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Vacuum Solutions of Classical Gravity on Cyclic Groups from Noncommutative Geometry*

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Abstract *Based on the observation that the moduli of a link variable on a cyclic group modify Connes' distance on this group, we construct several action functionals for this link variable within the framework of noncommutative geometry. After solving the equations of motion, we find that one type of action gives nontrivial vacuum solution for gravity on this cyclic group in a broad range of coupling constants and that such a solution can be expressed with Chebyshev's polynomials.*

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1 Introduction

It is a marvelous discovery of noncommutative geometry (NCG) that Dirac operator induces a metric onto a space.[1] Specifying this result to a cyclic group, we discussed recently at the end of Ref. [2] that the effect of a non-unitary 1D link variable on this ***Connes' distance*** is to modify a linear (Euclidean)-distance with the moduli of this link variable. This feature indicates that the general characteristics of NCG, gauge connection and metric are easily interwound on a noncommutative space. In this paper, we will explore the (classical) gravitational physics of cyclic groups in depth. We construct action functionals for gravitational fields from Dirac operator, deduce the equations of motion, then solve them and give a series of vacuum solutions. It will be shown that in a broad range of coupling constants of one action functional, there is a nontrivial vacuum solution that can be expressed by Chebyshev's polynomials and that breaks translation invariance. Of course, this model is just an exercise as a class of toy-models like;[3,4] however, it provides an easily-handling example to formulate gravity theory on discrete sets. Though being simple, it has shown a lot of general features of NCG approach to gravity problems. For a more abstract and general treatment of noncommutative gravity, one can refer to Majid's papers.[5,6]

This paper is organized as the following way. In Sec. 2, NCG on a cyclic group is formulated, especially Dirac operator is defined. Several action functionals are established in Sec. 3. Three types of vacua will be found out, in which only the last one is nontrivial. Some opening discussions are put into Sec. 4.

2 NCG of Cyclic Groups: Kinematics of Dirac Operator

Let $\mathcal{Z}_N$ be an N-order cyclic group

$$\mathcal{Z}_N = \{0, 1, 2, \ldots, N-1\}\,,$$

whose multiplication is just integer addition modulo N. Additions appearing below are understood as additions in $\mathcal{Z}_N$. We just consider $N > 2$ cases. $\mathcal{A}(\mathcal{Z}_N)$ is the algebra of complex functions on $\mathcal{Z}_N$ and there is a ***regular representation*** of $\mathcal{Z}_N$ on $\mathcal{A}(\mathcal{Z}_N)$ generated by

$$(T^+ f)(x) = f(x+1)\,, \qquad \forall f \in \mathcal{A}(\mathcal{Z}_N),\ x \in \mathcal{Z}_N\,.$$

For any finite N, $(T^+)^N = \mathbf{1}$ where $\mathbf{1}$ is identity transformation on $\mathcal{A}(\mathcal{Z}_N)$; hence

$$T^- := (T^+)^{-1} = (T^+)^{N-1}\,.$$

A spinor space $\mathcal{H}_s = \mathcal{C}^2$ is introduced to each point of $\mathcal{Z}_N$; a ***fermion field*** on $\mathcal{Z}_N$ is an element in

$$\mathcal{H} := \mathcal{A}(\mathcal{Z}_N) \otimes \mathcal{H}_s\,.$$

Under the standard inner product on $\mathcal{H}$, T^+ is a unitary operator. A ***Fredholm operator*** acting on $\mathcal{H}$ is defined as $F = \eta_+ + \eta_-$, in which $\eta_\pm = T^\pm \sigma_\pm$, $\sigma_\pm = (\sigma_1 \pm \mathrm{i}\sigma_2)/2$ and σ_i $(i = 1, 2, 3)$ are the Pauli matrices.[7] Note that $\eta_\pm$ fulfill Clifford algebra relation on 2D-Euclidean space $\eta_\pm \eta_\pm = 0$, $\{\eta_\pm, \eta_\mp\} = \mathbf{1}$; accordingly, $F^2 = \mathbf{1}$, s.t. $(\mathcal{H}, F)$ forms a ***Fredholm module***.[8] The fundamental ***noncommutativity*** in this formalism is $\eta_\pm f = (T^\pm f)\eta_\pm$. It is proved in Ref. [7] that Connes' distance on $\mathcal{Z}_N$ defined by

$$d_F(x, y) = \sup\{|f(x) - f(y)| : ||[F, f]|| \le 1\}\,,$$
$$\forall x, y \in \mathcal{Z}_N$$

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is just the conventional linear (Euclidean)-distance on

$$\mathcal{Z}_N : \ d(x,y) = \min\{|x-y|, N-|x-y|\}\,.$$

Dirac operator on $\mathcal{Z}_N$ is F twisted with a link-variable,

$$F(\omega) = \omega^\dagger \eta_+ + \eta_- \omega\,,$$

in which ω is an $End_{\mathcal{C}}(\mathcal{H}_c)$-valued function on $\mathcal{Z}_N$ and $\mathcal{H}_c$ is an internal Hilbert space. The triple $(\mathcal{A}(\mathcal{Z}_N), \widetilde{\mathcal{H}}, F(\omega))$ becomes a K***-cycle*** where $\widetilde{\mathcal{H}} = \mathcal{H}\otimes\mathcal{H}_c$.[9] A gauge transformation u is a $U(\mathcal{H}_c)$-valued function on $\mathcal{Z}_N$; $T^-\omega$, $\omega^\dagger T^+$ play the role of $U(\mathcal{H}_c)$ ***parallel transports***, providing that $\omega \to (T^+u)\omega u^\dagger$. Below only one-dimensional case $\mathcal{H}_c = \mathcal{C}$ will be considered, hence $\omega \in \mathcal{A}(\mathcal{Z}_N)$. Subsequently, $\omega = \rho\,\mathrm{e}^{\mathrm{i}\theta}$ where ρ, θ are real functions, $\rho(x) \geq 0$ and $\omega^\dagger = \overline{\omega}$, $\omega^\dagger\omega = |\omega|^2 = \rho^2$. We argued at the end of Ref. [2] that

$$d_{F(\omega)}(x,x+k) = \min\{\rho(x)^{-1} + \rho(x+1)^{-1} + \cdots + \rho(x+k-1)^{-1}, \rho(x+k)^{-1} + \rho(x+k+1)^{-1} + \cdots + \rho(x+N-1)^{-1}\},$$
$$\text{for all } x, k \in \mathcal{Z}_N\,,$$

Therefore, metric on $\mathcal{Z}_N$ is modified by the strength of ω in the sense that ρ^{-1} provides a varying lattice spacing. This point can be illustrated more clearly by some special examples.

(i) $\rho(x) = \rho_0 > 0$, then resulting metric differs from the "free" one by a lattice constant $1/\rho_0$;
(ii) $\rho(0) = 0$, then $d_{F(\omega)}(0,1) = \infty$, which can be interpreted that the points $x = 0$ and $x = 1$ are disconnected;
(iii) $\rho(0) \to +\infty$, then $d_{F(\omega)}(0,1) \to 0$, which is interpreted as a "black hole" by M. Hale in Ref. [4].

In the next section, dynamics of ω will be considered; cases (i) and (ii) will be shown to emerge from classical solutions to the equations of motion for ω. We deliver two identities at the end of this section

$$F(\omega)^2 = T^-(\omega\omega^\dagger)\eta_-\eta_+ + (\omega^\dagger\omega)\eta_+\eta_- = T^-(\rho^2)\sigma_-\sigma_+ + (\rho^2)\sigma_+\sigma_-\,, \tag{1}$$

$$F(\omega)\wedge F(\omega) := T^-(\omega\omega^\dagger)\eta_-\wedge\eta_+ + (\omega^\dagger\omega)\eta_+\wedge\eta_- = \frac{1}{2}(T^-(\rho^2) - \rho^2)[\sigma_-, \sigma_+] = -\frac{1}{2}\partial^-(\rho^2)\sigma_3\,, \tag{2}$$

where $\partial^\pm f := T^\pm f - f$.

3 Vacuum Solutions: Dynamics of Dirac Operator

Three action functionals containing only $F(\omega)$ will be considered below. What we mainly concern is whether there are nontrivial ***vacuum solutions*** in the corresponding equations of motion. By a "nontrivial vacuum", we mean that the translation invariance of $\mathcal{Z}_N$ is broken by this solution.

3.1 *Trivial Case:* $S[\omega] = \mathrm{Tr}\,(F(\omega)^{2k})$, $k = 1, 2, \ldots$

Notice Eq. (1), $S[\omega] = 2\sum\rho^{2k}$. This action admits only 0-solution obviously.

3.2 *Trivial Case:* $S[\omega] = \mathrm{Tr}\,[(F(\omega)\wedge F(\omega))(F(\omega)\wedge F(\omega))]$

By Eq. (2), $S[\omega] = \frac{1}{2}\sum(\partial^-(\rho^2))^2$. The equation of motion is deduced by $\delta_\omega S[\omega] = 0$,

$$\rho(x)[2\rho(x)^2 - \rho(x+1)^2 - \rho(x-1)^2] = 0\,, \quad \forall x \in \mathcal{Z}_N\,. \tag{3}$$

If $\rho(x) \neq 0, \forall x \in \mathcal{Z}$, let $\phi = \rho^2$, equation (3) takes the form

$$\partial^+\partial^-\phi = 0\,, \tag{4}$$

which is a discretized harmonic equation $\Delta\phi = 0$ on S^1. Equation (4) admits only constant solution $\rho(x) = \rho_0$, which can be understood schematically by the ***extremal value principle*** in commutative harmonic analysis.[10] The ***on-shell*** metric satisfies $d_{F(\omega)}(,) = (1/\rho_0)d_F(,)$, which corresponds to the case (i) in the last section, namely that lattice constant is modified from 1 to $1/\rho_0$. Else if there is one x_0 such that $\rho(x_0) = 0$, then the only consistent solution is $\rho(x) = 0$ for all x.

3.3 *Nontrivial Case*

Now consider

$$S[\omega] = \frac{1}{2}\ \mathrm{Tr}\left[(F(\omega)\wedge F(\omega))(F(\omega)\wedge F(\omega)) + \frac{\alpha}{4}F(\omega)^4 - \frac{\beta}{2}F(\omega)^2\right], \tag{5}$$

in which coupling constants α, β are real numbers and are required to be not equal to zero at the same time. One can check that

$$S[\omega] = \sum \Big[\frac{1}{4}(\partial^-(\rho^2))^2 + \frac{\alpha}{4}\rho^4 - \frac{\beta}{2}\rho^2\Big].$$

The equation of motion is given by

$$\rho[2t\rho^2 - T^+(\rho^2) - T^-(\rho^2) - \beta] = 0\,, \tag{6}$$

where $t := (2+\alpha)/2$.

3.3.1 *Nonsingular Case:* $\rho(x) > 0$, $\forall x \in \mathcal{Z}_N$

Remember the definition of ϕ, then equation (6) changes form as

$$-\phi(x-1) + 2t\phi(x) - \phi(x+1) = \beta, \quad \forall x \in \mathcal{Z}_N\,. \tag{7}$$

The cyclic symmetry in Eq. (7), $\phi(x) \to \phi(x+1)$, implies that $\phi(x) = \phi_0$, $\forall x \in \mathcal{Z}_N$ in which

$$\phi_0 = \frac{\beta}{2t-2} = \frac{\beta}{\alpha}\,.$$

Note that if $\alpha = 0, \beta \neq 0$, no solution exists. Upon this solution as a background, metric fulfills relation

$$d_{F(\omega)}(,) = \sqrt{\frac{\alpha}{\beta}}\, d_F(,)\,.$$

3.3.2 *Singular Case*

The remainder of this section will be devoted to one most interesting situation in which, without losing generality, $\rho(N-1) = 0$, $\rho(x) > 0$, $x = 0, 1, \ldots, N-2$. If we find out a solution to this case, then this solution breaks translation invariance, hence being a nontrivial vacuum. Some matrix algebra has to be prepared here. Introduce a matrix sequence $\{M_n(t):\ n = 1, 2, \ldots\}$ in which,

$$M_n(t) := \begin{pmatrix} 2t & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2t & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2t & -1 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \cdots & -1 & 2t & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2t \end{pmatrix}$$

is an $n \times n$ matrix, and define

$$U_n(t) = \det(M_n(t))\,, \tag{8}$$

one can prove this iterative relation easily

$$U_0(t) := 1\,, \quad U_1(t) = t\,, \qquad U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t)\,, \quad n = 2, 3, \ldots\,. \tag{9}$$

Equation (9) shows that $U_n(t)$ are ***Chebyshev's polynomials of the second kind***, hence

$$U_n(t) = \frac{z_+^{n+1} - z_-^{n+1}}{2\sqrt{t^2-1}}\,,$$

in which $z_\pm = t \pm \sqrt{t^2-1}$.[11] So equation (8) provides Chebyshev's polynomials with another interpretation. Note that all roots of $U_n(t)$ are real and lie in $(-1, 1)$ and that $U_n(t) > 0$, if $t > 1$, $n = 0, 1, 2, \ldots$. We set $t > 1$, i.e., $\alpha > 0$ from now on.

Now let

$$\Phi = (\phi(0), \phi(1), \ldots, \phi(N-2))^T\,, \quad \mathbf{1}_V = \underbrace{(1, 1, \ldots, 1)}_{N-1}{}^T\,,$$

then equation of motion (6) can be written as $M_{N-1}(t)\Phi = \beta\mathbf{1}_V$. Assume $\beta \neq 0$ and rescale $\phi = \beta v, \Phi = \beta V$, then

$$M_{N-1}(t)V = \mathbf{1}_V\,. \tag{10}$$

Introduce notations

$$f_{i,j}(t) = \frac{U_i(t)}{U_j(t)}\,, \qquad \Sigma_n(t) = \sum_{j=0}^{n} U_j(t)\,, \qquad \psi_n = \frac{\Sigma_n(t)}{U_{n+1}(t)}\,.$$

Formal solution to Eq. (10) is

$$v(x) = \sum_{y=0}^{N-2-x} f_{x,x+y}(t)\psi_{x+y}(t)\,, \quad x = 0, 1, \ldots, N-2\,. \tag{11}$$

Since we choose $t > 1$, there is no singularity in $v(x)$. If $\beta > 0$, due to the positivity of $U_n(t)$ when $t > 1$ ($\alpha > 0$), $\phi(x) > 0, \forall x = 0, 1, \ldots, N-2$, i.e. being physically acceptable. Solutions for $N = 3, 4, 5, 6$ are listed in Appendix.

4 Discussions

The geometric interpretation of the solutions in subsection 3.3.2 is that when coupling $\alpha > 0$ and $\beta > 0$, there are nontrivial vacuums and that the metrics appear to be

$$d_{F(\omega)}(x, y) = \beta^{-1/2}[v(x)^{-1/2} + v(x+1)^{-1/2} + \cdots + v(y-1)^{-1/2}]$$

for all $x, y = 0, 1, \ldots, N-2$, $x < y$. Moreover, this statement is valid in the limit $N \to \infty$.

Interesting topics along this line are interaction between link variable ω and matter fields, “black hole” solutions, quantization, $\dim(\mathcal{H}_c) > 1$ case and higher dimensional cases. Work on these aspects is in proceeding.

Appendix: Solutions to Eq. (10) for $N = 3, 4, 5, 6$

$$N = 3: \qquad v(0) = v(1) = \frac{1}{2t-1}\,;$$

$$N = 4: \qquad v(0) = v(2) = \frac{2t+1}{4t^2-2}\,, \qquad v(1) = \frac{t+1}{2t^2-1}\,;$$

$$N = 5: \qquad v(0) = v(3) = \frac{2t}{4t^2-2t-1}\,, \qquad v(1) = v(2) = \frac{2t+1}{4t^2-2t-1}\,;$$

$$N = 6: \qquad v(0) = v(4) = \frac{4t^2+2t-1}{2t(4t^2-3)}\,, \qquad v(1) = v(3) = \frac{2t+2}{4t^2-3}\,, \qquad v(2) = \frac{4t^2+4t+1}{2t(4t^2-3)}\,.$$

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