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Study Hankel Transforms and Properties of Bessel Function via Entangled State Representation Transformation in Quantum Mechanics*

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Abstract *In Phys. Lett. A* **313** *(2003) 343 we have found that the self-reciprocal Hankel transformation (HT) is embodied in quantum mechanics by a transform between two entangled state representations of continuum variables. In this work we study Hankel transforms and properties of Bessel function via entangled state representations' transformation in quantum mechanics.*

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1 Introduction

It is well-known that *Hankel transform* is very useful in solving many linear partial differential equations with appropriate initial-boundary-value problems which have obvious physical background, such as the free vibration of a large membrance, steady temperature distribution in a semi-infinite solid with a steady heat source, axisymmetric diffusion equation, acoustic radiation equation, axisymmetric Cauchy–Poisson water wave equation, etc. *Hankel transform* (HT) of a function $f(r)$ is defined by[1]

$$\mathcal{H}_n\{f(x)\} = \tilde{f}_n(k) = \int_0^\infty r\mathrm{J}_n(kr)f(r)\,\mathrm{d}r\,, \tag{1}$$

where $\mathrm{J}_n(kr)$ is the Bessel function of order n and the integral on the right-hand side is convergent. The *inverse Hankel transform* is

$$\mathcal{H}_n^{-1}[\tilde{f}_n(k)] = \int_0^\infty k\mathrm{J}_n(kr)\tilde{f}_n(k)\,\mathrm{d}k\,, \tag{2}$$

provided that the integral exists. In Ref. [2] we have answered the question: Is there any representation's mutual transformation in quantum mechanics which is just the Hankel transform? The question can also be posed as: what kind of representation's transformation in quantum mechanics does the Hankel transform correspond to? Although the fact that Fourier transform corresponds to coordinate-momentum representations' mutual transform in quantum mechanics has been known since the birth of quantum mechanics, it was not until 2003 that the above question was posed and solved. In Ref. [2] we have found that the self-reciprocal Hankel transformation (HT) is embodied in quantum mechanics by a transform between two entangled state representations which are deduced from two mutual conjugate Einstein–Podolsky–Rosen (EPR) entangled states of continuum variables. (The concept of quantum entanglement was initiated by EPR in 1935,[3] so we name the corresponding bipartite entangled states as EPR entangled states.) The properties of Bessel function and HT can now be studied by virtue of the properties of entangled states and the Bose operators' commutative relation, which seems to be an effective algebra method. In the following we shall show them as much as possible.

2 Brief Review of Bipartite EPR Entangled States of Continuum Variables and Their Deductive Entangled States

Einstein, Podolsky and Rosen used the zero-commutator of two particles' relative position and total momentum, $[(X_1 - X_2), (P_1 + P_2)] = 0$, to explain the quantum entanglement, based on which in Refs. [4] and [5] we have constructed the common eigenvector of $X_1 - X_2$ and $P_1 + P_2$,

$$\begin{aligned}(X_1 - X_2)|\eta\rangle &= \sqrt{2}\eta_1|\eta\rangle\,,\\ (P_1 + P_2)|\eta\rangle &= \sqrt{2}\eta_2|\eta\rangle\,,\end{aligned} \tag{3}$$

where $|\eta\rangle$ is the entangled state in two-mode Fock space,

$$\begin{aligned}|\eta\rangle &= \exp\Big\{-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\Big\}|00\rangle\,,\\ \eta &= \eta_1 + \mathrm{i}\,\eta_2\,,\end{aligned} \tag{4}$$

where $|00\rangle$ is the two-mode vacuum state, $a_i^\dagger$ $(i = 1, 2)$ are related to coordinate operator X_i and momentum operator P_i by $X_i = (a_i + a_i^\dagger)/\sqrt{2}$, $P_i = (a_i - a_i^\dagger)/\sqrt{2}\,\mathrm{i}$, $|\eta\rangle$ also obeys the eigenvalue equations

$$\begin{aligned}(a_1 - a_2^\dagger)|\eta\rangle &= \eta|\eta\rangle\,,\\ (a_2 - a_1^\dagger)|\eta\rangle &= \eta^*|\eta\rangle\,.\end{aligned} \tag{5}$$

On the other hand, the common eigenvector of $X_1 + X_2$ and $P_1 - P_2$ is $|\xi\rangle$,[6]

$$\begin{aligned}|\xi\rangle &= \exp\Big\{-\frac{1}{2}|\xi|^2 + \xi a_1^\dagger + \xi^* a_2^\dagger - a_1^\dagger a_2^\dagger\Big\}|00\rangle\,,\\ \xi &= \xi_1 + \mathrm{i}\,\xi_2\,,\end{aligned} \tag{6}$$

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$$(X_1+X_2)|\xi\rangle=\sqrt{2}\xi_1|\xi\rangle\,,\quad (P_1-P_2)|\xi\rangle=\sqrt{2}\xi_2|\xi\rangle\,. \tag{7}$$

$|\xi\rangle$ also satisfies the equations

$$(a_1+a_2^\dagger)|\xi\rangle=\xi|\xi\rangle,\quad (a_2+a_1^\dagger)|\xi\rangle=\xi^*|\xi\rangle\,. \tag{8}$$

Note that both $|\eta\rangle$ and $|\xi\rangle$ are complete and orthonormal,

$$\int\frac{\mathrm{d}^2\eta}{\pi}|\eta\rangle\langle\eta|=1\,,\quad \langle\eta|\eta'\rangle=\pi\delta(\eta-\eta')\delta(\eta^*-\eta'^*)\,,\quad \mathrm{d}^2\eta=\mathrm{d}\eta_1\,\mathrm{d}\eta_2\,, \tag{9}$$

$$\int\frac{\mathrm{d}^2\xi}{\pi}|\xi\rangle\langle\xi|=1\,,\quad \langle\xi|\xi'\rangle=\pi\delta(\xi-\xi')\delta(\xi^*-\xi'^*)\,, \tag{10}$$

so they are qualified to be two mutual conjugate representations. By "mutual conjugate" we imply the canonical commutative relation

$$[(X_1-X_2),\ (P_1-P_2)]=2\mathrm{i}\,,\quad [(X_1+X_2),\ (P_1+P_2)]=2\mathrm{i}\,, \tag{11}$$

and the overlap

$$\langle\eta|\xi\rangle=\frac{1}{2}\exp\Big[\frac{1}{2}(\eta^*\xi-\eta\xi^*)\Big]\,, \tag{12}$$

where $(\eta^*\xi-\eta\xi^*)$ is a pure imaginary. Thus in the "language" of quantum mechanics, writing $g(\eta)=\langle\eta|g\rangle$, $g(\xi)=\langle\xi|g\rangle$, the following representation transform is a 2-dimensional Fourier transform, i.e.

$$\langle\eta|g\rangle=\int\frac{\mathrm{d}^2\xi}{\pi}\langle\eta|\xi\rangle\langle\xi|g\rangle=\frac{1}{2}\int\frac{\mathrm{d}^2\xi}{\pi}\,\mathrm{e}^{(\eta^*\xi-\eta\xi^*)/2}\langle\xi|g\rangle\,. \tag{13}$$

In Ref. [7] we have introduced a new state vector $|q,r\rangle$ through the following U(1) phase integral over $|\eta=r\,\mathrm{e}^{\mathrm{i}\theta}\rangle$,

$$|q,r\rangle=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta|\eta=r\,\mathrm{e}^{\mathrm{i}\theta}\rangle\,\mathrm{e}^{-\mathrm{i}\,q\theta}\,. \tag{14}$$

Since it is deduced from $|\eta\rangle$, so we name it the deductive entangled states. It then turns out that $|q,r\rangle$ is the common eigenvector of the two-mode number-difference operator $a_1^\dagger a_1-a_2^\dagger a_2\equiv Q$ and $(a_1-a_2^\dagger)(a_1^\dagger-a_2)\equiv K$, $[Q,K]=0$,

$$(a_1^\dagger a_1-a_2^\dagger a_2)|q,r\rangle=q|q,r\rangle\,, \tag{15}$$

$$K|q,r\rangle=r^2|q,r\rangle\,. \tag{16}$$

(Note that in Ref. [7] $|q,r\rangle$ was denoted as $|q,k\rangle$, $\sqrt{k}=|\eta|$, which is a different convention). We have proved that $|q,r\rangle$ also makes up a representation,

$$\sum_{q=-\infty}^{\infty}\int_0^\infty \mathrm{d}(r^2)|q,r\rangle\langle q,r|=1\,, \tag{17}$$

$$\langle q,r|q',r'\rangle=\delta_{q,q'}\delta(r^2-r'^2)=\delta_{q,q'}\frac{1}{2r}\delta(r-r')\,. \tag{18}$$

Due to

$$[Q,(a_1^\dagger+a_2)(a_1+a_2^\dagger)]=0\,, \tag{19}$$

so from $|\xi\rangle$, which is conjugate to $|\eta\rangle$, we can also deduce

$$|s,r'\rangle=\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi|\xi=r'\,\mathrm{e}^{\mathrm{i}\varphi}\rangle\,\mathrm{e}^{-\mathrm{i}s\varphi}\,, \tag{20}$$

which simultaneously obeys

$$(a_1^\dagger a_1-a_2^\dagger a_2)|s,r'\rangle=s|s,r'\rangle\,,\quad (a_1^\dagger+a_2)(a_1+a_2^\dagger)|s,r'\rangle=r'^2|s,r'\rangle\,. \tag{21}$$

$|\xi\rangle$ is also complete

$$\sum_{s=-\infty}^{\infty}\int_0^\infty \mathrm{d}(r'^2)|s,r'\rangle\langle s,r'|=1\,,\quad \langle s,r'|s',r''\rangle=\delta_{s,s'}\frac{1}{2r'}\delta(r'-r'')\,. \tag{22}$$

Both $|q,r\rangle$ and $|s,r'\rangle$ are called charge-amplitude entangled states, as the operator $a_1^\dagger a_1-a_2^\dagger a_2$ is usually named as the charge operator when a positive (negative) charge quantum is assigned to a_1 (a_2). Note that both the integrals in Eq. (14) and in Eq. (20) can be performed thoroughly, and the explicit forms of $|s,r'\rangle$ and $|q,r\rangle$ can be obtained, but in this work we do not need to have these forms.

3 Matrix Element $\langle s,r'|q,r\rangle$ as Integral Kernel of Hankel Transform

Using the generating function of the Bessel function[8]

$$\mathrm{e}^{\mathrm{i}x\sin t}=\sum_{m=-\infty}^{\infty}\mathrm{J}_m(x)\,\mathrm{e}^{\mathrm{i}mt}\,, \tag{23}$$

where $\mathrm{J}_\nu(x)$ is the Bessel function of ν-order, and ν is an integer,

$$\mathrm{J}_m(x)=\sum_{k=0}^{\infty}\frac{(-1)^m}{k!(m+k)!}\Big(\frac{x}{2}\Big)^{m+2k}\,. \tag{24}$$

From Eqs. (14) and (20) we calculate $\langle s,r'|q,r\rangle$ by performing the following integrals:

$$\begin{aligned}\langle s,r'|q,r\rangle&=\frac{1}{4\pi^2}\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}s\varphi}\langle\xi=r'\,\mathrm{e}^{\mathrm{i}\varphi}|\int_0^{2\pi}\mathrm{d}\theta|\eta=r\,\mathrm{e}^{\mathrm{i}\theta}\rangle\,\mathrm{e}^{-\mathrm{i}\,q\theta}\\&=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\mathrm{e}^{-\mathrm{i}\,q\theta}\,\mathrm{e}^{\mathrm{i}s\varphi}\exp[\mathrm{i}rr'\sin(\theta-\varphi)]\,\mathrm{d}\,\theta\,\mathrm{d}\varphi\\&=\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}s\varphi}\,\mathrm{e}^{-\mathrm{i}\,q\theta}\sum_{m=-\infty}^{\infty}\mathrm{J}_m(rr')\exp[\mathrm{i}m(\theta-\varphi)]\\&=\frac{1}{2}\sum_{m=-\infty}^{\infty}\delta_{m,q}\delta_{m,s}\mathrm{J}_m(rr')=\frac{1}{2}\delta_{s,q}\mathrm{J}_s(rr')\,.\end{aligned} \tag{25}$$

So the matrix element $\langle s,r'|q,r\rangle$ is just proportional to the Bessel function, which is a remarkable thing. When $s=\nu$, $r'=1$, $q=\nu$, $r=x$, equation (25) becomes

$$\langle\nu,1|\nu,x\rangle=\frac{1}{2}\,\mathrm{J}_\nu(x)\,, \tag{26}$$

which is the standard Bessel function. By defining

$$\langle q,r|g\rangle=g(q,r),\qquad \langle s,r'|g\rangle=\mathcal{G}(s,r')\,, \tag{27}$$

and using Eq. (22) we see

$$\mathcal{G}(s,r')\equiv\langle s,r'|g\rangle=\sum_{q=-\infty}^{\infty}\int_0^\infty\mathrm{d}(r^2)\langle s,r'|q,r\rangle\langle q,r|g\rangle=\frac{1}{2}\int_0^\infty\mathrm{d}(r^2)\mathrm{J}_s(rr')g(s,r)\equiv\mathcal{H}[\,g(s,r)]\,, \tag{28}$$

which is just the Hankel transform of $g(q,r)$ as mathematically defined in Eq. (6). The inverse transform of Eq. (28) is

$$\langle q,r|g\rangle=\sum_{s=-\infty}^{\infty}\int_0^\infty\mathrm{d}(r'^2)\langle q,r|s,r'\rangle\langle s,r'|g\rangle=\frac{1}{2}\int_0^\infty\mathrm{d}(r'^2)\mathrm{J}_s(rr')g(q,r')\equiv\mathcal{H}^{-1}[\,g(q,r')]\,. \tag{29}$$

Thus equations (28) and (29) are just the mutual Hankel transform. Now we can make full use of entangled state representation in quantum mechanics to study Hankel transforms.

4 Study Hankel Transforms via Entangled State Representations and Bose Operator Algebra

From Eqs. (15) and (21) we have

$$\langle\nu,1|(a_1^\dagger+a_2)(a_1+a_2^\dagger)(a_1-a_2^\dagger)(a_1^\dagger-a_2)|\nu,x\rangle=x^2\langle\nu,1|\nu,x\rangle\,, \tag{30}$$

$$\langle\nu,1|(a_1^\dagger a_1-a_2^\dagger a_2)^2|\nu,x\rangle=v^2\langle\nu,1|\nu,x\rangle\,. \tag{31}$$

Next, we recall that the two-mode squeezing operator $\exp[\lambda(a_1^\dagger a_2^\dagger-a_1a_2)]$[9] multiplied by e^λ has a neat representation[5]

$$S(\lambda)\equiv\exp[\lambda(a_1^\dagger a_2^\dagger-a_1a_2+1)]=\int\frac{\mathrm{d}^2\eta}{\pi}|\eta/\mu\rangle\langle\eta|,\quad \mu=\mathrm{e}^\lambda\,, \tag{32}$$

which means that $S(\lambda)$ actually squeezes the $|\eta\rangle$ state (or the $|\xi\rangle$ state in an opposite way), $S(\lambda)|\eta\rangle=|\eta/\mu\rangle$. Thus from Eq. (14) we see

$$S(\lambda)|q,r\rangle=|q,r/\mu\rangle\quad\text{or}\quad S(\lambda)|\nu,x\rangle=|\nu,x/\mu\rangle\,. \tag{33}$$

It then follows that

$$\begin{aligned}(a_1^\dagger a_2^\dagger-a_1a_2+1)^2|\nu,x\rangle&=\frac{\partial^2}{\partial\lambda^2}\,\mathrm{e}^{\lambda(a_1^\dagger a_2^\dagger-a_1a_2+1)}|_{\lambda=0}|\nu,x\rangle\\&=\frac{\partial^2}{\partial\lambda^2}S(\lambda)|\nu,x\rangle|_{\lambda=0}=\frac{\partial^2}{\partial\lambda^2}|\nu,x\,\mathrm{e}^{-\lambda}\rangle|_{\lambda=0}=\frac{\partial}{\partial\lambda}\Big[-x\,\mathrm{e}^{-\lambda}\frac{\partial}{\partial(x\,\mathrm{e}^{-\lambda})}|\nu,x\,\mathrm{e}^{-\lambda}\rangle\Big]|_{\lambda=0}\\&=\Big[(x\,\mathrm{e}^{-\lambda})^2\frac{\partial^2}{\partial(x\,\mathrm{e}^{-\lambda})^2}+x\,\mathrm{e}^{-\lambda}\frac{\partial}{\partial(x\,\mathrm{e}^{-\lambda})}\Big]|\nu,x\,\mathrm{e}^{-\lambda}\rangle|_{\lambda=0}=\Big(y^2\frac{\mathrm{d}^2}{\mathrm{d}y^2}+y\frac{\mathrm{d}}{\mathrm{d}y}\Big)|\nu,y\rangle|_{y=x\,\mathrm{e}^{-\lambda},\,\lambda=0}\\&=\Big(x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\frac{\mathrm{d}}{\mathrm{d}x}\Big)|\nu,x\rangle\,.\end{aligned} \tag{34}$$

Note the operator identity

$$(a_1^\dagger a_2^\dagger-a_1a_2+1)^2-(a_1^\dagger a_1-a_2^\dagger a_2)^2=-(a_1^\dagger+a_2)(a_1+a_2^\dagger)(a_1-a_2^\dagger)(a_1^\dagger-a_2)\,, \tag{35}$$

and from Eqs. (30), (34), and (31) we have

$$(a_1^\dagger+a_2)(a_1+a_2^\dagger)|\nu,x\rangle=-\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{\nu^2}{x^2}\Big)|\nu,x\rangle\,. \tag{36}$$

Now if we want to know the Hankel transform of

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{\nu^2}{x^2}\Big)|\nu,x\rangle\equiv K(x)\,,$$

whose definition is $\int_0^\infty K(x)J_\nu(x'x)x\,\mathrm{d}x$, we do not need to perform this integration by parts as usual, because this requires the recursive relations of Bessel function and quite a few derivation steps. Instead, we can use Eq. (36), the second equation of Eqs. (15) and (17) to write

$$\begin{aligned}\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{\nu^2}{x^2}\Big)|\nu,x\rangle&=-(a_1^\dagger+a_2)(a_1+a_2^\dagger)\sum_{s=-\infty}^{\infty}\int_0^\infty 2x'\,\mathrm{d}x'|s,x')(s,x'|\nu,x\rangle\\&=-\sum_{s=-\infty}^{\infty}\int_0^\infty 2x'\,\mathrm{d}x'x'^2\frac{1}{2}\delta_{s,\nu}\mathrm{J}_s(xx')|s,x')=-\int_0^\infty x'\,\mathrm{d}x'\mathrm{J}_\nu(xx')x'^2|\nu,x')\,.\end{aligned} \tag{37}$$

The right-hand side is just the Hankel transform of $-x'^2|\nu,x')$, thus its reciprocal is

$$\int_0^\infty \Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{\nu^2}{x^2}\Big)|\nu,x\rangle \mathrm{J}_\nu(x'x)x\,\mathrm{d}x=-x'^2|\nu,x')\,. \tag{38}$$

We have known that

$$\Big(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}-\frac{\nu^2}{x^2}\Big)|\nu,x\rangle$$

and $-x'^2|\nu,x')$ are mutually Hankel transforms. Hence our method leads to simplification in calculating the Hankel transform, this transform is often used to significantly simplify procedures in solving the Laplace equation, say, the heat conduction equation or the electrostatic potential equation, in which the Bessel equation is involved. Equations (35) and (36) also lead to the standard Bessel equation[10]

$$\begin{aligned}&(\nu,1|[(a_1^\dagger a_2^\dagger-a_1a_2+1)^2+(a_1^\dagger+a_2)(a_1+a_2^\dagger)(a_1-a_2^\dagger)(a_1^\dagger-a_2)-(a_1^\dagger a_1-a_2^\dagger a_2)^2]|\nu,x\rangle\\&\quad=\Big[x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2}+x\frac{\mathrm{d}}{\mathrm{d}x}+(x^2-\nu^2)\Big](\nu,1|\nu,x\rangle=0\,,\end{aligned} \tag{39}$$

where $(\nu,1|\nu,x\rangle$ is just the Bessel function, so Bessel equation automatically appears as the matrix element of the operator identity (35), and we do not need to solve it.

From the new definition of Hankel transform in terms of quantum mechanical representation transforms, one can easily notice the consanguineous relations between the Hankel transform and the Bessel function. The new form of Bessel function in Eq. (25) can simplify some integral calculations which usually need much work.

4.1 *Deduction of Closure Equation of Bessel Function*

Now using the above results we derive the Bessel function's **Closure** equation.[11] Note

$$\mathrm{J}_\nu(\alpha\rho)=2(\nu,\alpha|\nu,\rho\rangle=2\sum_{\nu'=-\infty}^{\infty}(\nu,\alpha|\nu',\rho\rangle\,, \tag{40}$$

where α, α', ρ are real numbers, so $\mathrm{J}_\nu(\alpha\rho)$ is also a real function, thus

$$\mathrm{J}_{\nu'}(\alpha'\rho)=[\mathrm{J}_{\nu'}(\alpha'\rho)]^*=2\sum_{\nu''=-\infty}^{\infty}\langle\nu'',\rho|\nu,\alpha')\,. \tag{41}$$

Using Eqs. (40) and (41) we examine

$$\int_0^\infty \mathrm{J}_\nu(\alpha\rho)\mathrm{J}_\nu(\alpha'\rho)\rho\,\mathrm{d}\,\rho=2\int_0^\infty\sum_{\nu'=-\infty}^{\infty}(\nu,\alpha|\nu',\rho\rangle\sum_{\nu''=-\infty}^{\infty}\langle\nu'',\rho|\nu,\alpha')\delta_{v''v'}2\rho\rho \tag{42}$$

$$=2(\nu,\alpha|\Big[\int_0^\infty 2\rho\mathrm{d}\,\rho\sum_{\nu'=-\infty}^{\infty}\sum_{\nu''=-\infty}^{\infty}|\nu',\rho\rangle\langle\nu'',\rho|\delta_{v''v'}\Big]|\nu,\alpha') \tag{43}$$

$$=2(\nu,\alpha|\nu,\alpha'\rangle=\frac{1}{\alpha}\delta(\alpha-\alpha')\,, \tag{44}$$

which is just the **Closure** equation of Bessel function, where we have used

$$\int_0^\infty 2\rho\,\mathrm{d}\rho\sum_{\nu'=-\infty}^{\infty}\sum_{\nu''=-\infty}^{\infty}|\nu',\rho\rangle\langle\nu'',\rho|\delta_{v''v'}=1\,. \tag{45}$$

Equation (42) is useful in developing Green's functions in cylindrical coordinates, where the eigenfunctions are Bessel functions. Comparing with the pure mathematical deduction, our method is much more convenient.

4.2 *Derivation of Bessel Function's Wronskian Formulas*

We now derive some recurrence relations of the Bessel function with the introduction of the two kinds of entangled states. Here we prove

$$\mathrm{J}_{n-1}(x)=\frac{n}{x}\mathrm{J}_n(x)+\mathrm{J}_n'(x),\qquad \mathrm{J}_{n+1}(x)=\frac{n}{x}\mathrm{J}_n(x)-\mathrm{J}_n'(x) \tag{46}$$

with the help of the formulas

$$\begin{aligned}&(n,1|(a_1^\dagger+a_2)=(n-1,1|,\quad (a_1-a_2^\dagger)|n,x\rangle=x|n-1,x\rangle,\quad (a_1^\dagger a_1-a_2^\dagger a_2)|n,x\rangle=n|n,x\rangle\,,\\&(n,1|(a_1^\dagger a_2^\dagger-a_1a_2+1)|n,x\rangle=-\frac{1}{2}x\frac{\partial}{\partial x}\mathrm{J}_n(x)\,,\end{aligned} \tag{47}$$

and sandwiching the operator identity

$$(a_1^\dagger+a_2)(a_1-a_2^\dagger)=(a_1^\dagger a_1-a_2^\dagger a_2)-(a_1^\dagger a_2^\dagger-a_1a_2+1) \tag{48}$$

between $(n,1|\cdots|n,x\rangle$, it follows that

$$(n,1|(a_1^\dagger+a_2)(a_1-a_2^\dagger)|n,x\rangle=\frac{1}{2}x\mathrm{J}_{n-1}(x)=(n,1|[(a_1^\dagger a_1-a_2^\dagger a_2)-(a_1^\dagger a_2^\dagger-a_1a_2+1)]|n,x\rangle$$
$$=\frac{1}{2}n\mathrm{J}_n(x)+\frac{1}{2}x\frac{\partial}{\partial x}\mathrm{J}_n(x)\,, \tag{49}$$

which is just Eq. (44). Similarly, using

$$(a_1+a_2^\dagger)(a_1^\dagger-a_2)=(a_1^\dagger a_1-a_2^\dagger a_2)+(a_1^\dagger a_2^\dagger-a_1a_2+1)\,, \tag{50}$$

we can prove Eq. (45). Using the same method we can also prove

$$\mathrm{J}_n(x)=(-1)^n x^n\Big(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\Big)^n\mathrm{J}_0(x)\,, \tag{51}$$

By now, we have seen that Bessel function's recurrence relations correspond to some operator identities' entangled state matrix elements in the frame of quantum mechanics. The fact once again convinces us that the Bessel function can be understood not only as a special function, but also can be recognized as an overlap of two particular quantum mechanical representations and that its properties are just natural behavior in the representations.

4.3 *Wronskian Formulas for Bessel Functions*

Among the properties of Bessel function, there are two important equations, which are called *Wronskian Formulas*,[11]

$$\mathrm{J}_v(x)\mathrm{J}_{-v+1}(x)+\mathrm{J}_{-v}(x)\mathrm{J}_{v-1}(x)=\frac{2\sin v\pi}{\pi x},\quad \mathrm{J}_v(x)\mathrm{J}_{-v-1}(x)+\mathrm{J}_{-v}(x)\mathrm{J}_{v+1}(x)=-\frac{2\sin v\pi}{\pi x}\,. \tag{52}$$

They can also be proved with our method. We pay a little attention to the case of $x=0$, because of the uncertainty of $0/0$ as well as the simplicity of $\mathrm{J}_v(0)$, ($\mathrm{J}_v(0)=0$, when $v\neq0$; $\mathrm{J}_0(0)=1$). We only consider the case of $x\neq0$, where the *Wronskian Formulas* become

$$\mathrm{J}_v(x)\mathrm{J}_{-v+1}(x)+\mathrm{J}_{-v}(x)\mathrm{J}_{v-1}(x)=0,\qquad \mathrm{J}_v(x)\mathrm{J}_{-v-1}(x)+\mathrm{J}_{-v}(x)J_{v+1}(x)=0\,, \tag{53}$$

because v is an integer. Let

$$A=a_1^\dagger+a_2,\qquad B=a_1+a_2^\dagger,\qquad C=a_1-a_2^\dagger,\qquad D=a_1^\dagger-a_2\,,$$
$$W=a_1^\dagger a_2^\dagger-a_1a_2+1,\qquad V=a_1^\dagger a_1-a_2^\dagger a_2,\qquad BD=V+W,\qquad AC=V-W\,. \tag{54}$$

Operating them on the entangled states, we get

$$A|v,r')=r'|v+1,r'),\qquad B|v,r')=r'|v-1,r'),\qquad C|v,r\rangle=r|v-1,r\rangle,\qquad D|v,r\rangle=r|v+1,r\rangle\,,$$
$$W|\nu,r\rangle=-r\frac{\mathrm{d}}{\mathrm{d}r}|\nu,r\rangle,\qquad V|\nu,r')=v|\nu,r')\,. \tag{55}$$

If $r\neq0$, $r'\neq0$, we have

$$(v-1,1|v-1,r\rangle=(v,1|\frac{AC}{r}|v,r\rangle,\qquad(-v+1,1|-v+1,r\rangle=(-v,1|\frac{BD}{r}|-v,r\rangle\,. \tag{56}$$

Then replacing r with x we obtain

$$\begin{aligned}\mathrm{J}_v(x)\mathrm{J}_{-v+1}(x)+\mathrm{J}_{-v}(x)\mathrm{J}_{v-1}(x)&=4(v,1|v,x\rangle(-v,1|\frac{BD}{x}|-v,x\rangle+4(v,1|\frac{AC}{x}|v,x\rangle(-v,1|-v,x\rangle\\&=4(v,1|v,x\rangle(-v,1|\frac{1}{x}(V+W)|-v,x\rangle+4(v,1|\frac{1}{x}(V-W)|v,x\rangle(-v,1|-v,x\rangle\\&=4(v,1|v,x\rangle(-v,1|\frac{1}{x}\Big(-v-x\frac{\mathrm{d}}{\mathrm{d}x}\Big)|-v,x\rangle+4(v,1|\frac{1}{x}\Big(v+x\frac{\mathrm{d}}{\mathrm{d}x}\Big)|v,x\rangle(-v,1|-v,x\rangle\\&=\frac{4}{x}\Big(-v-x\frac{\mathrm{d}}{\mathrm{d}x}\Big)(v,1|v,x\rangle(-v,1|-v,x\rangle+\frac{4}{x}\Big(v+x\frac{\mathrm{d}}{\mathrm{d}x}\Big)(v,1|v,x\rangle(-v,1|-v,x\rangle\\&=\frac{4}{x}\Big[\Big(-v-x\frac{\mathrm{d}}{\mathrm{d}x}\Big)+\Big(v+x\frac{\mathrm{d}}{\mathrm{d}x}\Big)\Big](v,1|v,x\rangle(-v,1|-v,x\rangle=0\,.\end{aligned} \tag{57}$$

Similarly, we can derive the second Wronskian Formula.

5 Other Applications

Beginning with the definition (1), we perform the following Hankel transform:

$$\begin{aligned}\mathcal{H}_1\Big\{\frac{\exp(-zr)}{r}\Big\}&=2\Big(\frac{1}{2\pi}\Big)^2\int_0^\infty \mathrm{e}^{-zr}\Big[\int_0^{2\pi}\mathrm{d}\varphi(\xi=k\,\mathrm{e}^{\mathrm{i}\varphi}|\,\mathrm{e}^{\mathrm{i}\varphi}\int_0^{2\pi}\mathrm{d}\theta|\eta=r\,\mathrm{e}^{\mathrm{i}\theta}\rangle\,\mathrm{e}^{-\mathrm{i}\theta}\Big]\mathrm{d}r\\&=\Big(\frac{1}{2\pi}\Big)^2\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\frac{\mathrm{e}^{-\mathrm{i}(\theta-\varphi)}}{z-\mathrm{i}k\sin(\theta-\varphi)}\,.\end{aligned}$$

On the other hand, we have known that[9]

$$\mathcal{H}_1\Big\{\frac{\exp(-zr)}{r}\Big\} = \int_0^\infty \mathrm{e}^{-zr}\mathrm{J}_1(kr)\mathrm{d}r = \frac{1}{k}\Big(1 - \frac{z}{\sqrt{k^2+z^2}}\Big)\,. \tag{58}$$

Comparing Eqs. (57) and (58) we get a new integral formula,

$$\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\frac{\mathrm{e}^{-\mathrm{i}(\theta+\varphi)}}{z-\mathrm{i}k\sin(\theta-\varphi)} = \frac{4\pi^2}{k}\Big(1-\frac{z}{\sqrt{k^2+z^2}}\Big)\,. \tag{59}$$

More generally, we have

$$\mathcal{H}_v\Big\{\frac{\exp(-zr)}{r}\Big\} = \Big(\frac{1}{2\pi}\Big)^2\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\frac{\mathrm{e}^{-\mathrm{i}v(\theta-\varphi)}}{z-\mathrm{i}k\sin(\theta-\varphi)}\,, \tag{60}$$

where $\mathcal{H}_v\{\exp(-zr)/r\}$ has been computed in Ref. [11],

$$\mathcal{H}_v\Big\{\frac{\exp(-zr)}{r}\Big\} = \frac{1}{\sqrt{z^2+k^2}}\Big(\frac{k}{z+\sqrt{z^2+k^2}}\Big)^v\,. \tag{61}$$

Therefore we gain a complicated integral formula,

$$\Big(\frac{1}{2\pi}\Big)^2\int_0^{2\pi}\mathrm{d}\varphi\int_0^{2\pi}\mathrm{d}\theta\frac{\mathrm{e}^{-\mathrm{i}v(\theta-\varphi)}}{z-\mathrm{i}k\sin(\theta-\varphi)} = \frac{1}{\sqrt{z^2+k^2}}\Big(\frac{k}{z+\sqrt{z^2+k^2}}\Big)^v\,. \tag{62}$$

Using the entangled states we can also derive Bessel's integral, i.e

$$\mathrm{J}_v(x) = \frac{1}{\pi}\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{-\mathrm{i}v\theta}\sum_{v'=-\infty}^{\infty}\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}v'\varphi}\frac{1}{2}\,\mathrm{e}^{\mathrm{i}x\sin(\theta-\varphi)} \tag{63}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{-\mathrm{i}v\theta}\sum_{v'=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\mathrm{J}_m(x)\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\varphi\,\mathrm{e}^{\mathrm{i}(v'-m)\varphi}\,\mathrm{e}^{\mathrm{i}m\theta} \tag{64}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\,\mathrm{e}^{-\mathrm{i}v\theta}\sum_{v'=-\infty}^{\infty}\mathrm{J}_{v'}(x)\,\mathrm{e}^{\mathrm{i}v'\theta} = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{e}^{\mathrm{i}(x\sin\theta-v\theta)}\mathrm{d}\theta \tag{65}$$

$$= \frac{1}{2\pi\mathrm{i}}\int_c \mathrm{e}^{(x/2)(t-1/t)}t^{-v-1}\mathrm{d}t\,, \tag{66}$$

where the integral path C is a unit counterclockwise circle. Some generalized Bessel equations were obtained in Ref. [12]

6 Conclusion

Based on the identification of $\mathrm{J}_\nu(x)$ as the transform matrix element between two mutual conjugate entangled state representations $(\nu, 1|\nu, x\rangle/2$, we have re-derived various properties of Bessel function such as Closure equation, recurrence relations and Wronskian Formula with the help of Bose operator identities and entangled state representations, which once more convinces, the validity of our method. The Hankel transform now can be studied by virtue of the entangled state representations, with which we have obtained a series of integral formulas as by-products. All these again show the importance and usefulness of the entangled states.

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