng and Y.S. Li, Chin. Ann. Math. B **18** (1997) 457.

[33] W.X. Ma and X.G. Geng, “Bäcklund Transformation of Soliton Systems From Symmetry Constraints”, in *Proceedings of the AARMS-CRM Workshop on Bäcklund and Darboux Transformation: The Geometry of Soliton Theory*, Halifax, Canada (1999).